Unified Hierarchy of Fluctuation-Response Identities & Bounds in Nonequilibrium Markovian Dynamics

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Main reference: Kwon, Chun, Park, JSL, PRL 135, 097101 (2025)

Chun, Kwon, Park, JSL, arXiv:2601.16387

Kwon, Chun, Park, JSL, arXiv:2605.05038

in collaboration with Hyunggyu Park (KIAS)

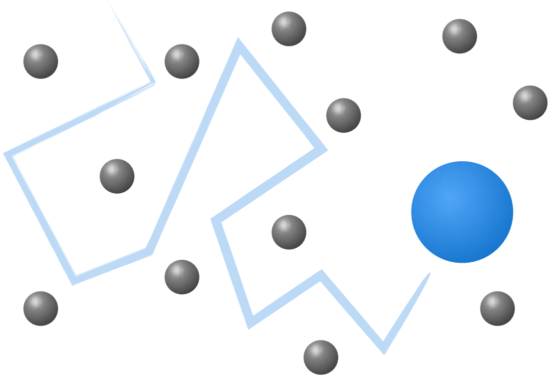
Hyun-Myung Chun (KIAS)

Euijoon Kwon (KIAS)

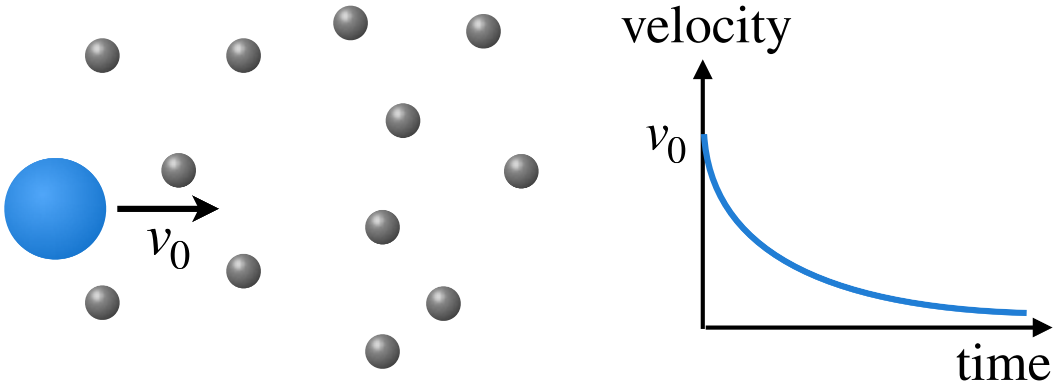
Fluctuation - Dissipation - Response Relations

Effects of a thermal environment

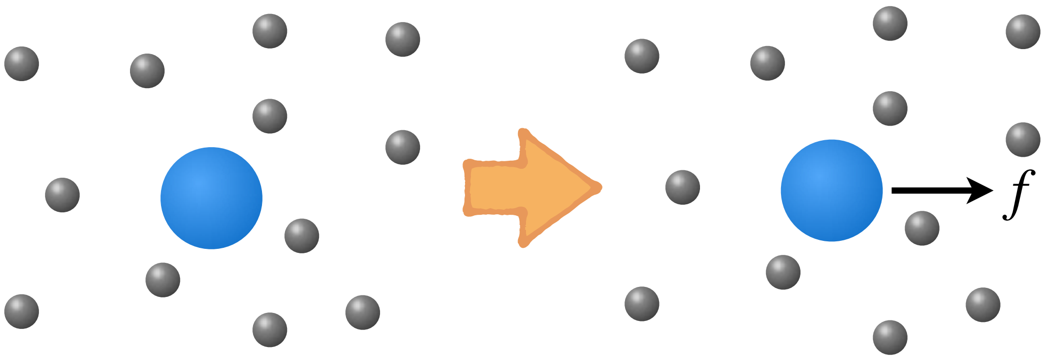
1. Stochastic motion (fluctuation)



2. Dissipation



3. Response to a perturbation



$\langle v \rangle_{ss} = 0$ $\qquad$ $\langle v \rangle_{ss} = \mu f$

μ : mobility, determined from bath properties

Fluctuation - Dissipation - Response Relations

Effects of a thermal environment

1. Stochastic motion (fluctuation)

2. Dissipation

3. Response to a perturbation → These have the same origin.

There must be relations between them.

In equilibrium

Their mathematical relationships (linear response) are well established.

fluctuation

response $$\mu = \frac{1}{k_B T}\int_0^\infty \langle v(t)v(0)\rangle_{eq} dt = \frac{1}{\gamma}$$ dissipation

Green-Kubo

For a free Brownian particle in Langevin dynamics

$$\langle J\rangle = Lf \qquad L = \alpha\int_0^\infty \langle J(t)J(0)\rangle_{eq} dt$$

Fluctuation-Response relation

: provides important insight into the response properties of complex systems.

Fluctuation - Dissipation - Response Relations in Nonequilibrium

Previous studies

1. Harada-Sasa relation: $\langle J \rangle_0 = \gamma \left\{ v_s^2 + \int_{-\infty}^{\infty} \left[\tilde{C}(\omega) - 2TR'(\omega) \right] \frac{d\omega}{2\pi} \right\}$

Harada and Sasa, PRL 95, 130602 (2005)

$\langle J \rangle_0$ ↑ **dissipation**; $\tilde{C}(\omega)$ ↑ **fluctuation**; $R'(\omega)$ ↑ **response**

→ (violation of FDT) ∝ (heat dissipation)

2. Nonequilibrium FDT   Baiesi, Maes, Wynants, PRL 103, 010602 (2009)

$(t > s)$

$$R_{QV}(t,s) = \frac{\delta \langle Q(t) \rangle}{\delta h_s} = \frac{\beta}{2} \frac{d}{ds} \langle V(s) Q(t) \rangle - \frac{1}{2} \langle [L^\dagger V(s)] Q(t) \rangle = \beta \frac{d}{ds} \langle V(s) Q(t) \rangle$$

$[L^\dagger V(s)]$: average of $\frac{dV}{ds}$ at x_s

equilibrium limit → (equilibrium FDT)

$U \to U - h_s V$ (small potential perturbation)

$Q(t)$: observable at time t

3. TUR (Thermodynamic Uncertainty Relation)

$$\frac{\mathrm{Var}[\Theta_\tau]}{\langle \Theta_\tau \rangle^2} \langle \Delta S_{\mathrm{tot}} \rangle \geq 2k_B$$

fluctuation → $\mathrm{Var}[\Theta_\tau]$; **Dissipation** → $\langle \Delta S_{\mathrm{tot}} \rangle$

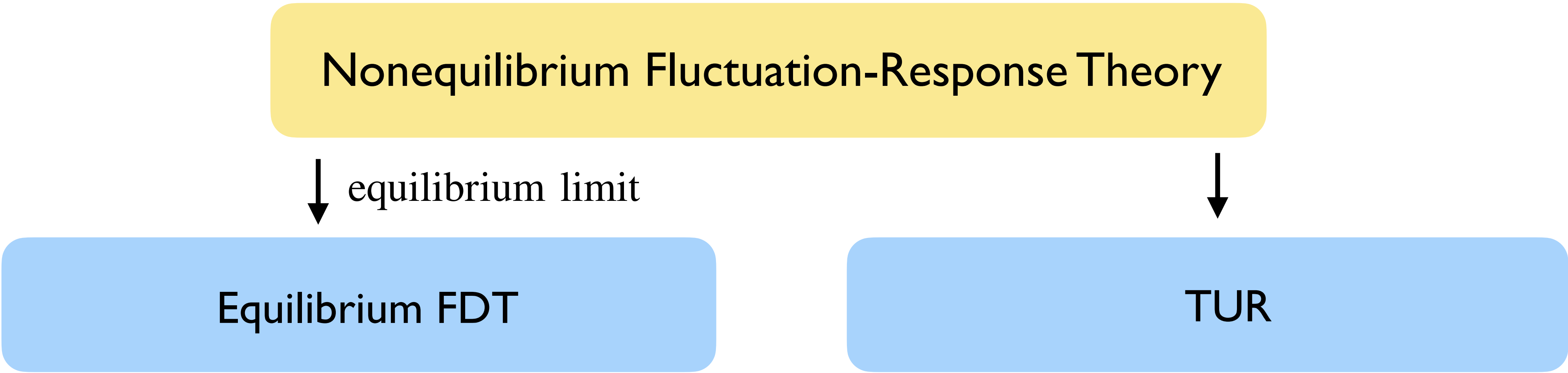
ΔS_{tot} : entropy production

Θ_t : observable at time t

→ These theories have been studied independently.
A clear link between them has not been found.

Goal of This Study

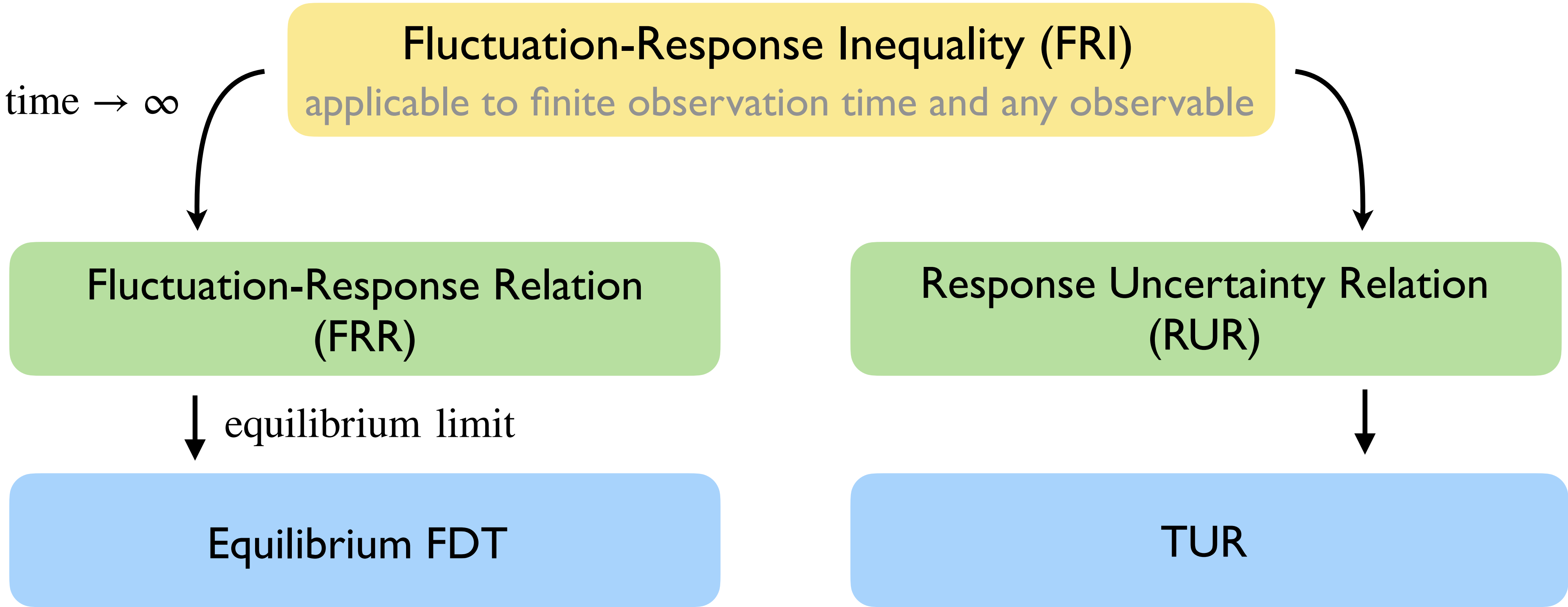
→ Establish such a unified theoretical framework in nonequilibrium systems



Summary of the Main Results

Kwon, Chun, Park, JSL, PRL 135, 097101 (2025)
Chun, Kwon, Park, JSL, arXiv:2601.16387

1. Classical Stochastic System for both Markov-jump & Langevin systems



2. Extension to Open Quantum System

: Quantum Fluctuation-Response Inequality (QFRI)

Setup

Continuous-time Markov jump System

Master equation

$$\dot{p}_i(t) = \sum_{j(\neq i)} \left(W_{ij} p_j(t) - W_{ji} p_i(t) \right) \quad \Rightarrow \quad \lim_{t\to\infty} p_i(t) = \pi_i : \text{steady-state probability}$$

Transition rate: $W_{ij} = \exp\left[B_{ij}(\epsilon) + \frac{F_{ij}(\eta)}{2} \right]$

$B_{ij} = B_{ji}$: symmetric part

$F_{ij} = -F_{ji}$: antisymmetric part

Observable $\Theta(\tau) = \int_0^\tau dt \left[\sum_i g_i h_i(t) + \sum_{i\neq j} \Lambda_{ij} \dot{N}_{ij}(t) \right]$

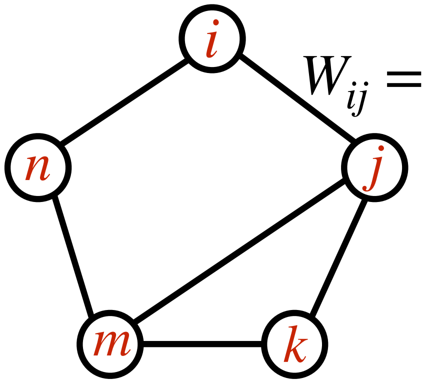
$h_i(t) = \delta_{s(t),i}$ $\quad s(t)$: state of the system at time t

$N_{ij}(t)$: accumulated number of jumps $j \to i$

If $g_i = 0$ and $\Lambda_{ij} = -\Lambda_{ji}$, Θ is a current-like observable.

If $\Lambda_{ij} = 0$, Θ is a state-dependent observable.

Two Ways of Perturbing W_{ij}



$$W_{ij} = \exp\left[B_{ij}(\epsilon) + \frac{F_{ij}(\eta)}{2}\right]$$

$B_{ij} = B_{ji}$: symmetric part

$F_{ij} = -F_{ji}$: antisymmetric part

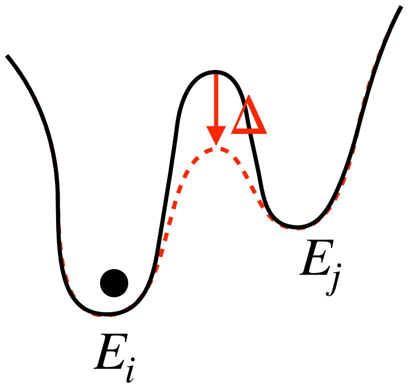
1. Kinetic perturbation: Perturb B_{ij}

2. Entropic perturbation: Perturb F_{ij}

$B_{ij} \to B_{ij} + \Delta$

$W_{ij} \to W_{ij} e^{\Delta}$

$W_{ji} \to W_{ij} e^{\Delta}$



perturbing energy barrier

Effect of B_{ij} perturbation in equilibrium

$W_{ij}\pi_j^{\mathrm{eq}} - W_{ji}\pi_i^{\mathrm{eq}} = 0$ → (perturbation) → $e^{\Delta}W_{ij}\pi_j^{\mathrm{eq}} - e^{\Delta}W_{ji}\pi_i^{\mathrm{eq}} = 0$

detailed balance — same equilibrium state

Steady state does not change. $\to \partial_{B_{ij}}\langle\Theta\rangle = 0$

Only kinetic property varies.

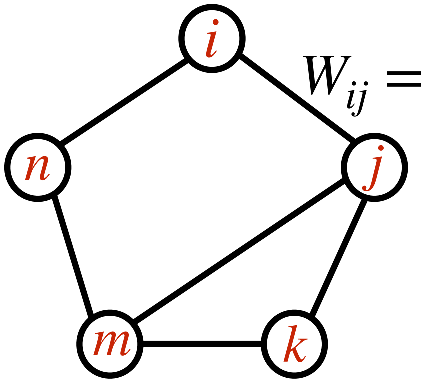
Not need to consider B_{ij} perturbation.

Effect of B_{ij} perturbation in nonequilibrium

B_{ij} perturbation changes the steady state. $\to \partial_{B_{ij}}\langle\Theta\rangle \neq 0$

Need to consider B_{ij} perturbation.

Two Ways of Perturbing W_{ij}



$$W_{ij} = \exp\left[B_{ij}(\epsilon) + \frac{F_{ij}(\eta)}{2} \right]$$

$B_{ij} = B_{ji}$: symmetric part

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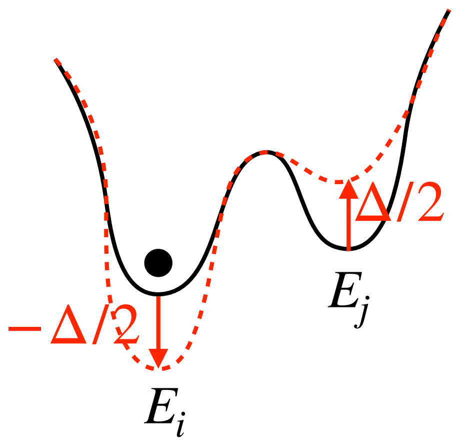
1. Kinetic perturbation: Perturb B_{ij}

2. Entropic perturbation: Perturb F_{ij}

$F_{ij} \to F_{ij} + \Delta$

$W_{ij} \to W_{ij} e^{\Delta/2}$

$W_{ji} \to W_{ij} e^{-\Delta/2}$



tilting energy barrier
(applying a driving force)

F_{ij} perturbation drives the system out of equilibrium.

$\to$ entropy is produced

F_{ij} perturbation changes the steady state and current.

$\to \partial_{F_{ij}} \langle \Theta \rangle \neq 0$

Need to consider F_{ij} perturbation.

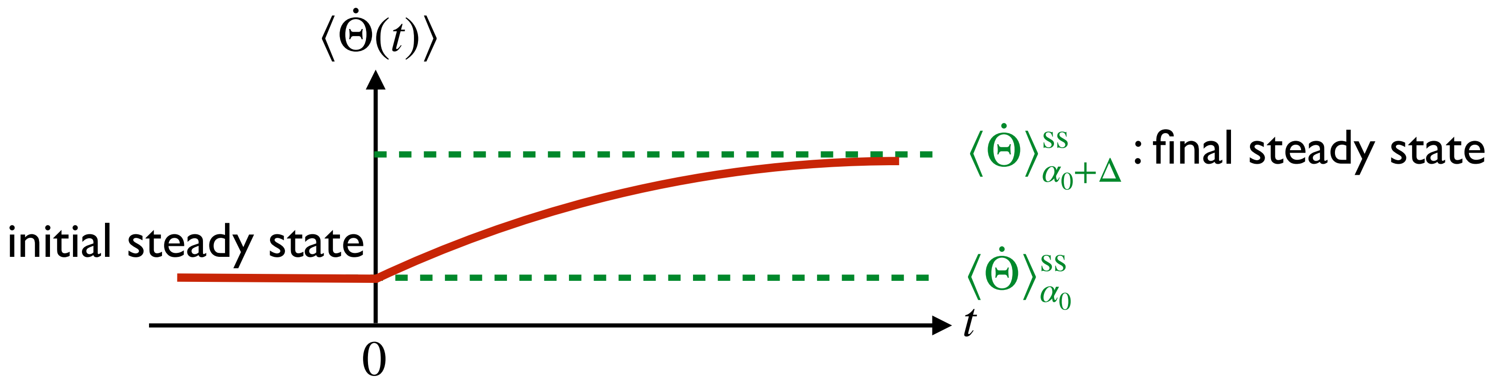
Response to the Perturbation

How observable responds to the B_{ij} or F_{ij} perturbation

Observable: $\Theta(\tau) = \int_0^\tau dt \left[\sum_i g_i h_i(t) + \sum_{i \neq j} \Lambda_{ij} \dot{N}_{ij}(t) \right]$ $= \dot{\Theta}(t)$

Protocol of Perturbing α $\quad (\alpha = B_{ij}, F_{ij})$

- initial condition: steady state with $\alpha(t) = \alpha_0 \quad (t < 0)$
- perturbing α at $t = 0$: $\alpha_0 \to \alpha_0 + \Delta \quad (t = 0)$
- relaxation to new steady state $t > 0$ with $\alpha(t) = \alpha_0 + \Delta \quad (0 < t)$



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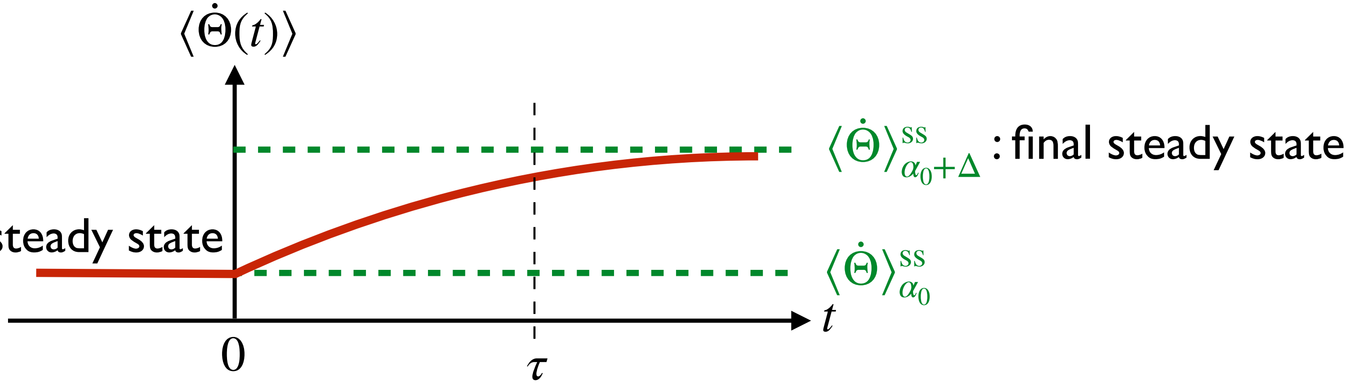


Static response: $\partial_\alpha \langle \dot{\Theta} \rangle^{\mathrm{ss}} = \lim_{\Delta \to 0} \dfrac{\langle \dot{\Theta} \rangle^{\mathrm{ss}}_{\alpha+\Delta} - \langle \dot{\Theta} \rangle^{\mathrm{ss}}_{\alpha}}{\Delta}$ (obtained from infinite-time measurement)

Dynamic response: $\partial_\alpha \langle \Theta(\tau) \rangle = \lim_{\Delta \to 0} \dfrac{\langle \Theta(\tau) \rangle_{\alpha+\Delta} - \langle \Theta(\tau) \rangle_{\alpha}}{\Delta}$

(obtained from finite-time measurement)

note : $\partial_\alpha \langle \dot{\Theta} \rangle^{\mathrm{ss}} = \lim_{\tau \to \infty} \partial_\alpha \langle \Theta(\tau) \rangle / \tau$

Fluctuation-Response Inequalities (FRI)

Cramér-Rao inequality with multiple perturbation parameters (B_{ij} or F_{ij})

$$\sum_{\alpha,\beta} R_\alpha(\tau)[\mathbb{I}^{-1}(\tau)]_{\alpha,\beta} R_\beta(\tau) \le \mathrm{Var}[\Theta(\tau)]$$

$R_\alpha(\tau) \equiv \partial_\alpha \langle \Theta(\tau) \rangle$: dynamic response $\quad (\alpha = B_{ij}, F_{ij})$

$[\mathbb{I}]_{\alpha,\beta} = -\langle \partial_\alpha \partial_\beta \ln \mathscr{P}[\Gamma] \rangle$: Fisher information matrix

1. B_{ij} perturbation $\forall (i,j)$

$[\mathbb{I}]_{B_{ij},B_{i'j'}}(\tau) = \tau \delta_{ii'} \delta_{jj'} a_{ij}$: diagonal

$(a_{ij} = W_{ij}\pi_j + W_{ji}\pi_i)$

$$\sum_{i<j} \frac{R^2_{B_{ij}}(\tau)}{\tau a_{ij}} \le \mathrm{Var}[\Theta(\tau)]$$

2. F_{ij} perturbation $\forall (i,j)$

$[\mathbb{I}]_{F_{ij},F_{i'j'}}(\tau) = \frac{1}{4} \tau \delta_{ii'} \delta_{jj'} a_{ij}$: diagonal

$$\sum_{i<j} \frac{4R^2_{F_{ij}}(\tau)}{\tau a_{ij}} \le \mathrm{Var}[\Theta(\tau)]$$

FRIs are valid for finite-time observations and arbitrary observables!

Dynamic response: $\partial_\alpha \langle \Theta(\tau) \rangle = \lim_{\Delta \to 0} \frac{\langle \Theta(\tau) \rangle_{\alpha+\Delta} - \langle \Theta(\tau) \rangle_\alpha}{\Delta}$

Fluctuation-Response Inequalities (FRI)

$$\sum_{i<j} \frac{R^2_{B_{ij}}(\tau)}{\tau a_{ij}} \leq \mathrm{Var}[\Theta(\tau)]$$

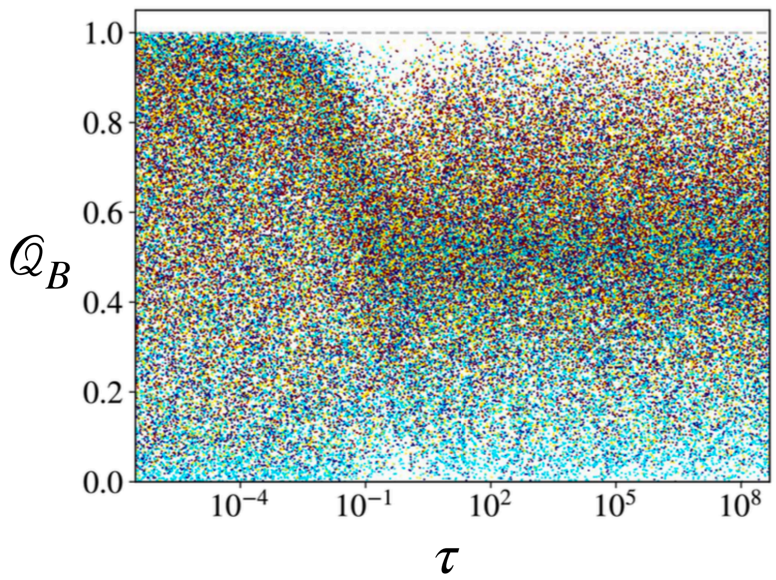
$$\rightarrow \mathcal{Q}_B \leq 1$$

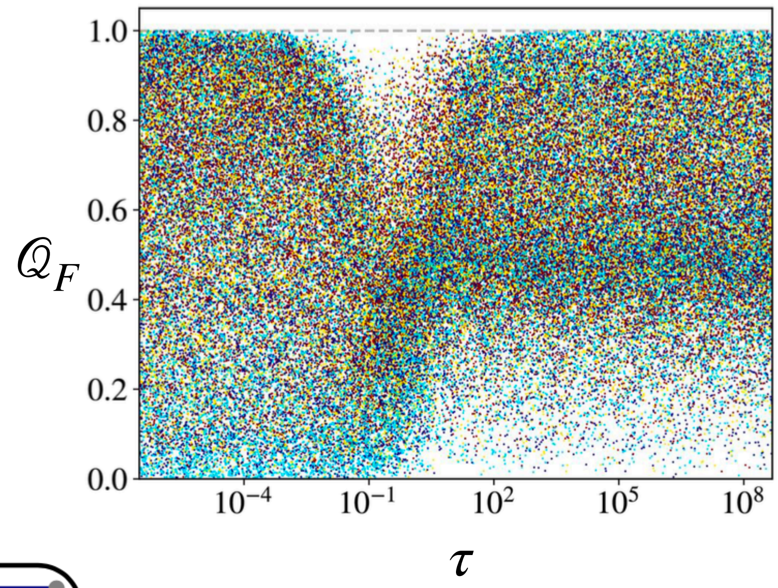
$$\mathcal{Q}_B \equiv \sum_{i<j} \frac{R^2_{B_{ij}}(\tau)}{\tau a_{ij} \mathrm{Var}\{\Theta(\tau)\}}$$

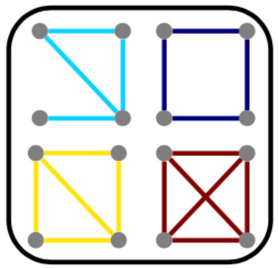
$$\sum_{i<j} \frac{4R^2_{F_{ij}}(\tau)}{\tau a_{ij}} \leq \mathrm{Var}[\Theta(\tau)]$$

$$\rightarrow \mathcal{Q}_F \leq 1$$

$$\mathcal{Q}_F \equiv \sum_{i<j} \frac{4R^2_{F_{ij}}(\tau)}{\tau a_{ij} \mathrm{Var}[\Theta(\tau)]}$$







FRI → FRR

$$\sum_{i<j} \frac{R_{B_{ij}}^2(\tau)}{\tau a_{ij}} \leq \mathrm{Var}[\Theta(\tau)]$$

$$\sum_{i<j} \frac{4R_{F_{ij}}^2(\tau)}{\tau a_{ij}} \leq \mathrm{Var}[\Theta(\tau)]$$

relation btw B_{ij} and F_{ij} responses

$$\frac{R_{B_{ij}}(\tau)}{R_{F_{ij}}(\tau)} = \frac{2J_{ij}}{a_{ij}}$$

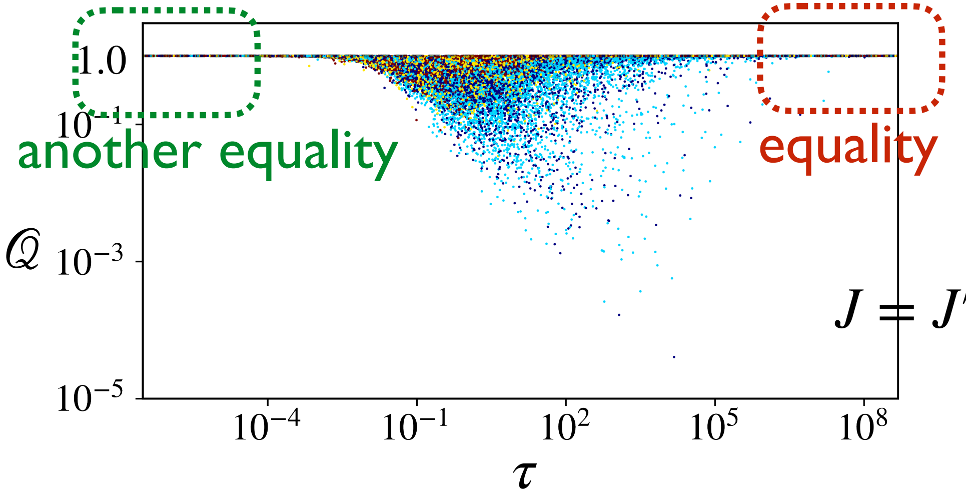
$$a_{ij} = W_{ij}\pi_j + W_{ji}\pi_i$$

$$J_{ij} = W_{ij}\pi_j - W_{ji}\pi_i$$

$$\sum_{i<j} \frac{a_{ij}R_{B_{ij}}^2(\tau)}{\tau J_{ij}^2} \leq \mathrm{Var}[\Theta(\tau)]$$

dynamic

$$\mathcal{Q} \equiv \sum_{i<j} \frac{a_{ij}R_{B_{ij}}^2(\tau)}{\tau J_{ij}^2 \mathrm{Var}[\Theta(\tau)]}$$



(Θ : current-like observable)

$$\to \lim_{\tau\to\infty} \sum_{i<j} \frac{a_{ij}R_{B_{ij}}^2(\tau)/\tau^2}{J_{ij}^2} = \lim_{\tau\to\infty} \mathrm{Var}[\Theta(\tau)]/\tau$$

static

$J = J'$

Fluctuation-Response Relation (infinite time)

$$\sum_{i<j} \frac{a_{ij}}{J_{ij}^2} \partial_{B_{ij}} \langle J \rangle^{\mathrm{ss}} \partial_{B_{ij}} \langle J' \rangle^{\mathrm{ss}} = \mathrm{Cov}(J, J')$$

Aslyamov et al., PRL 134, 157101 (2025)

FRI → FRR

I. Classical Stochastic System

Fluctuation-Response Inequality (FRI)

applicable to finite-time observation and any observable

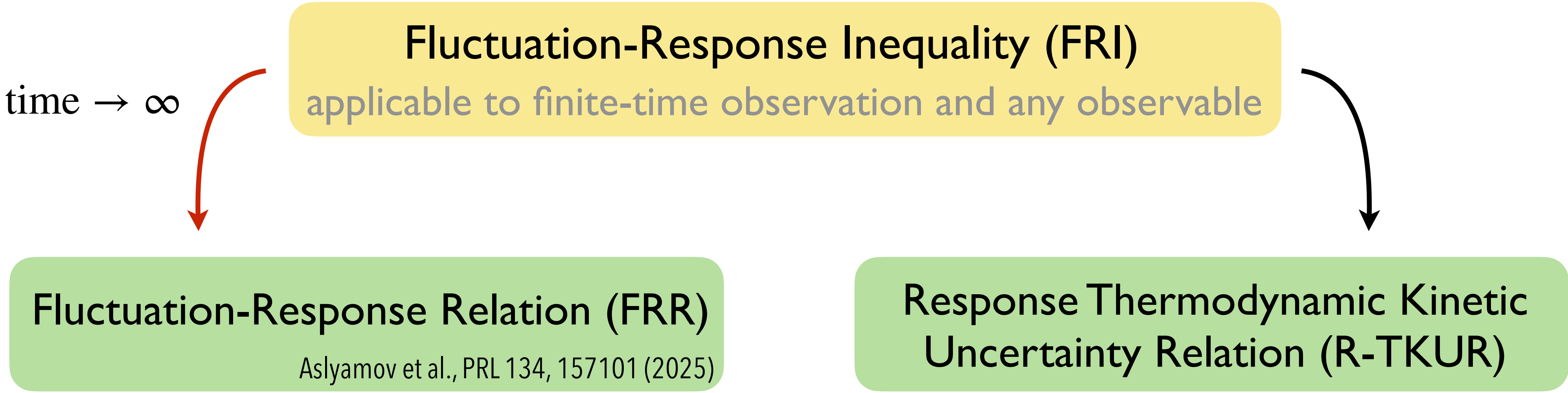
time → ∞

Fluctuation-Response Relation (FRR)

Aslyamov et al., PRL 134, 157101 (2025)

FRI → R-TKUR

I. Classical Stochastic System



FRI → R-TKUR

Fluctuation-Response Inequality

$$\sum_{i<j} \frac{a_{ij} R_{B_{ij}^2(\tau)}}{\tau J_{ij}^2} \leq \mathrm{Var}[\Theta(\tau)] \quad \rightarrow \quad \frac{R_\epsilon^2(\tau)}{\mathrm{Var}[\Theta(\tau)]} \leq \frac{b_{\max}^2}{2} \Sigma_{\mathrm{ps}} \leq \frac{b_{\max}^2}{2} \min\{\Sigma,\ 2A\}$$

Cauchy-Schwartz

Σ_{ps} : pseudo EP $\quad (\Sigma_{\mathrm{ps}} \leq \Sigma,\ 2A)$

Σ : total EP

A : total dynamical activity

Transition rate: $W_{ij} = \exp\left[B_{ij}(\epsilon) + \frac{F_{ij}(\eta)}{2}\right]$

Response: $R_\epsilon(\tau) \equiv \partial_\epsilon \langle \Theta(\tau) \rangle = \sum_{i<j} \partial_\epsilon B_{ij} \cdot \partial_{B_{ij}} \langle \Theta(\tau) \rangle = \sum_{i<j} b_{ij} R_{B_{ij}} \quad (b_{ij} \equiv \partial_\epsilon B_{ij})$

$b_{\max} = \max\{\,|b_{ij}|\,\}$

Response Thermodynamic Kinetic Uncertainty Relation (R-TKUR)

$$\frac{R_\epsilon^2(\tau)}{\mathrm{Var}[\Theta(\tau)]} \leq \frac{b_{\max}^2}{2} \min\{\Sigma,\ 2A\}$$

FRI → R-TKUR

I. Classical Stochastic System

Fluctuation-Response Inequality (FRI)

applicable to finite-time observation and any observable

time $\rightarrow \infty$

Fluctuation-Response Relation (FRR)

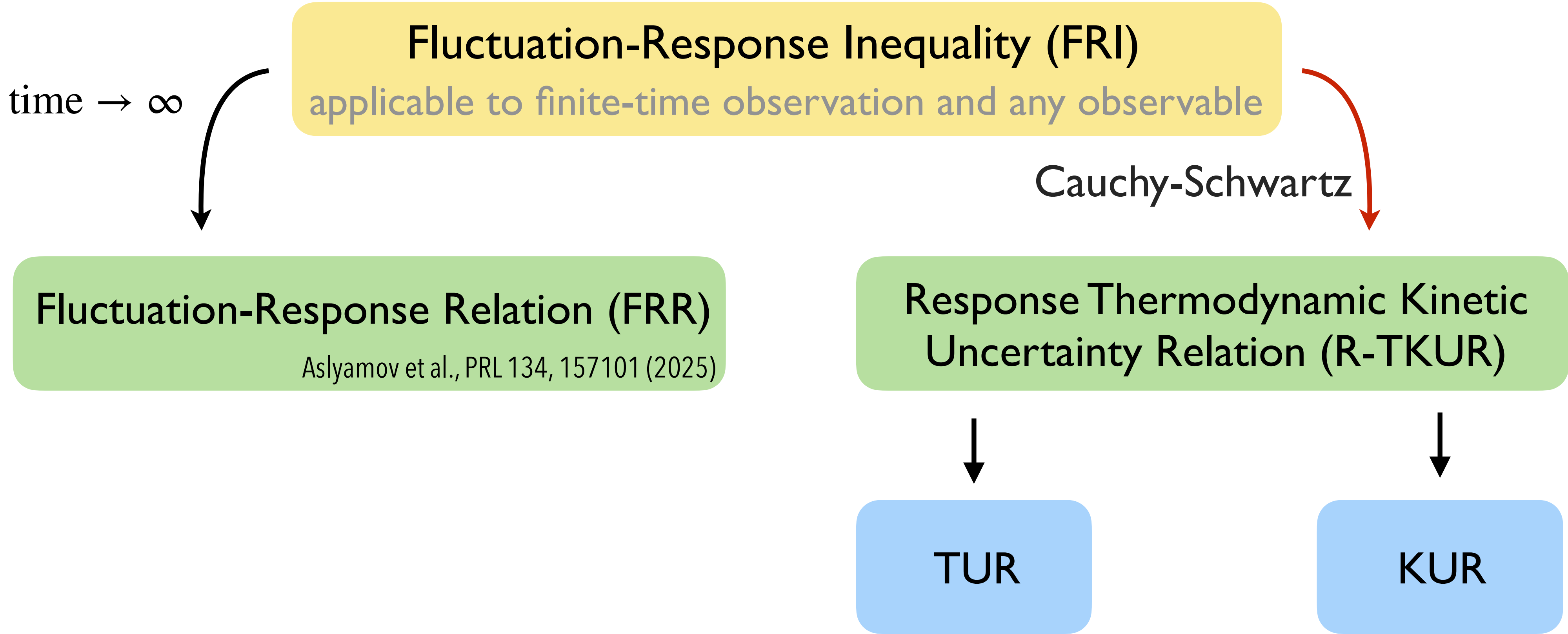
Aslyamov et al., PRL 134, 157101 (2025)

Cauchy-Schwartz

Response Thermodynamic Kinetic Uncertainty Relation (R-TKUR)

R-TKUR → TUR & KUR

1. Classical Stochastic System



R-TKUR → TUR & KUR

Response Thermodynamic Kinetic Uncertainty Relation (R-TKUR)

$$\frac{R_\epsilon^2(\tau)}{\mathrm{Var}[\Theta(\tau)]} \leq \frac{b_{\max}^2}{2}\min\{\Sigma,\ 2A\}$$

Σ : total EP

A : total dynamical activity

$b_{ij} = b = b_{\max} \quad \forall\ ij$: uniform perturbation

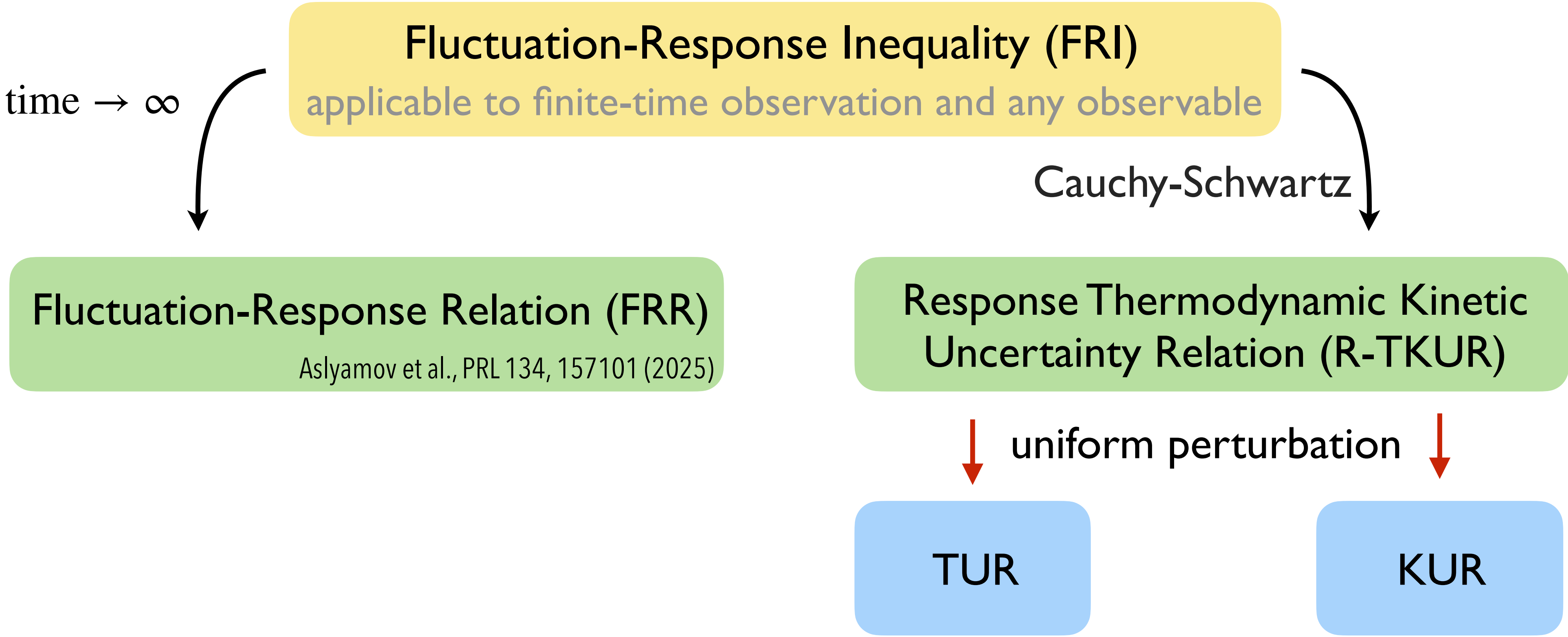
$\rightarrow R_\epsilon(\tau) = b\langle\Theta(\tau)\rangle$

Thermodynamic, Kinetic Uncertainty Relation (TUR, KUR)

$$\rightarrow \frac{\langle\Theta(\tau)\rangle^2}{\mathrm{Var}[\Theta(\tau)]} \leq \frac{1}{2}\min\{\underbrace{\Sigma}_{\text{TUR}},\ \underbrace{2A}_{\text{KUR}}\}$$

R-TKUR → TUR & KUR

1. Classical Stochastic System



FRR → FDT (Equilibrium)

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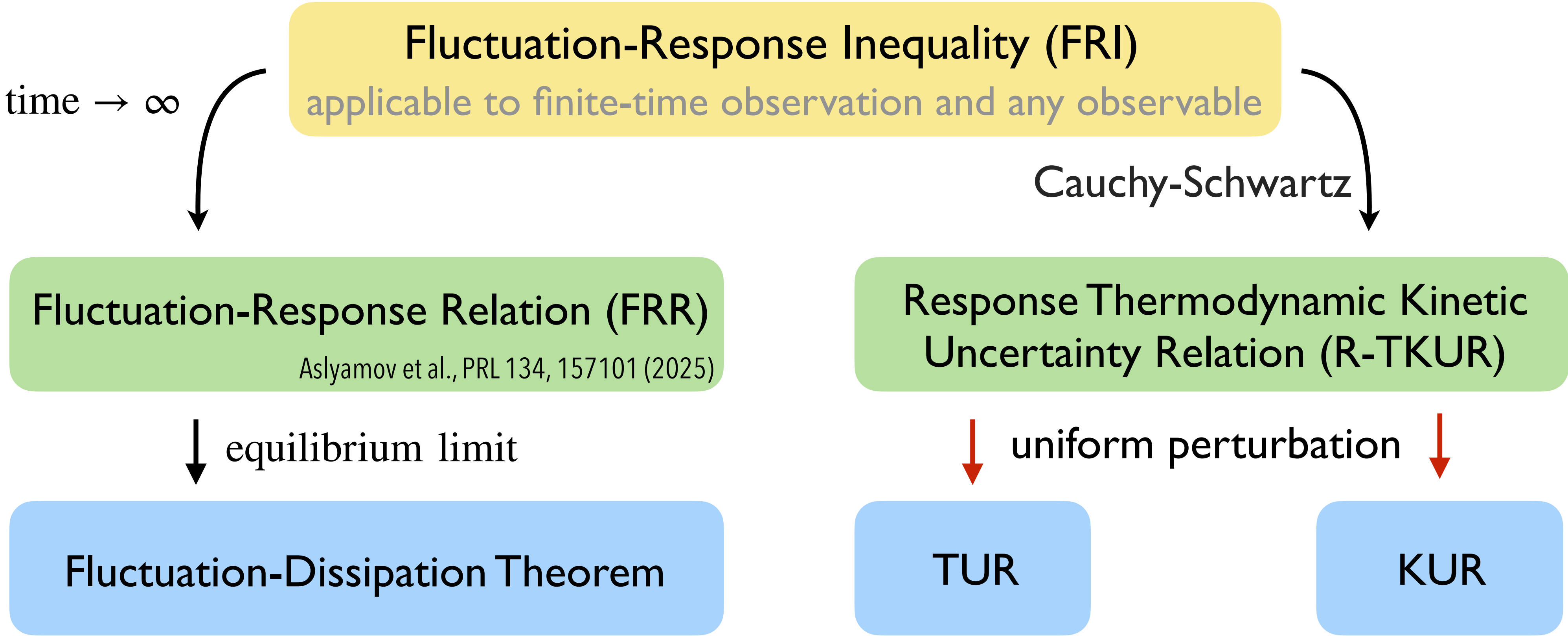

FRR → FDT (Equilibrium)

Fluctuation-Response Relation

$$\mathrm{Cov}(J, J') = \sum_{i>j} \frac{4}{a_{ij}} \partial_{F_{ij}} \langle J \rangle^{\mathrm{ss}} \partial_{F_{ij}} \langle J' \rangle^{\mathrm{ss}} \xrightarrow{J = J'} \underbrace{\mathrm{Var}(J)}_{\text{fluctuation}} = \sum_{i>j} \frac{4}{a_{ij}} \underbrace{\left(\partial_{F_{ij}} \langle J \rangle^{\mathrm{ss}}\right)^2}_{\text{response}}$$

In equilibrium, fluctuation ∝ response

Empirical edge current: $j_{ij}(t) = \dot{N}_{ij}(t) - \dot{N}_{ji}(t) \quad \rightarrow \langle j_{ij}(t) \rangle = W_{ij} p_j(t) - W_{ji} p_i(t) = J_{ij}$

$N_{ij}(t)$: accumulated number of jumbs $j \rightarrow i$

Edge response: $L_{(ij),(mn)} \equiv \dfrac{dJ_{ij}}{dF_{mn}}$: how J_{ij} responds to external driving F_{mn} perturbation

Edge current fluctuation: $C_{(ij),(mn)} \equiv \langle\langle j_{ij}, j_{mn} \rangle\rangle = \mathrm{Cov}[j_{ij}, j_{mn}]$

$$\mathrm{Var}(J) = \mathbf{\Lambda}^{\mathrm{T}} \mathbb{C} \mathbf{\Lambda} = \sum_{i>j} \frac{4}{a_{ij}} \left(\partial_{F_{ij}} \langle J \rangle^{\mathrm{ss}}\right)^2 = \mathbf{\Lambda}^{\mathrm{T}} 2\mathbb{L} \mathbf{\Lambda} \longrightarrow \mathbb{C} = 2\mathbb{L}$$

: FDT in equilibrium

$J(t) = \sum_{i \neq j} \Lambda_{ij} \dot{N}_{ij}(t)$: current-like observable

Aslyamov et al., PRL 134, 157101 (2025)

Vroylandt et al., JStatMech 054002 (2019)

Summary of the Main Results

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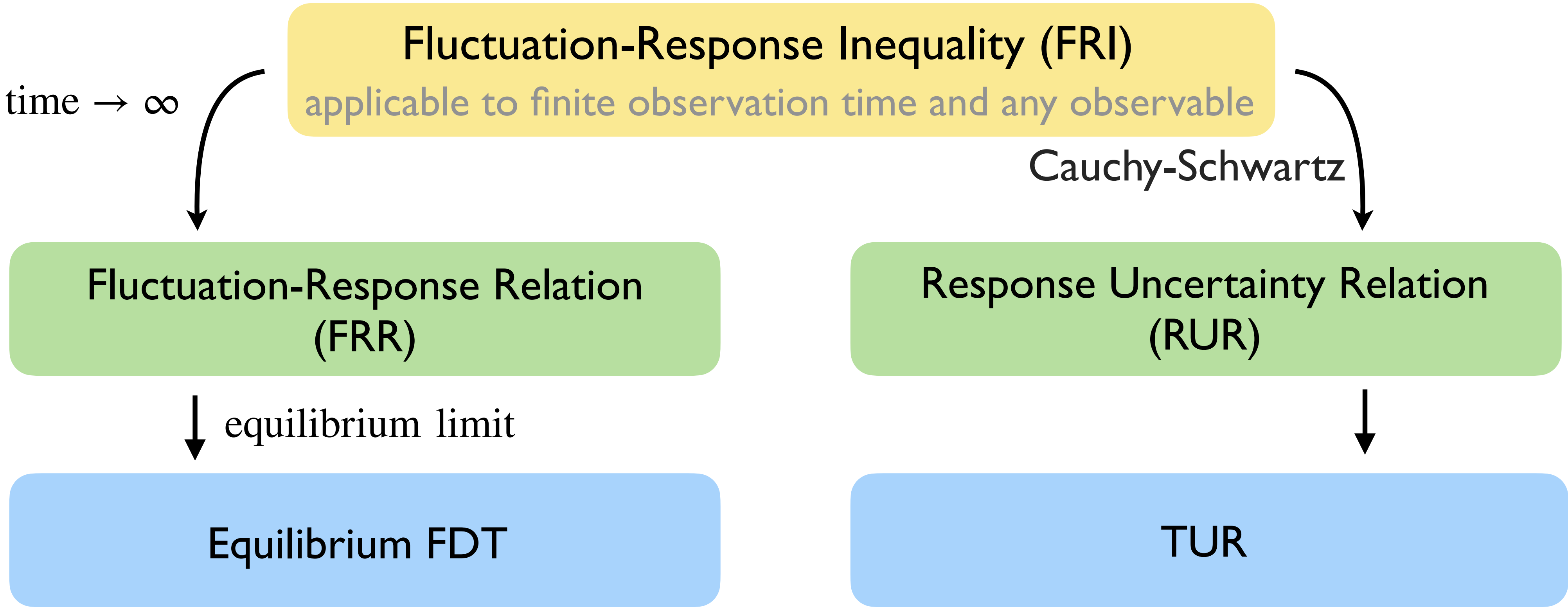


Setup for Overdamped Langevin Systems

Langevin equation

$$\dot{x}_t = \mu(x_t)F(x_t) + \sqrt{2\mu(x_t)T(x_t)} \circledast \xi_t$$

$\circledast$: anti-Ito product

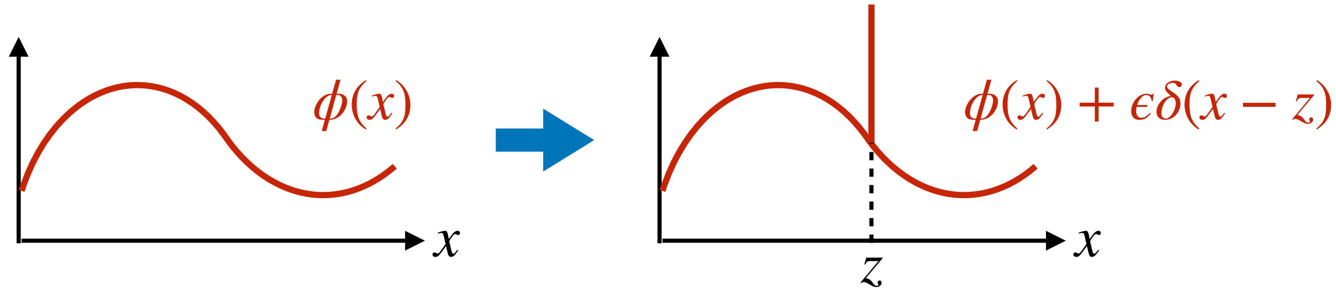
$\mu(x)$: mobility

$F(x)$: external force

$T(x)$: temperature

Time-averaged observable: $\Theta(\tau) = \frac{1}{\tau}\int_0^\tau [f(x_t) + \dot{x}_t \circ g(x_t)]dt$

Local perturbation at $x = z$: $\phi(x) \to \phi(x) + \epsilon\delta(x - z)$ $\quad \phi \in \{\mu, F, T\}$



$$\langle\Theta\rangle_{\rm ss} \to \langle\Theta\rangle_{\rm ss} + \epsilon\frac{\delta\langle\Theta\rangle_{\rm ss}}{\delta\phi(z)}$$

Response to the perturbation at $x = z$: $\frac{\delta\langle\Theta\rangle_{\rm ss}}{\delta\phi(z)}$

Hierarchy for Langevin Systems

Chun, Kwon, Park, JSL, arXiv:2601.16387

Fluctuation-Response Inequality (FRI)

applicable to finite observation time and any observable

$$\mathrm{Var}[\Theta(\tau)] \ge \int dz \int_0^\tau ds \, \frac{2p(z,s)D(z)}{[\tilde{N}_\phi(z,s)]^2}\left(\frac{\delta\langle\Theta(\tau)\rangle}{\delta\phi(z,s)}\right)^2$$

time $\to \infty$ ↓   ↓ Cauchy – Schwarz

Fluctuation-Response Relation (FRR)

$$C_{\Theta_1,\Theta_2} = \int \frac{2\pi(z)D(z)}{N_{\phi_1}(z)N_{\phi_2}(z)}\,\frac{\delta\langle\Theta_1\rangle_{ss}}{\delta\phi_1(z)}\,\frac{\delta\langle\Theta_2\rangle_{ss}}{\delta\phi_2(z)}dz$$

$$C_{\Theta_1,\Theta_2} \equiv \lim_{\tau\to\infty} \tau\,\mathrm{Cov}\{\Theta_1(\tau),\Theta_2(\tau)\}$$

↓ equilibrium limit

Equilibrium FDT

$$C^{\mathrm{eq}}_{J(x),J(y)} = 2T\left[\frac{\delta\langle J(x)\rangle}{\delta F(y)}\right]_{\mathrm{eq}}$$

Response Uncertainty Relation (RUR)

$$\frac{[\delta_\phi\langle\Theta(\tau)\rangle]^2}{\mathrm{Var}[\Theta(\tau)]} \le \frac{\psi_{\max}^2}{2}\int dz \int_0^\tau ds \, \frac{[\tilde{N}_\phi(z,s)]^2}{p(z,s)D(z)}$$

↓ uniform perturbation

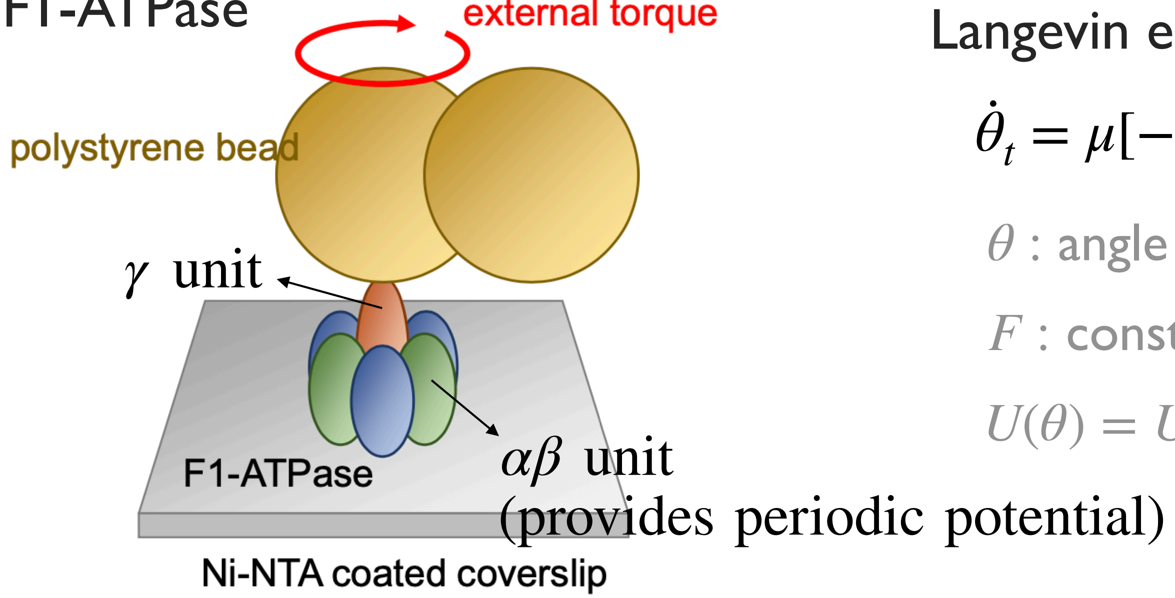
TUR

$$\frac{\langle(1+\tau\partial_\tau)J(\tau)\rangle^2}{\mathrm{Var}[J(\tau)]} \le \Sigma_\tau/2$$

Numerical Confirmation: Giant-Diffusion Phenomenon

diffusion coefficient of a Brownian particle moving in a tilted periodic potential can be enhanced by several orders of magnitude near the critical tilt of the potential

F1-ATPase



Langevin equation for a duplex bead

$$\dot{\theta}_t = \mu[-\partial_\theta U(\theta)|_{\theta=\theta_t} + F] + \sqrt{2\mu T}\xi_t$$

Hayashi et al., PRL 114, 248101 (2015)

θ : angle of the bead

F : constant torque

$U(\theta) = U_0 \cos(3\theta)$: potential with period 120°

Numerical Confirmation: Giant-Diffusion Phenomenon

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Observable

$$\bar{\omega}(\tau) = \tau^{-1} \int_0^\tau \dot{\theta}_t dt$$

time-averaged angular velocity

Diffusion coefficient

$$D_\infty = \lim_{\tau \to \infty} \tau \mathrm{Var}[\bar{\omega}(\tau)]/2$$

Perturbations	Responses	Response Uncertainty Relation
$F \mapsto F + \varepsilon h(t)$	$\delta_F \langle \bar{\omega} \rangle_{\mathrm{ss}}$	$D_\infty \geq \frac{T(\partial_F \omega)^2}{\mu}$
$\ln \mu \mapsto \ln \mu + \varepsilon h(t)$	$\delta_{\ln \mu} \langle \bar{\omega} \rangle_{\mathrm{ss}}$	$D_\infty \geq \frac{(\partial_{\ln \mu} \omega)^2}{\sigma}$
$T \mapsto T + \varepsilon h(t)$	$\delta_T \langle \bar{\omega} \rangle_{\mathrm{ss}}$	$D_\infty \geq \frac{T(\partial_T \omega)^2}{\mu I}$

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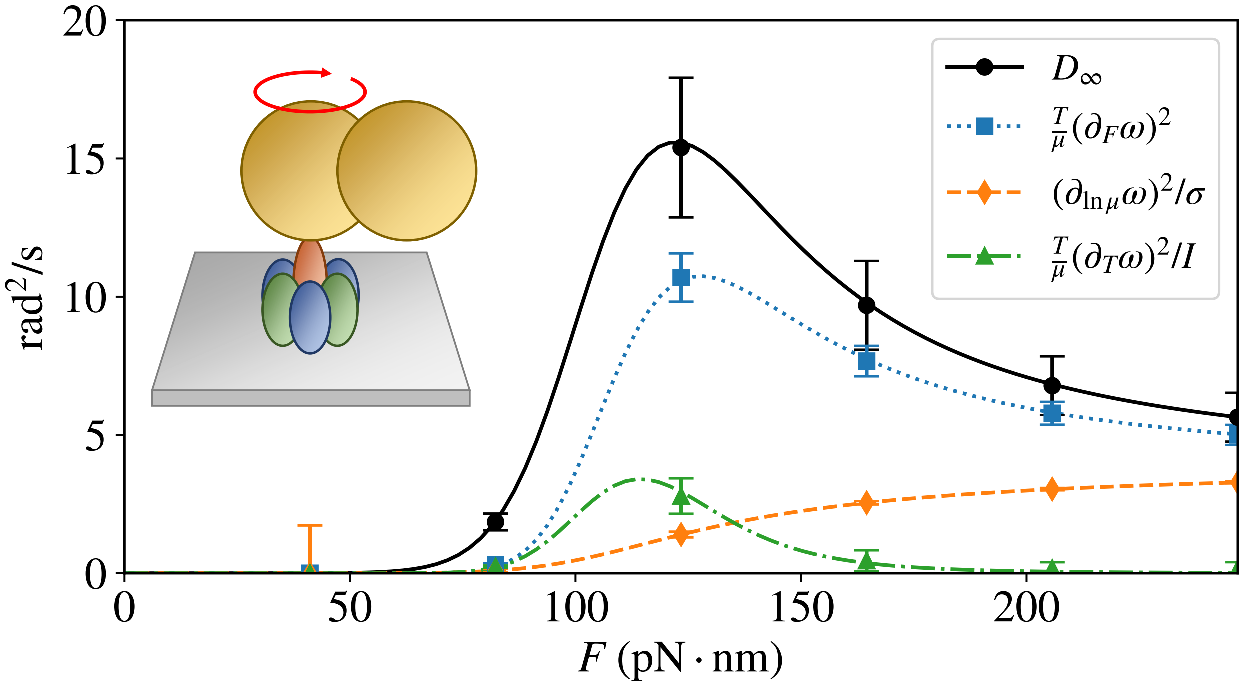

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$$D_\infty \geq \frac{T(\partial_F\omega)^2}{\mu}$$

$$D_\infty \geq \frac{(\partial_{\ln\mu}\omega)^2}{\sigma}$$

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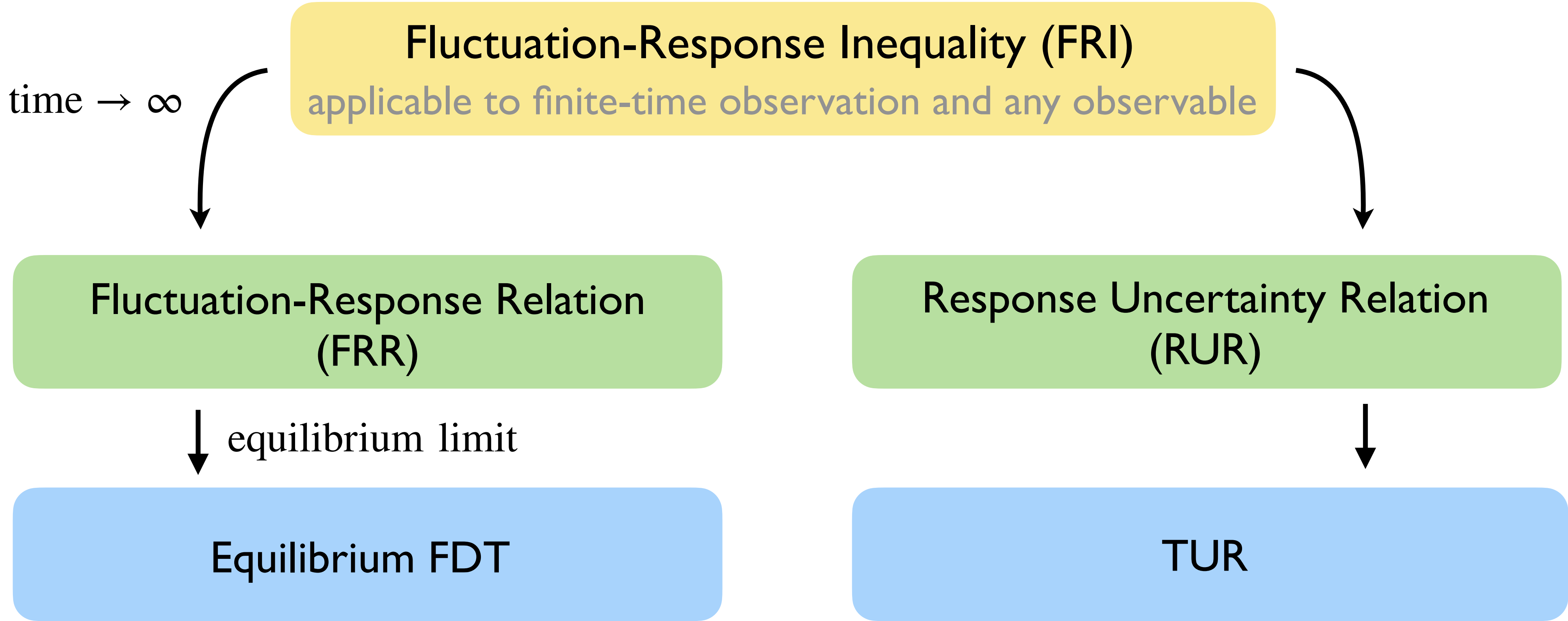
Summary of the Main Results

Kwon, Chun, Park, JSL, PRL 135, 097101 (2025)
Chun, Kwon, Park, JSL, arXiv:2601.16387

1. Classical Stochastic System

for both Markov-jump & Langevin systems



2. Extension to Open Quantum System

: Quantum Fluctuation-Response Inequality (QFRI)

Quantum Fluctuation-Response Inequality (QFRI)

Setup

Lindblad master equation:

$$\dot{\rho}(t) = -i[H, \rho(t)] + \sum_{k=1}^{K} \underbrace{\left(L_k \rho(t) L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho(t) - \frac{1}{2} \rho(t) L_k^\dagger L_k \right)}_{\mathscr{D}(L_k)\rho(t)} \equiv \mathscr{L}\rho(t)$$

Jump operator: $L_k(\theta_k) = L_k^{\theta_k} = e^{\theta_k/2} L_k$ K : number of jump operators L_k

Observable: $\Theta(\tau) = \sum_{k=1}^{K} \Lambda_k N_k(\tau)$ $N(\tau)$: accumlated number of jumps via channel k

Protocol of Perturbing θ_k

- initial condition: steady state with $\theta_k(t) = \theta_k \quad (t < 0)$
- perturbing θ_k at $t = 0$: $\theta_k \to \theta_k + \Delta \quad (t = 0)$
- relaxation to new steady state $t > 0$ with $\theta_k(t) = \theta_k + \Delta \quad (0 < t)$

Quantum Fluctuation-Response Inequality (QFRI)

Quantum FRI

Quantum Cramér-Rao inequality with multiple perturbation parameters θ_k

$$\sum_{\alpha,\beta} R_{\theta_\alpha}(\tau)[\mathbb{I}_Q^{-1}(\tau)]_{\theta_\alpha\theta_\beta} R_{\theta_\beta}(\tau) \leq \mathrm{Var}[\Theta(\tau)] \qquad (\alpha, \beta = 1, \cdots, K)$$

$R_{\theta_\alpha}(\tau) \equiv \partial_{\theta_\alpha}\langle\Theta(\tau)\rangle$: dynamic response

$[\mathbb{I}_Q(\tau)]_{\theta_\alpha,\theta_\beta}$: Quantum Fisher information matrix

$= \tau\delta_{\alpha\beta}a_\alpha$ $\qquad a_\alpha = \mathrm{tr}[L_\alpha^{\theta_k}\rho(L_\alpha^{\theta_k})^\dagger]$: traffic

Quantum FRI

Quantum R-KUR

$$\sum_k \frac{R_{\theta_k}^2(\tau)}{\tau a_k} \leq \mathrm{Var}\{\Theta(\tau)\} \xrightarrow[\text{Cauchy-Schwartz}]{\theta_k = \theta_k(\epsilon)} \frac{R_\epsilon^2(\tau)}{\mathrm{Var}\{\Theta(\tau)\}} \leq \tau(\Delta\theta_{\max})^2\dot{A}$$

$\Delta\theta_{\max} = \max_k\{\,|d_\epsilon\theta_k|\,\}$

$\dot{A} = \sum_k \alpha_k$: dynamical activity

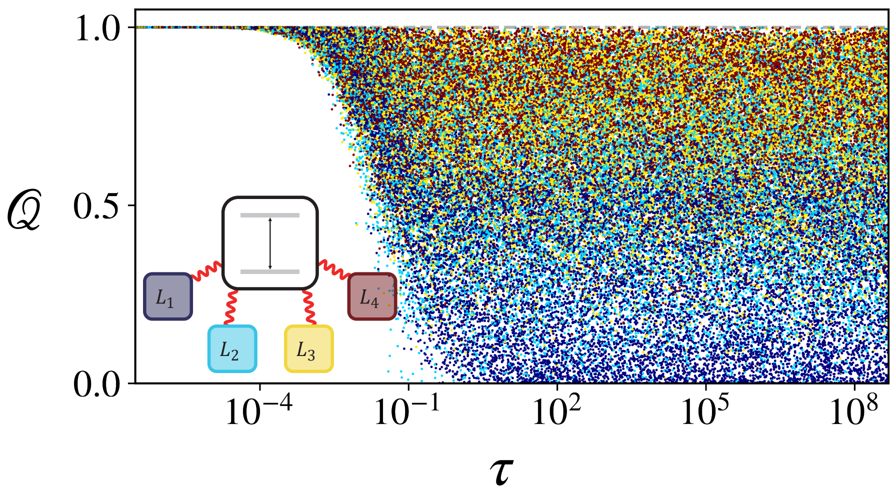
FRI valid for finite-time observations

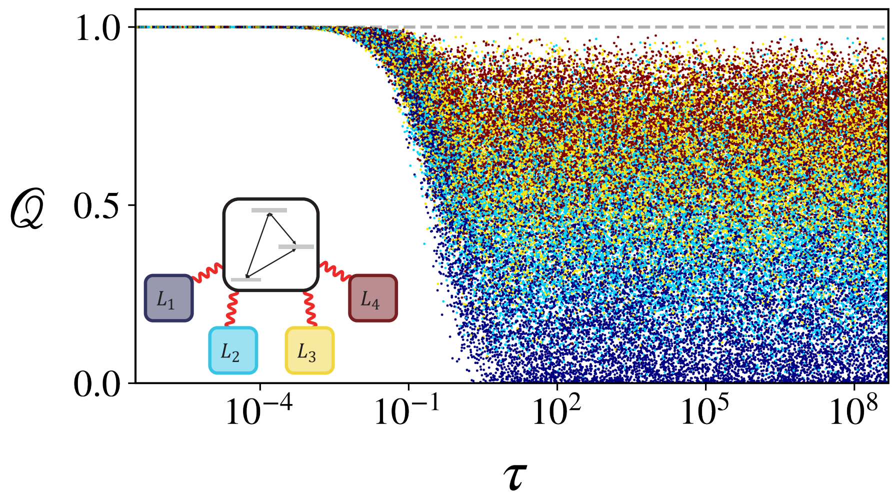
Quantum Fluctuation-Response Inequality (QFRI)

Numerical Confirmation

Quantum FRI

$$\sum_k \frac{R_{\theta_k}^2(\tau)}{\tau a_k} \leq \mathrm{Var}\{\Theta(\tau)\} \;\rightarrow\; \mathcal{Q} \equiv \sum_k \frac{R_{\theta_k}^2(\tau)}{\tau a_k \mathrm{Var}\{\Theta(\tau)\}} \leq 1$$





$\mathcal{Q}$ does not converge to 1 (equality not satisfied) in the large τ regime.

Summary

1. We find unified hierarchy of various fluctuation-response theories (FRT) for both Markov-jump & Langevin.

Chun, Kwon, Park, JSL, arXiv:2601.16387 Kwon, Chun, Park, JSL, PRL 135, 097101 (2025)

2. We extend the theory to Open Quantum System

and found the Quantum Fluctuation-Response Inequality (QFRI).

Chun, Kwon, Park, JSL, arXiv:2601.16387

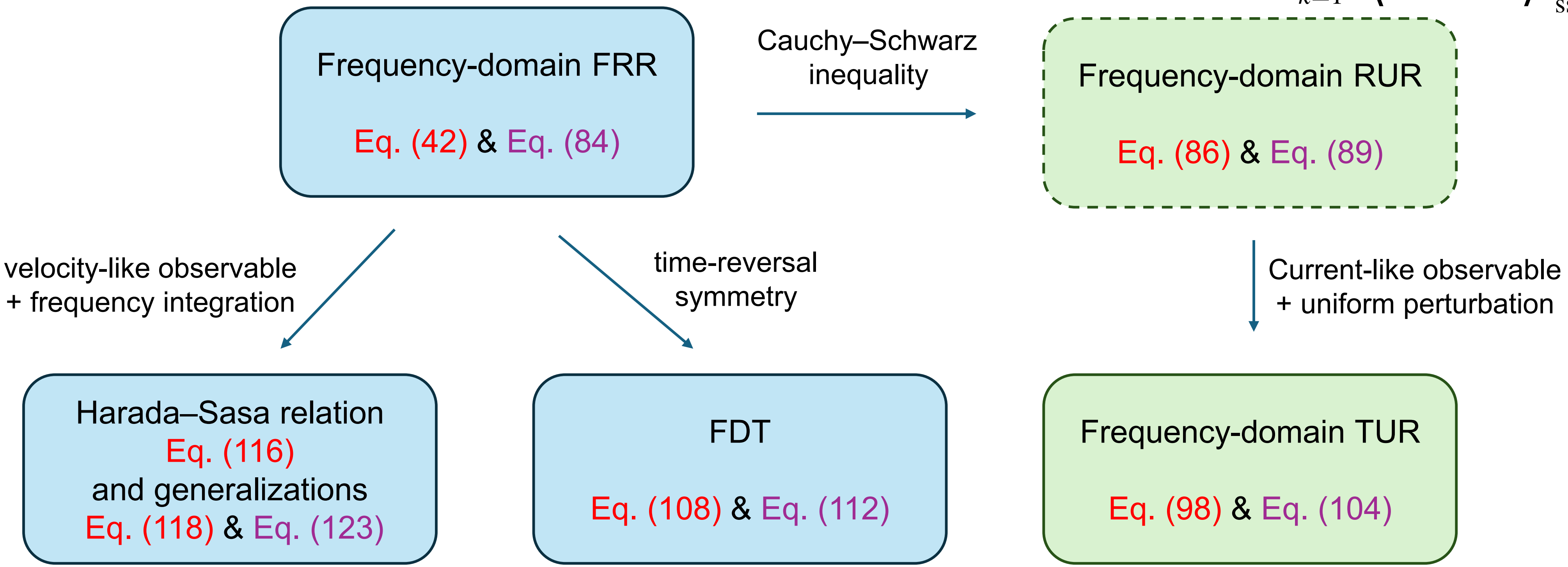
Final Remark

Recently, we also found the same hierarchy for FRT in the frequency domain. Kwon, Chun, Park, JSL, arXiv:2605.05038

Dechant, arXiv:2510.15228

$$\left[R^{\boldsymbol{\theta}}_{\boldsymbol{F}\mapsto\boldsymbol{F}+\epsilon\boldsymbol{\psi}}(\omega)\right]^{\dagger} C_{\boldsymbol{\theta},\boldsymbol{\theta}^{\top}}(\omega)^{-1} R^{\boldsymbol{\theta}}_{\boldsymbol{F}\mapsto\boldsymbol{F}+\epsilon\boldsymbol{\psi}}(\omega) \le \sum_{k=1}^{N} \left\langle \frac{\mu_k |\psi_k|^2}{2T_k} \right\rangle_{\mathrm{ss}}$$

$$C_{\boldsymbol{\theta},\boldsymbol{\theta}^{\top}}(\omega) = \sum_{k,l=1}^{N} \int d\boldsymbol{z}\, R^{\boldsymbol{\theta}}_{\phi_k(\boldsymbol{z})}(\omega) \left[A^{\phi}(\boldsymbol{z})^{-1}\right]_{kl} \left[R^{\boldsymbol{\theta}}_{\phi_l(\boldsymbol{z})}(\omega)\right]^{\dagger}$$



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Euijoon Kwon (SNU)

Main reference: Kwon, Chun, Park, JSL, PRL 135, 097101 (2025)

Chun, Kwon, Park, JSL, arXiv:2601.16387

Kwon, Chun, Park, JSL, arXiv:2605.05038