String condensation
and
many phases in QCD at finite temperature

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Based on many old papers + recent discussions with Jack Holden (Tsinghua U.), Sameer Murthy (King's College London), Bo Sundborg (Stockholm U.)

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The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction.

Sidney Coleman

Gauged Gaussian Two Matrix Model

$$\hat{H} = \frac{1}{2}\mathrm{Tr}\left(\hat{P}_X^2 + \hat{X}^2 + \hat{P}_Y^2 + \hat{Y}^2\right)$$

in the large-N limit

(Essentially, re-interpretation of Sundborg 1999; Anarony et al. 2003)

$$\hat{H} = \frac{1}{2}\mathrm{Tr}\left(\underline{\hat{P}_X^2 + \hat{X}^2} + \underline{\hat{P}_Y^2 + \hat{Y}^2}\right)$$

$$\hat{A}, \hat{A}^\dagger \qquad \hat{B}, \hat{B}^\dagger$$

$$\underbrace{\mathrm{Tr}\left(\hat{A}^\dagger \hat{A}^\dagger \hat{B}^\dagger \hat{A}^\dagger \cdots\right)}_{L} |0\rangle$$

$E = L$ (up to zero-pt energy)

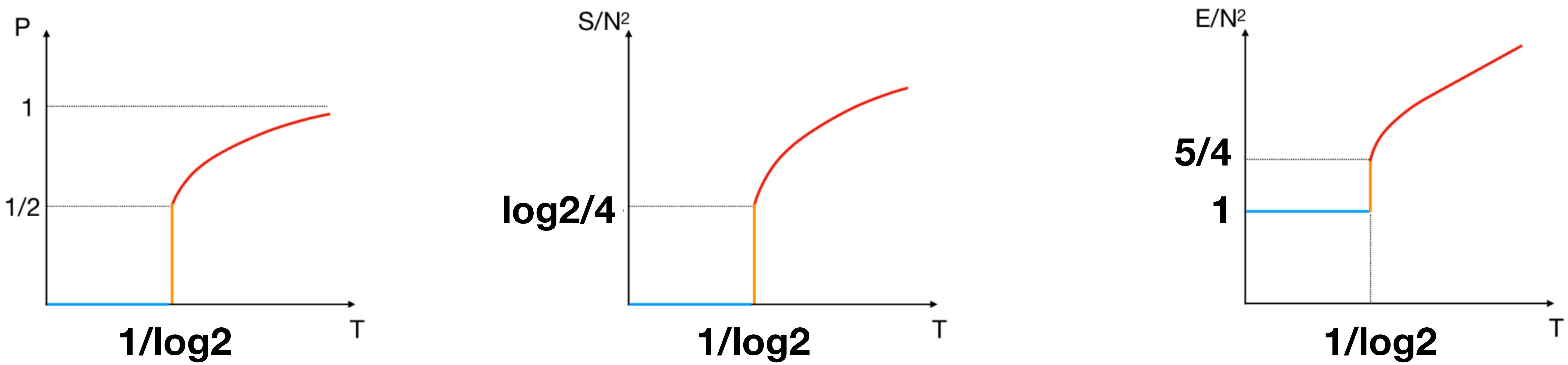
$S = L \log 2$ (# of states ~ 2^L)

(valid at $L \ll N^2$)

$$F = E - TS = L(1 - T\log 2)$$

(up to zero-pt energy; valid at L ≪ N^2)

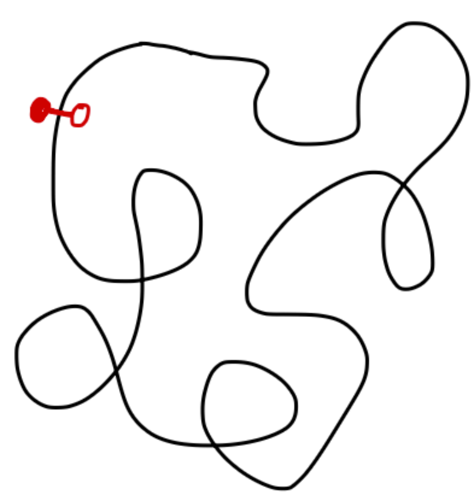
$$F = 0 \;\; @ \;\; T = \frac{1}{\log 2}$$



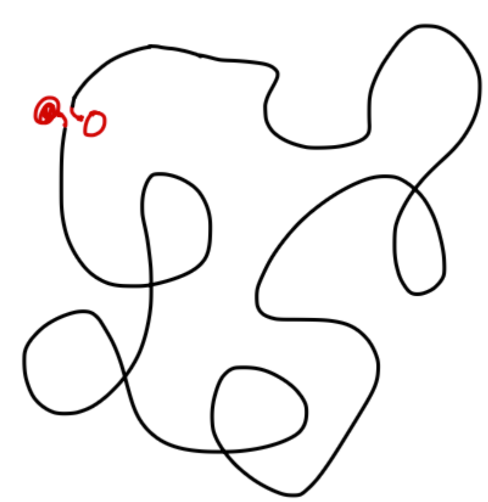
Hagedorn String

Deconfinement = string condensation

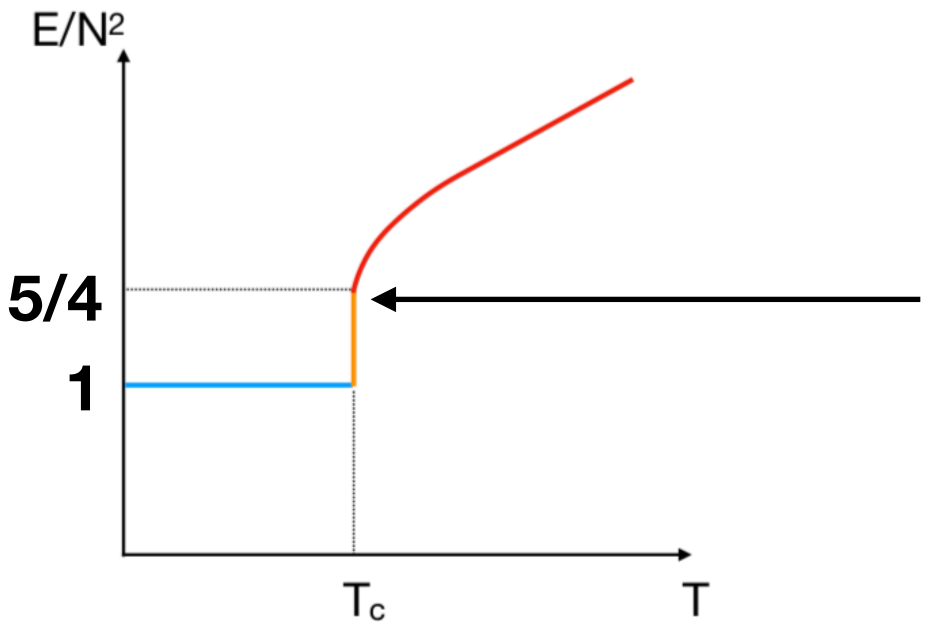
- String condensation can take place regardless of the details of the theories.
- Energy E, free energy E $\sim N^2$ in the ’t Hooft limit
- String tension vanishes (when there is “space”)



long closed string
+
short open string



long open string



Hagedorn growth ends because of trace relations

$$\mathrm{Tr}(XYX\cdots) \sim \sum \mathrm{Tr}(XY\cdots)\cdot\mathrm{Tr}(X\cdots)$$

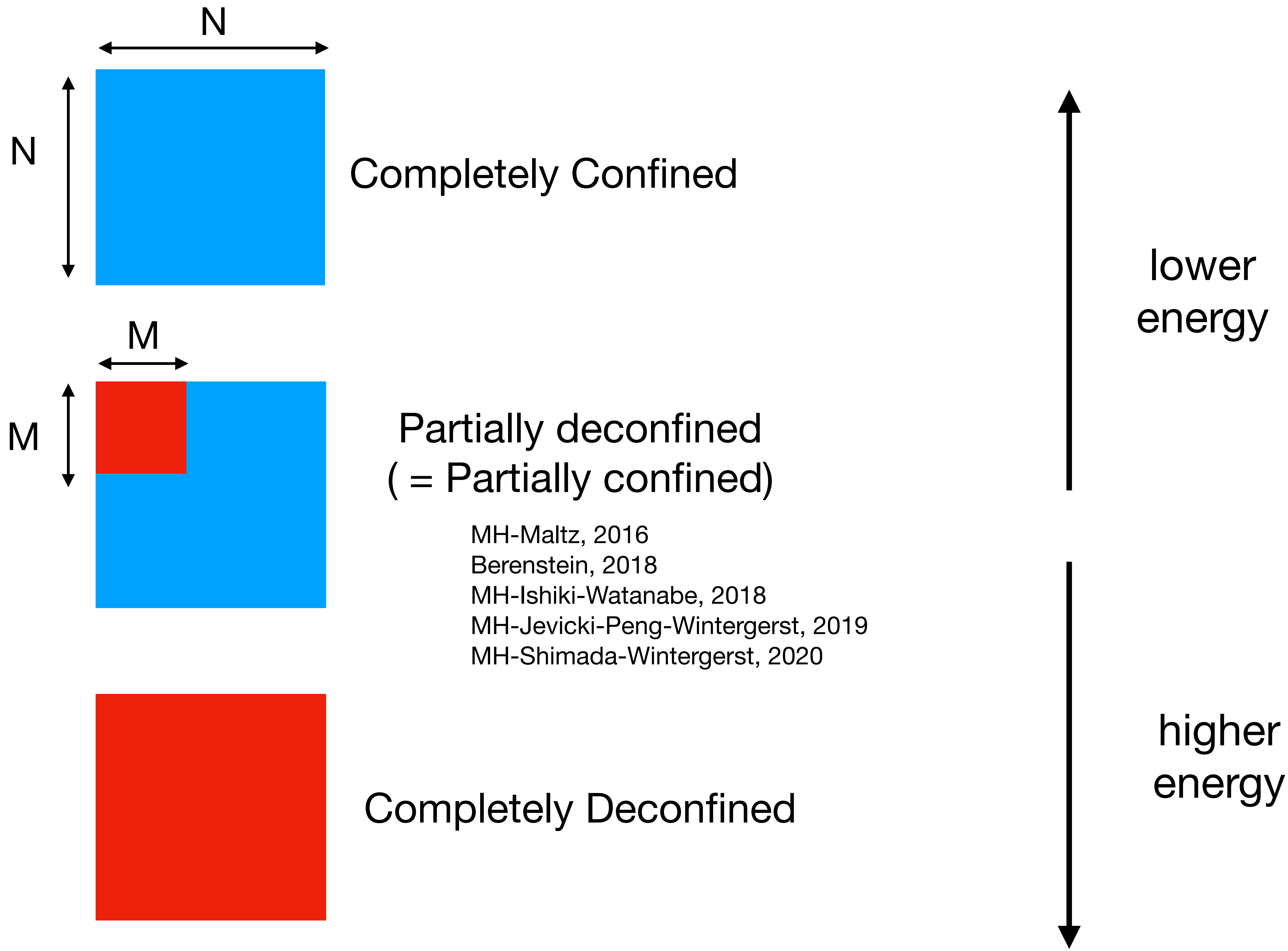
- Many trace relations set in at $E = N^2/4$
- Trace relations → Slower growth rate of density of states

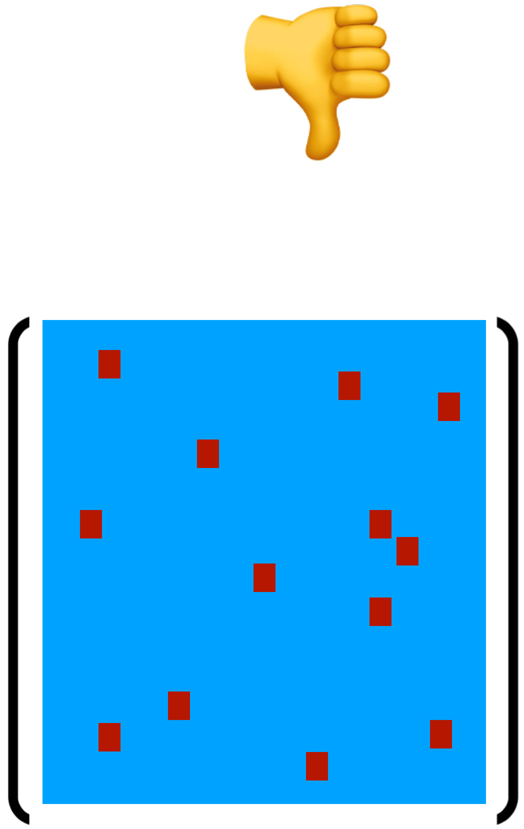
For $L = M^2/4 < N^2/4$:

$$|W, W', \cdots\rangle \text{ in } \mathrm{SU}(M) \quad \longleftrightarrow \quad |W, W', \cdots\rangle \text{ in } \mathrm{SU}(N)$$

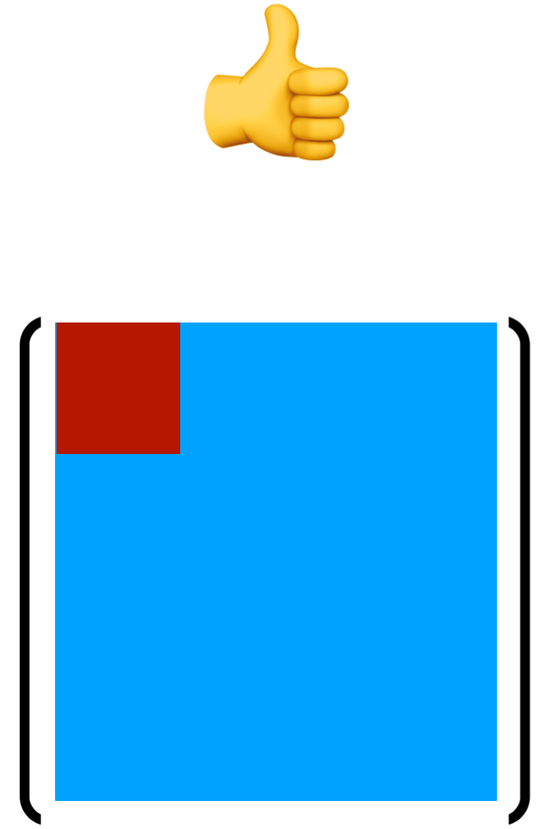
(almost) one-to-one correspondence

SU(N) theory looks like SU(M) theory

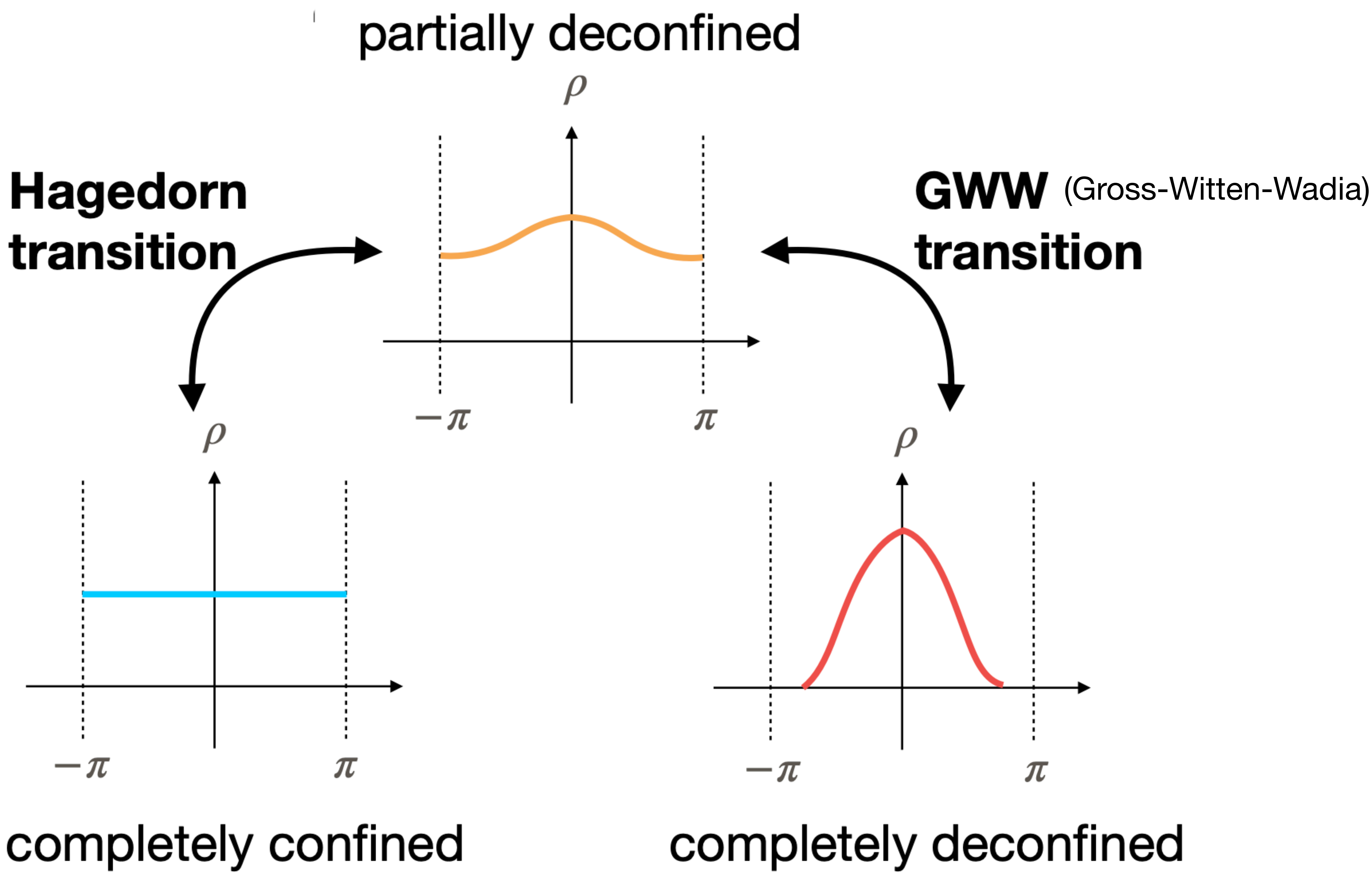

N
N
Completely Confined
M
M
Partially deconfined
(= Partially confined)
MH-Maltz, 2016
Berenstein, 2018
MH-Ishiki-Watanabe, 2018
MH-Jevicki-Peng-Wintergerst, 2019
MH-Shimada-Wintergerst, 2020
Completely Deconfined
lower
energy
higher
energy



no symmetry

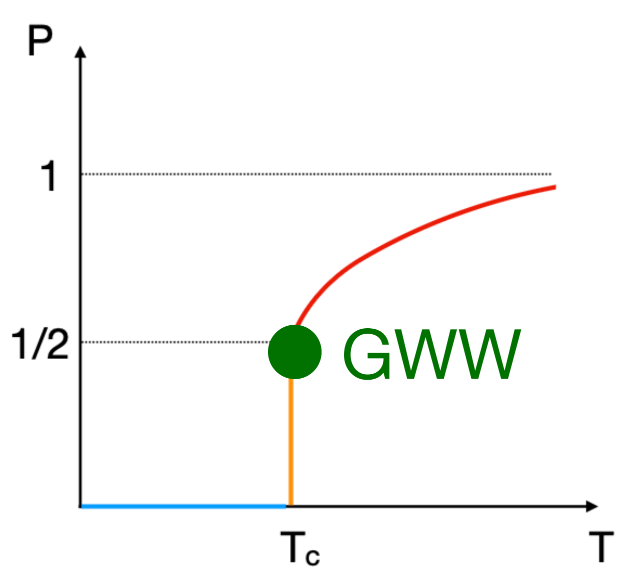


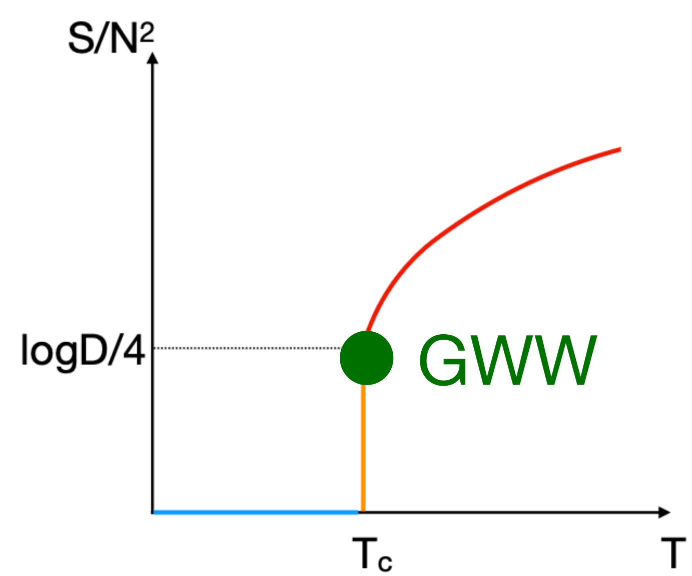
Polyakov line phase distribution

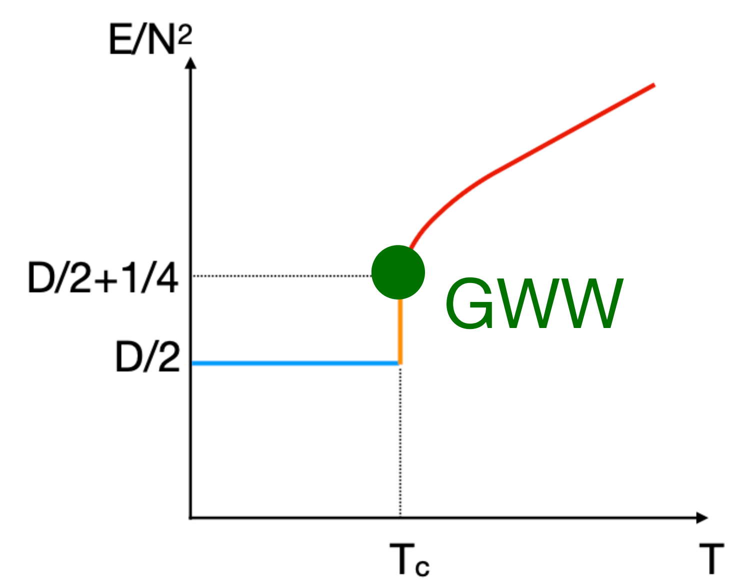


Gaussian Matrix Model

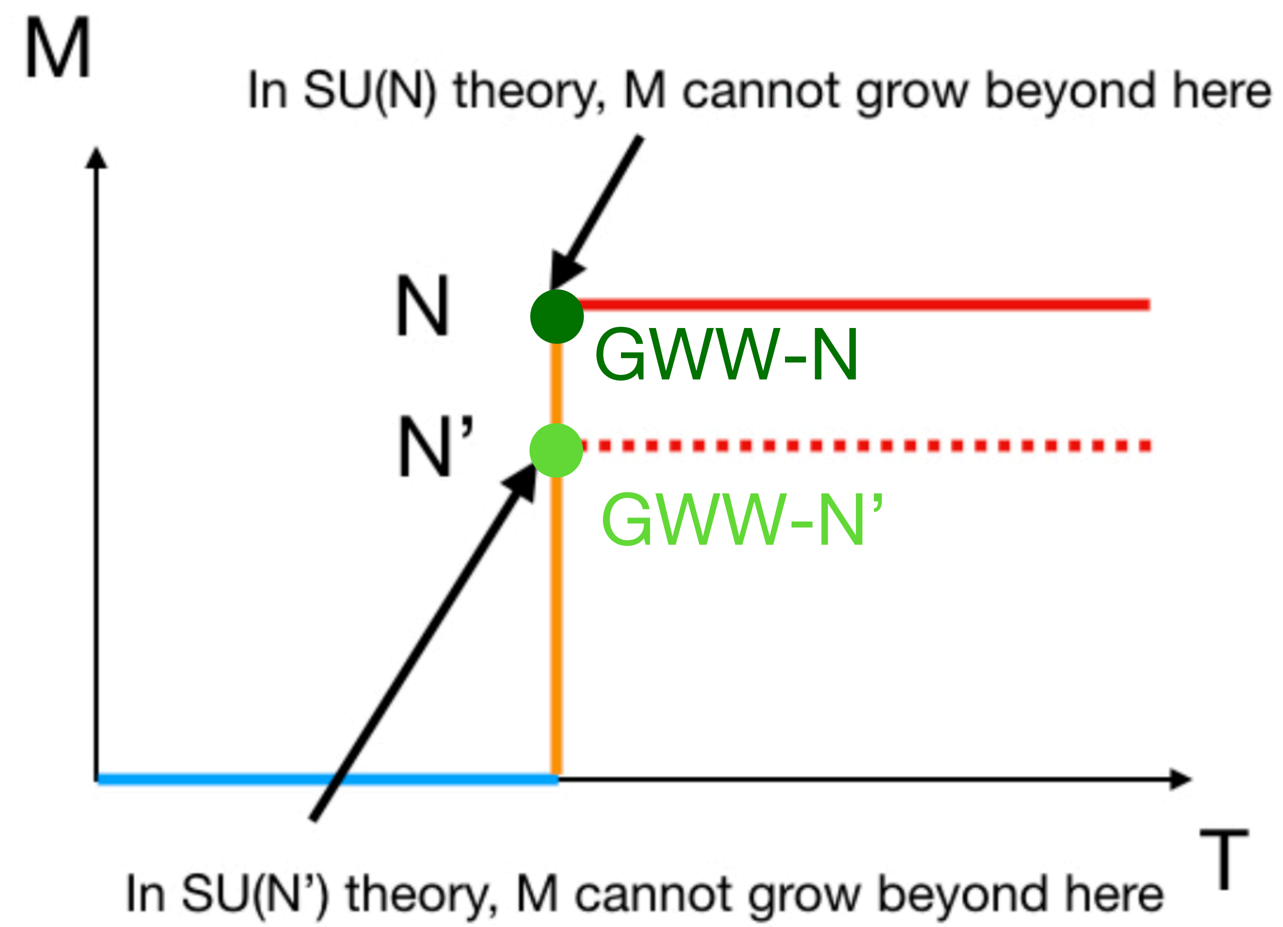
$$\hat{H}_{\text{free}} = \text{Tr} \sum_{I=1}^{D} \left(\frac{1}{2} \hat{P}_I^2 + \frac{1}{2} \hat{X}_I^2 \right) = \text{Tr} \sum_{I=1}^{D} \left(\hat{A}_I^{\dagger} \hat{A}_I \right) + \frac{DN^2}{2}$$

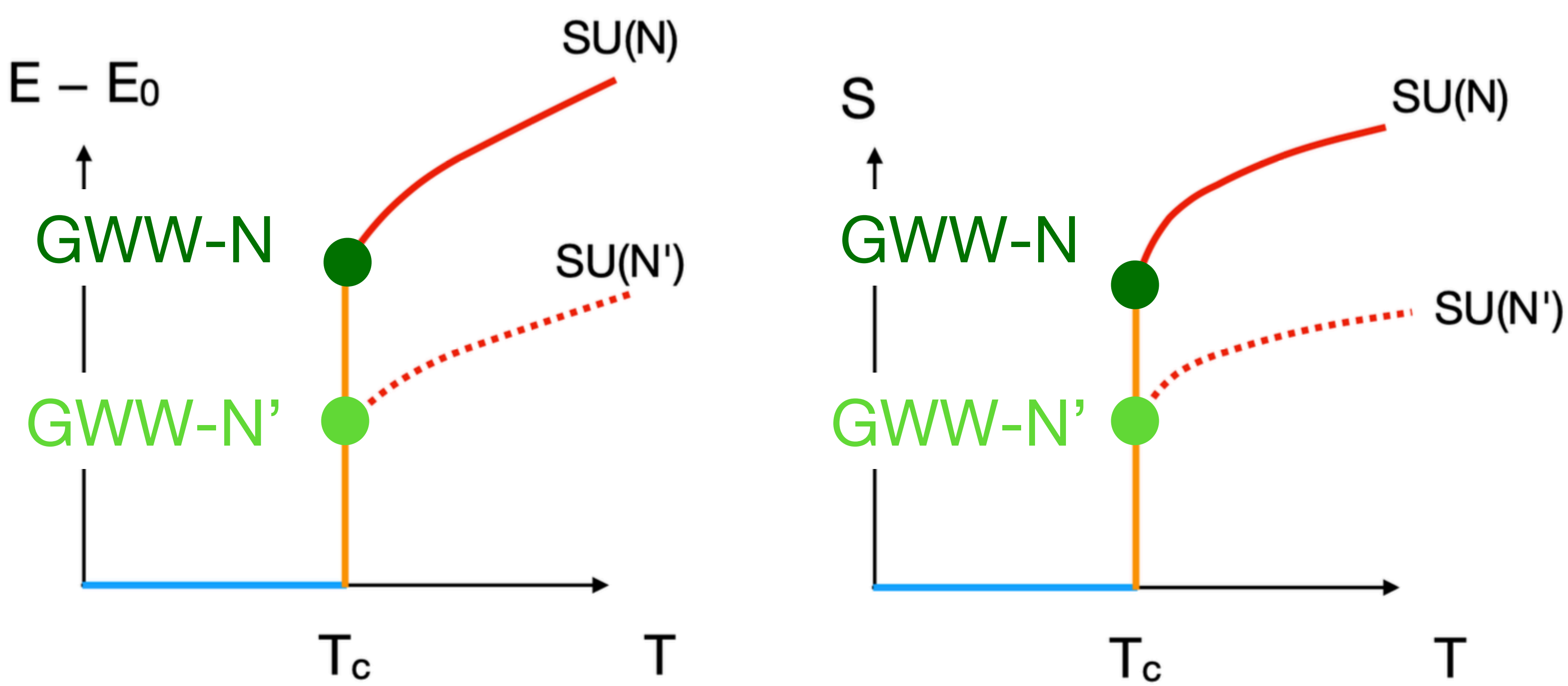




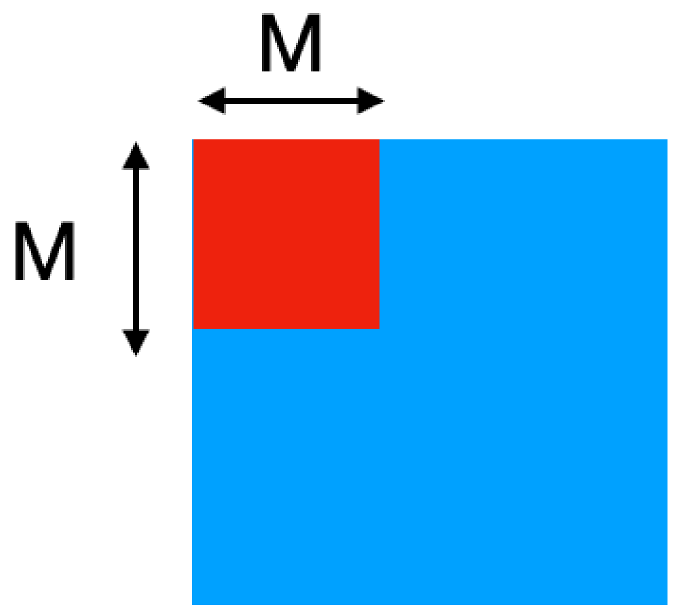


GWW transition


M
In SU(N) theory, M cannot grow beyond here
N
GWW-N
N’
GWW-N’
T
In SU(N’) theory, M cannot grow beyond here


SU(N)
E – E0
GWW-N
SU(N')
GWW-N’
Tc
T
S
SU(N)
GWW-N
SU(N')
GWW-N’
Tc
T

$$P = \frac{M}{2N}$$



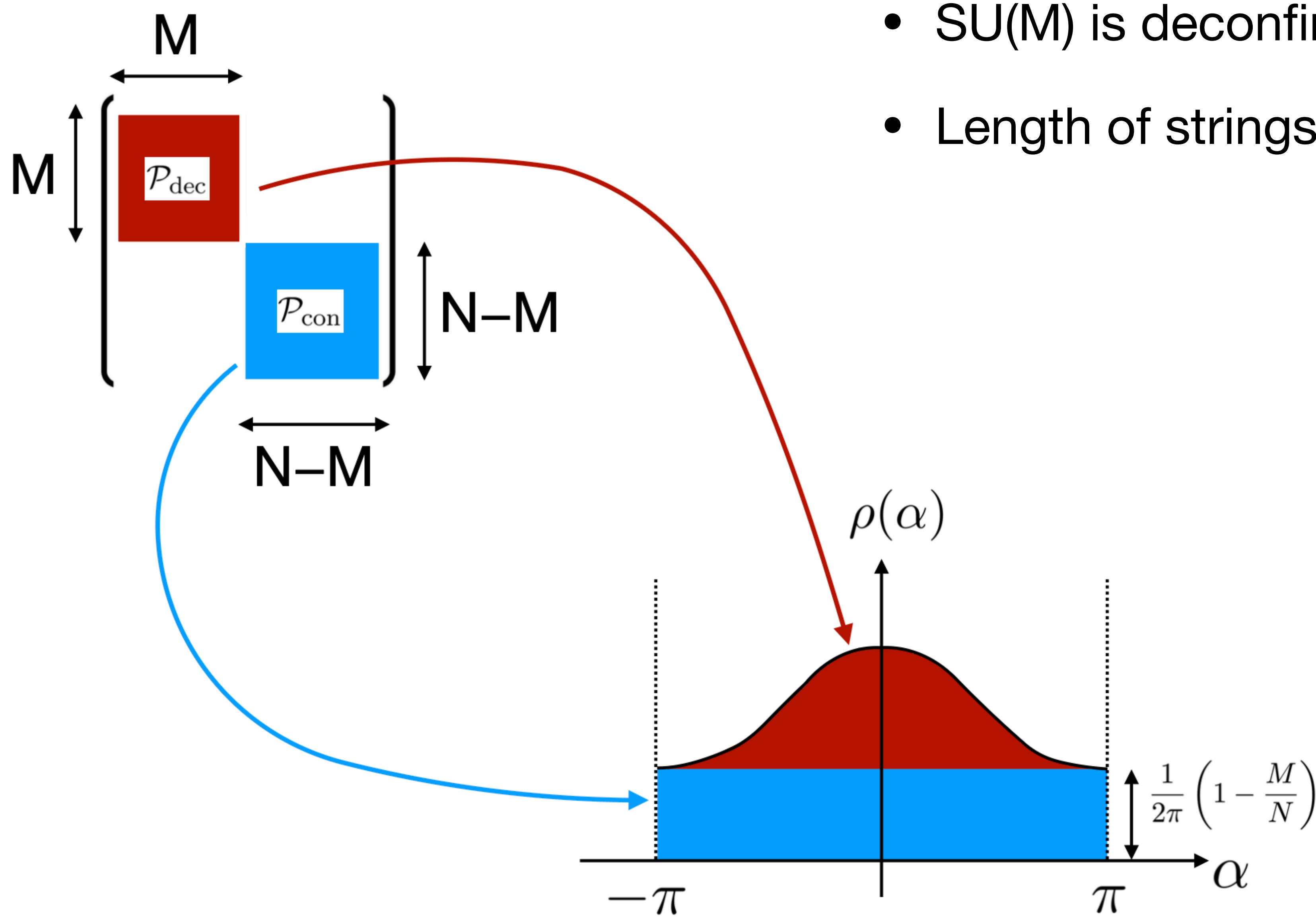
$$E(T = T_{\rm c}, P = \frac{M}{2N}, N) = \frac{D}{2} \cdot (N^2 - M^2) + \left(\frac{D}{2} + \frac{1}{4}\right) \cdot M^2 \,,$$

$$S(T = T_{\rm c}, P = \frac{M}{2N}, N) = 0 \cdot (N^2 - M^2) + \frac{\log D}{4} \cdot M^2 \,,$$

$$\rho(\theta)|_{T=T_{\rm c}, P=\frac{M}{2N}} = \frac{1}{2\pi} \cdot \left(1 - \frac{M}{N}\right) + \frac{1}{2\pi}(1 + \cos\theta) \cdot \frac{M}{N} \,.$$

SU(M)-deconfined state in SU(N) theory = GWW point of SU(M) theory

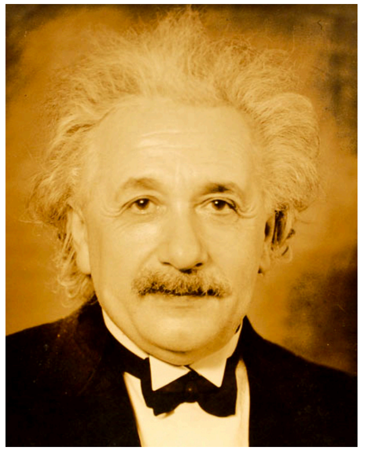
Meaning of Polyakov Loop



- SU(M) is deconfined
- Length of strings = $M^2/4$

(A derivation will be given shortly)

Bose



Einstein

N indistinguishable bosons

historically the first example of non-Abelian gauge theory
in the large-N limit

N bosons in 3d harmonic trap

1st quantization picture

$$\hat{H} = \sum_{i=1}^{N} \left(\frac{\hat{\vec{p}}_i^2}{2m} + \frac{m\omega^2}{2} \hat{\vec{x}}_i^2 \right)$$

$$\hat{\vec{x}}_i = (\hat{x}_i, \hat{y}_i, \hat{z}_i)$$

$$\hat{\vec{p}}_i = (\hat{p}_{x,i}, \hat{p}_{y,i}, \hat{p}_{z,i})$$

Fock states $|\vec{n}_1, \vec{n}_2, \cdots, \vec{n}_N\rangle \equiv \prod_{i=1}^{3} \frac{\hat{a}_{i1}^{\dagger n_{i1}}}{\sqrt{n_{i1}!}} \frac{\hat{a}_{i2}^{\dagger n_{i2}}}{\sqrt{n_{i2}!}} \cdots \frac{\hat{a}_{iN}^{\dagger n_{iN}}}{\sqrt{n_{iN}!}} |0\rangle$

States related by S_N permutation are identical.

→ S_N permutation is gauged.

Summation over singlet states $Z(T) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{inv}}}(e^{-\hat{H}/T})$

Summation over all states & projection to singlet states

$$Z(T) = \frac{1}{\mathrm{vol}(G)} \int_G dg \mathrm{Tr}_{\mathcal{H}_{\mathrm{ext}}}(\hat{g} e^{-\hat{H}/T})$$

$G = \mathrm{SU}(N)$ + adjoint fields → Yang-Mills, Matrix Model

$G = \mathrm{S}_N$ + fundamental fields → N indistinguishable bosons

For Yang-Mills and Matrix Model:

$$Z(T) = \int [dA_t][dX]e^{-S[A_t,X]}$$

Feynman's method

$$Z(T) = \frac{1}{\mathrm{vol}G}\int_G dg \mathrm{Tr}_{\mathcal{H}_{\mathrm{ext}}}\left(\hat{g}e^{-\hat{H}/T}\right)$$

Polyakov loop

$$Z(T) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{inv}}}(e^{-\hat{H}/T})$$

$$(\text{Analogy: } \mathcal{H}_{\mathrm{ext}} \sim \mathrm{Ker}\hat{Q}_{\mathrm{BRST}},\ \mathcal{H}_{\mathrm{inv}} \sim \mathrm{Ker}\hat{Q}_{\mathrm{BRST}}/\mathrm{Im}\hat{Q}_{\mathrm{BRST}}\ )$$

Non-interacting bosons * N

$$\hat{H} = \sum_{i=1}^{N} \left(\frac{\hat{\vec{p}}_i^2}{2m} + \frac{m\omega^2}{2} \hat{\vec{x}}_i^2 \right)$$

S_N gauge symmetry

Non-interacting bosons * N^2

$$\hat{H} = \frac{1}{2} \sum_I \mathrm{Tr} \left(\hat{P}_I^2 + \hat{X}_I^2 \right)$$

SU(N) gauge symmetry

Non-interacting bosons * N

$$\hat{H} = \sum_{i=1}^{N} \left(\frac{\hat{\vec{p}}_i^{\,2}}{2m} + \frac{m\omega^2}{2} \hat{\vec{x}}_i^{\,2} \right)$$

S_N gauge symmetry

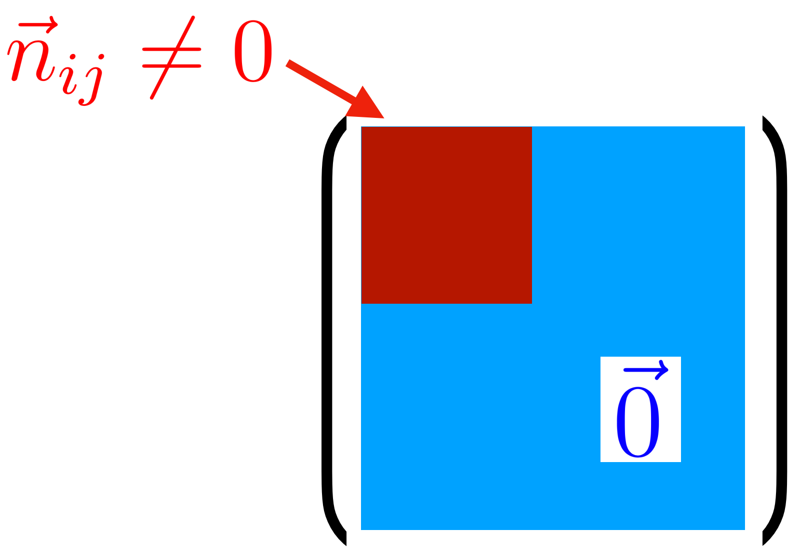
Bose-Einstein Condensation

$$|\vec{n}_1, \cdots, \vec{n}_M, \vec{0}, \cdots, \vec{0}\rangle$$

Non-interacting bosons * N^2

$$\hat{H} = \frac{1}{2} \sum_I \mathrm{Tr} \left(\hat{P}_I^2 + \hat{X}_I^2 \right)$$

SU(N) gauge symmetry



N bosons in 3d harmonic trap

$$\hat{H} = \sum_{i=1}^{N} \left(\frac{\hat{\vec{p}}_i^2}{2m} + \frac{m\omega^2}{2} \hat{\vec{x}}_i^2 \right)$$

$$\hat{\vec{x}}_i = (\hat{x}_i, \hat{y}_i, \hat{z}_i)$$

$$\hat{\vec{p}}_i = (\hat{p}_{x,i}, \hat{p}_{y,i}, \hat{p}_{z,i})$$

Fock states $|\vec{n}_1, \vec{n}_2, \cdots, \vec{n}_N\rangle \equiv \prod_{i=1}^{3} \frac{\hat{a}_{i1}^{\dagger n_{i1}}}{\sqrt{n_{i1}!}} \frac{\hat{a}_{i2}^{\dagger n_{i2}}}{\sqrt{n_{i2}!}} \cdots \frac{\hat{a}_{iN}^{\dagger n_{iN}}}{\sqrt{n_{iN}!}} |0\rangle$

$$Z(T) = \frac{1}{N!} \sum_{\sigma \in \mathrm{S}_N} \sum_{\vec{n}_1, \cdots, \vec{n}_N} \langle \vec{n}_1, \cdots, \vec{n}_N | \hat{\sigma} e^{-\hat{H}/T} | \vec{n}_1, \cdots, \vec{n}_N \rangle$$

$$= \frac{1}{N!} \sum_{\vec{n}_1, \cdots, \vec{n}_N} e^{-(E_{\vec{n}_1} + \cdots + E_{\vec{n}_N})/T} \left(\sum_{\sigma \in \mathrm{S}_N} \langle \vec{n}_1, \cdots, \vec{n}_N | \vec{n}_{\sigma(1)}, \cdots, \vec{n}_{\sigma(N)} \rangle \right)$$

measures the amount of redundancy

京都　三十三間堂

Sanjusangendo, Kyoto

N=1001

(Einstein visited Kyoto in 1922)

$$Z(T) = \frac{1}{N!} \sum_{\sigma \in \mathrm{S}_N} \sum_{\vec{n}_1, \cdots, \vec{n}_N} \langle \vec{n}_1, \cdots, \vec{n}_N | \hat{\sigma} e^{-\hat{H}/T} | \vec{n}_1, \cdots, \vec{n}_N \rangle$$

$$= \frac{1}{N!} \sum_{\vec{n}_1, \cdots, \vec{n}_N} e^{-(E_{\vec{n}_1} + \cdots + E_{\vec{n}_N})/T} \underline{\left(\sum_{\sigma \in \mathrm{S}_N} \langle \vec{n}_1, \cdots, \vec{n}_N | \vec{n}_{\sigma(1)}, \cdots, \vec{n}_{\sigma(N)} \rangle \right)}$$

$|\vec{0}, \vec{0}, \cdots, \vec{0}\rangle$ $\quad N!$

$|\vec{n}_1, \cdots, \vec{n}_N\rangle$ $\quad 1$

(all of them are different)

$|\vec{n}_1, \cdots, \vec{n}_M, \vec{0}, \cdots, \vec{0}\rangle$ $\quad (N-M)!$

Enhancement factor → BEC

Einstein, 1924
Feynman, 1953

$$Z(T) = \frac{1}{\mathrm{vol}(G)} \int_G dg \mathrm{Tr}_{\mathcal{H}_{\mathrm{ext}}}(\hat{g} e^{-\hat{H}/T})$$

$G = \mathrm{S}_N$ + fundamental fields → N indistinguishable bosons

The enhancement factor $N! = \mathrm{vol}(\mathrm{S}_N)$ triggers BEC.

(Einstein, 1924; Feynman, 1953)

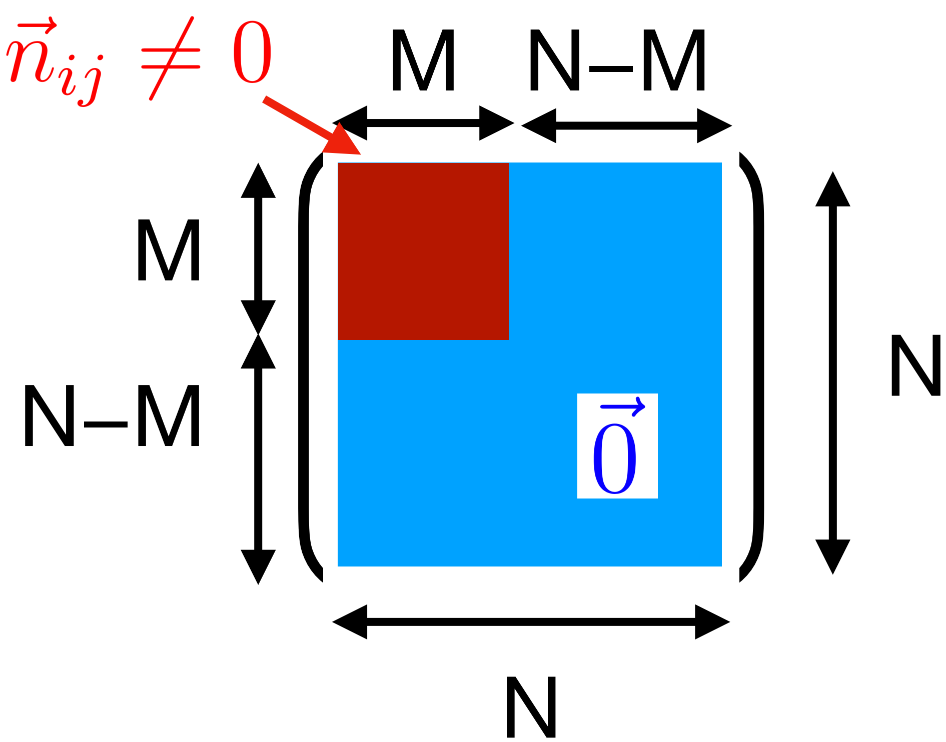
$G = \mathrm{SU}(N)$ + adjoint fields → Yang-Mills, Matrix Model

The enhancement factor $\mathrm{vol}(\mathrm{SU}(N)) \sim e^{N^2}$ triggers confinement.

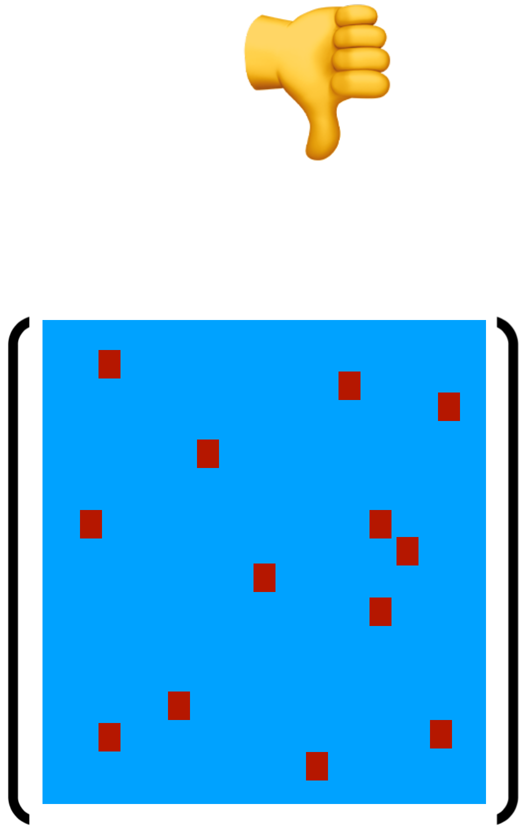
Partially-BEC state

$$|\vec{n}_1, \cdots, \vec{n}_M, \vec{0}, \cdots, \vec{0}\rangle \quad (N-M)!$$

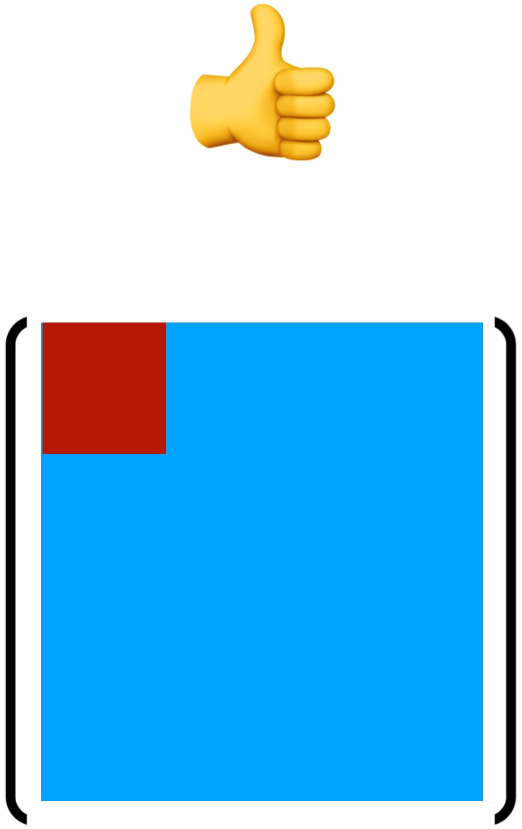
Partially-confined state



$$\mathrm{vol}(\mathrm{SU}(N-M)) \sim e^{(N-M)^2}$$



no symmetry



Larger enhancement factor (volume of SU(N-M))

Non-interacting bosons * N

$$\hat{H} = \sum_{i=1}^{N} \left(\frac{\hat{\vec{p}}_i^2}{2m} + \frac{m\omega^2}{2} \hat{\vec{x}}_i^2 \right)$$

S_N gauge symmetry

Bose-Einstein Condensation

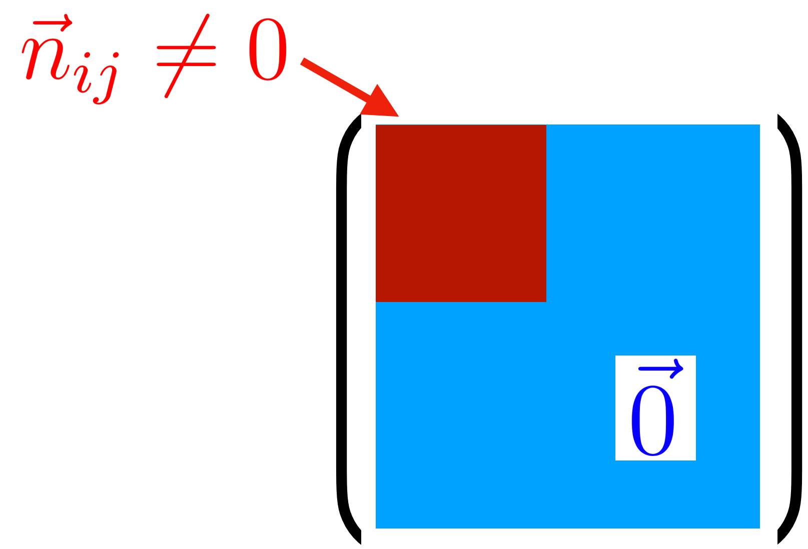
$$\left| \vec{n}_1, \cdots, \vec{n}_M, \vec{0}, \cdots, \vec{0} \right\rangle$$

Non-interacting bosons * N²

$$\hat{H} = \frac{1}{2} \sum_I \mathrm{Tr} \left(\hat{P}_I^2 + \hat{X}_I^2 \right)$$

SU(N) gauge symmetry

Partial confinement

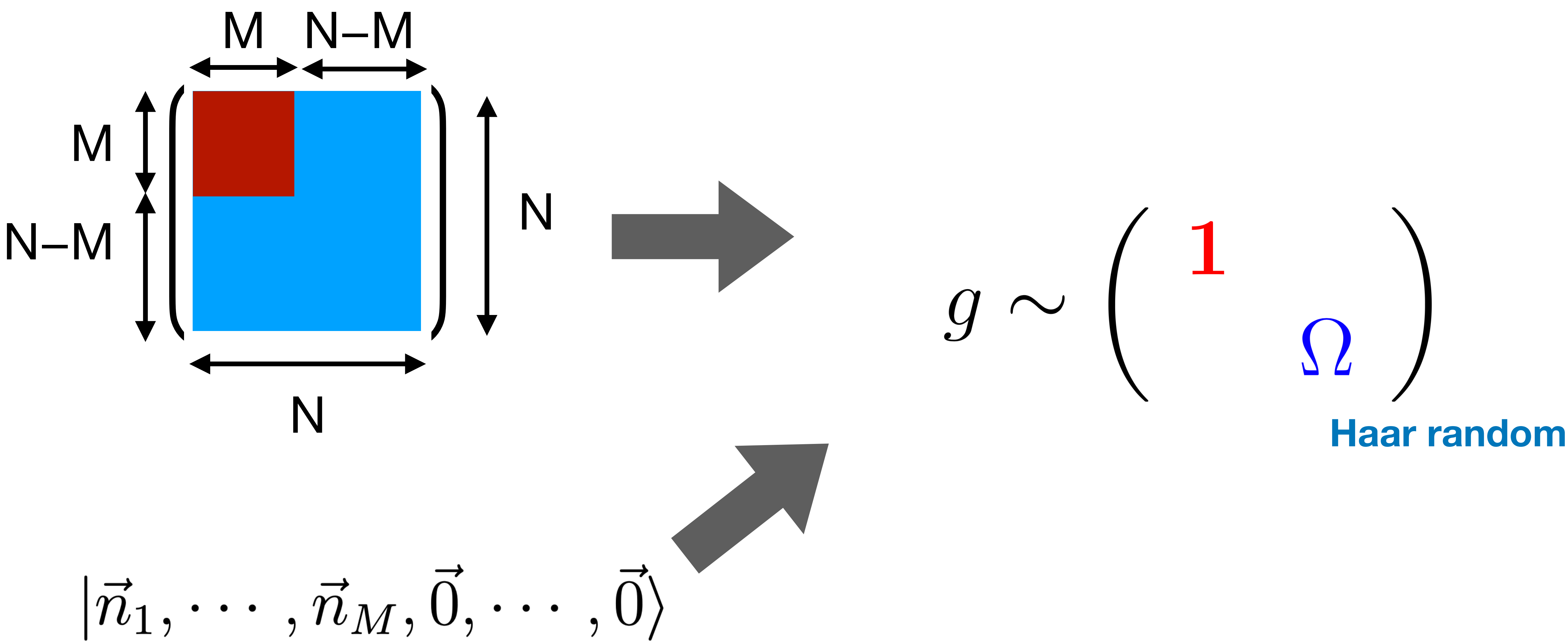


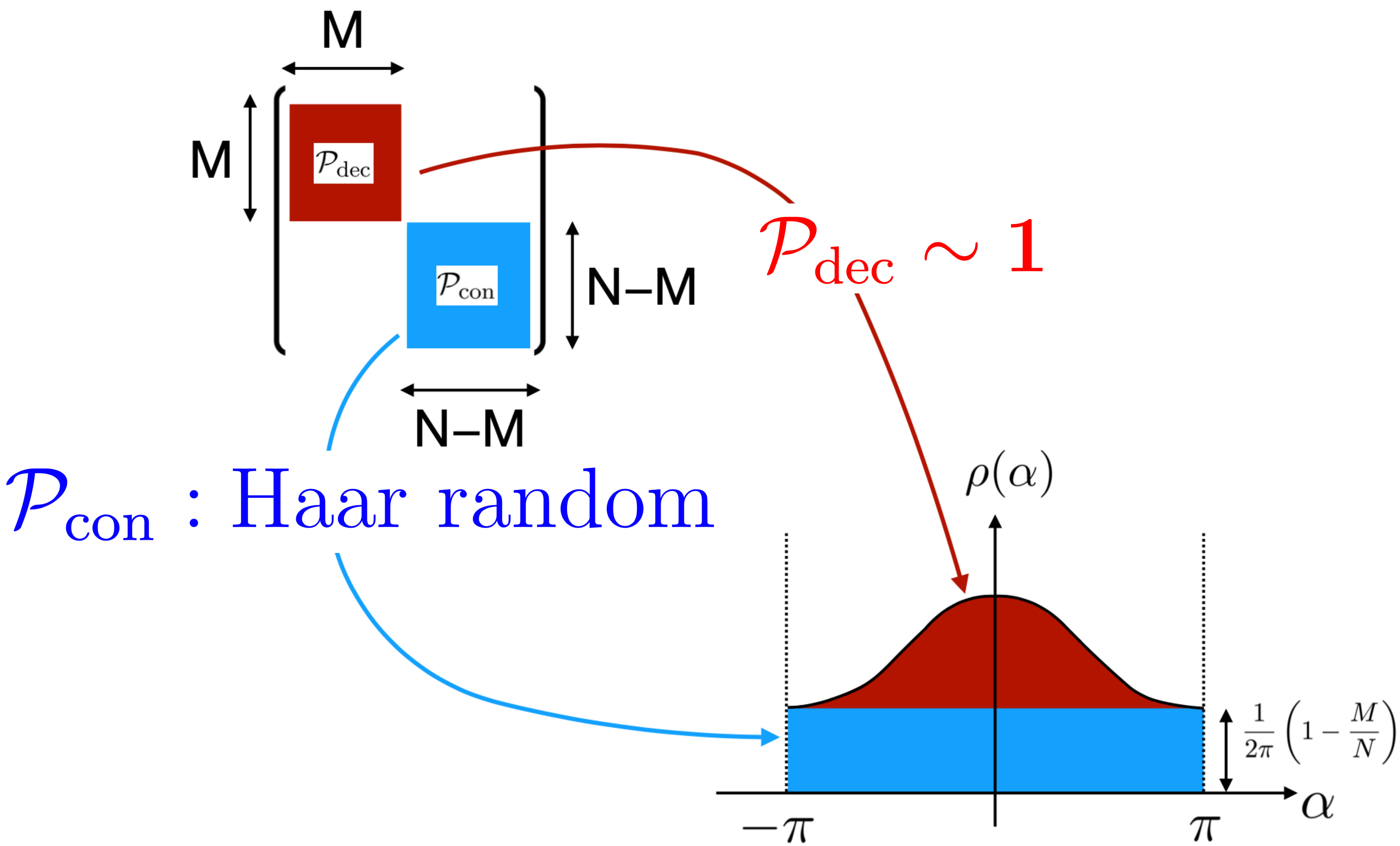
$$Z(T) = \frac{1}{\mathrm{vol}G}\int_G dg \mathrm{Tr}_{\mathcal{H}_{\mathrm{ext}}}\left(\hat{g}e^{-\hat{H}/T}\right)$$

$$\sim \frac{1}{\mathrm{vol}G}e^{-E_{\mathrm{typical}}/T}\int_G dg\langle\mathrm{typical}|\hat{g}|\mathrm{typical}\rangle$$

Polyakov loop

Typical $\hat{g}$'s which leave $|\mathrm{typical}\rangle$ unchanged dominate the phase distribution



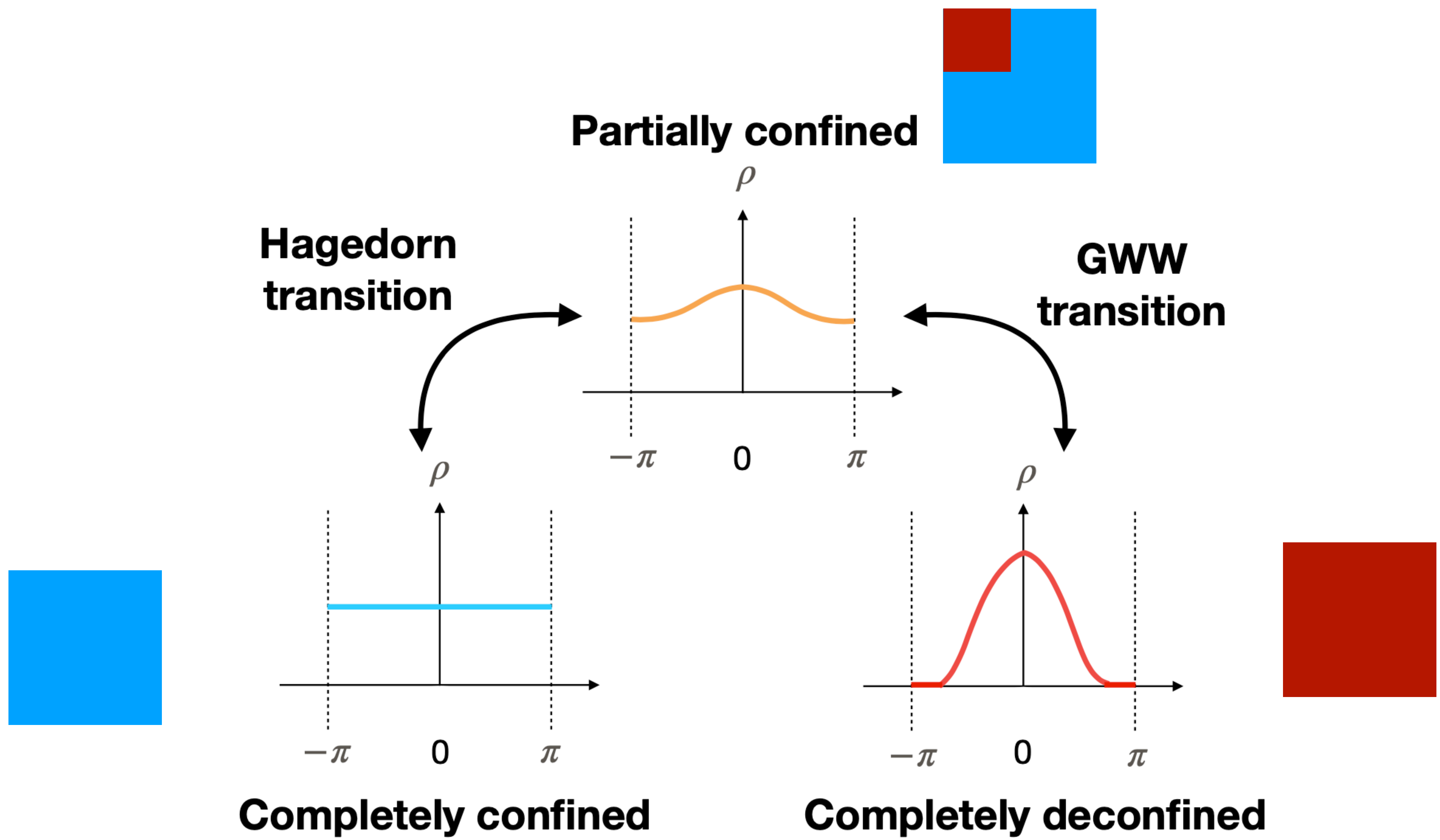


MH-Shimada-Wintergerst, 2020

(Essentially, Feynman found this n 1953)

Generalization to QFT & finite N: MH-Watanabe, 2023

Polyakov Loop $P = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$



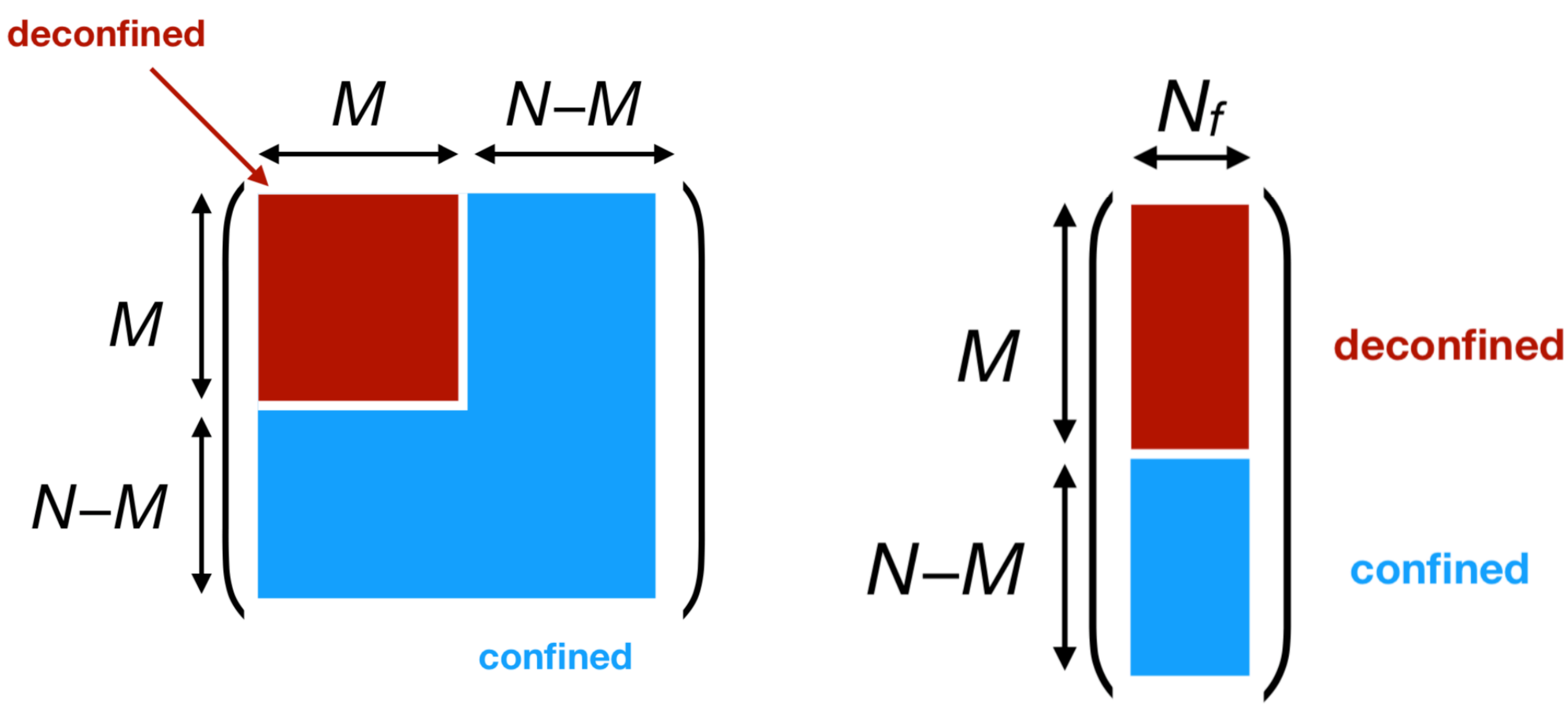
MH-Maltz, 2016 (JHEP)
MH-Ishiki-Watanabe, 2018 (JHEP)
MH-Shimada-Wintergerst, 2020 (JHEP)

No problem with interactions

$$|\text{ground state}\rangle = \hat{U}\,|\mathbf{0}\rangle$$

$$\hat{A}_{I,ij}^{(\text{dressed})} = \hat{U}\,\hat{A}_{I,ij}^{(\text{free})}\hat{U}^{-1}$$

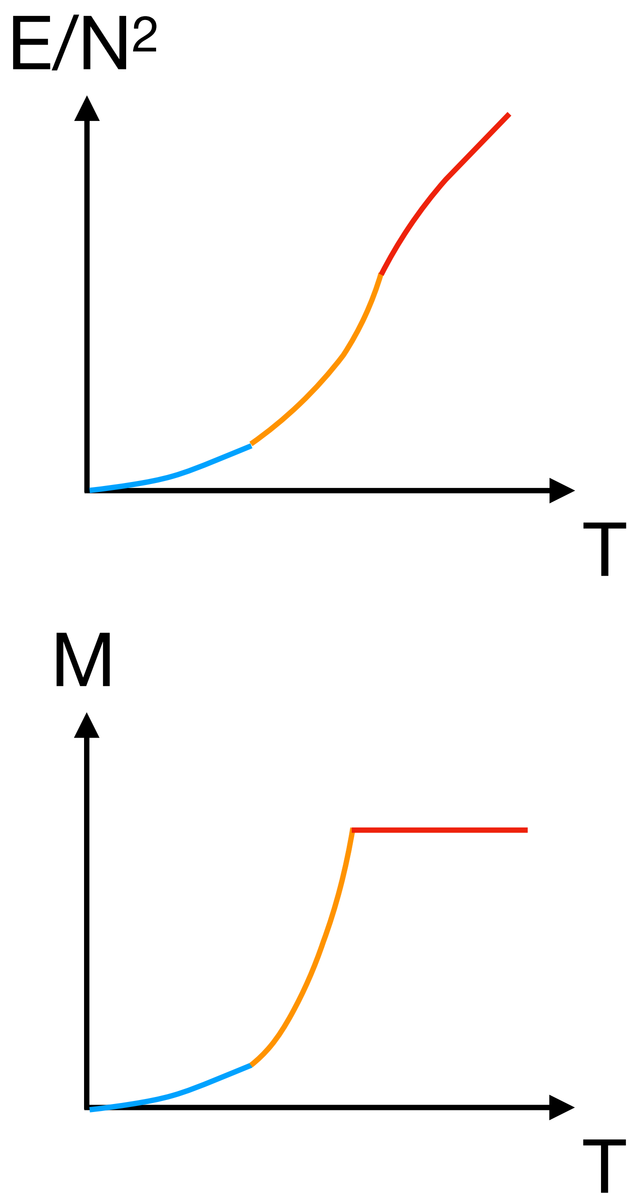
QCD phase transition



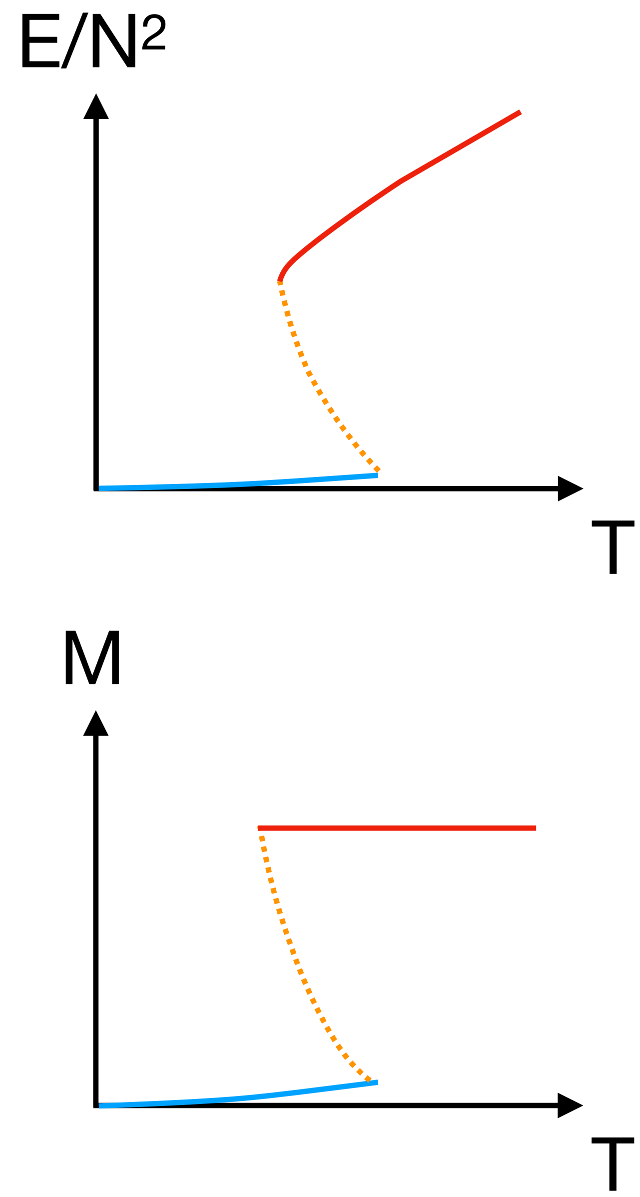
Weak-coupling analysis: MH-Robinson 2019
(Essentially, re-interpretation of Schnitzer 2004)

Finite N: MH-Ohata-Shimada-Watanabe 2023

QCD phase transition



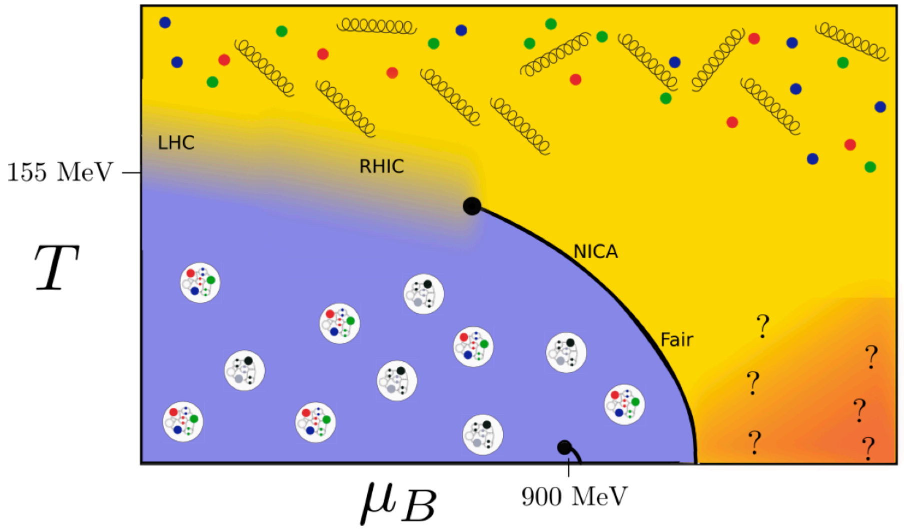
Light quark mass



Heavy quark mass

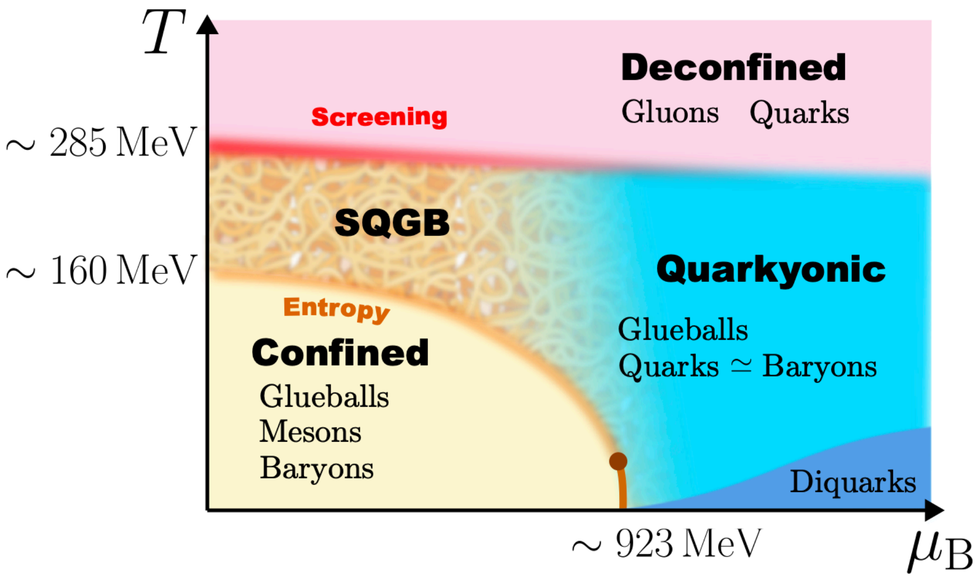
QCD phase diagram

(speculation)



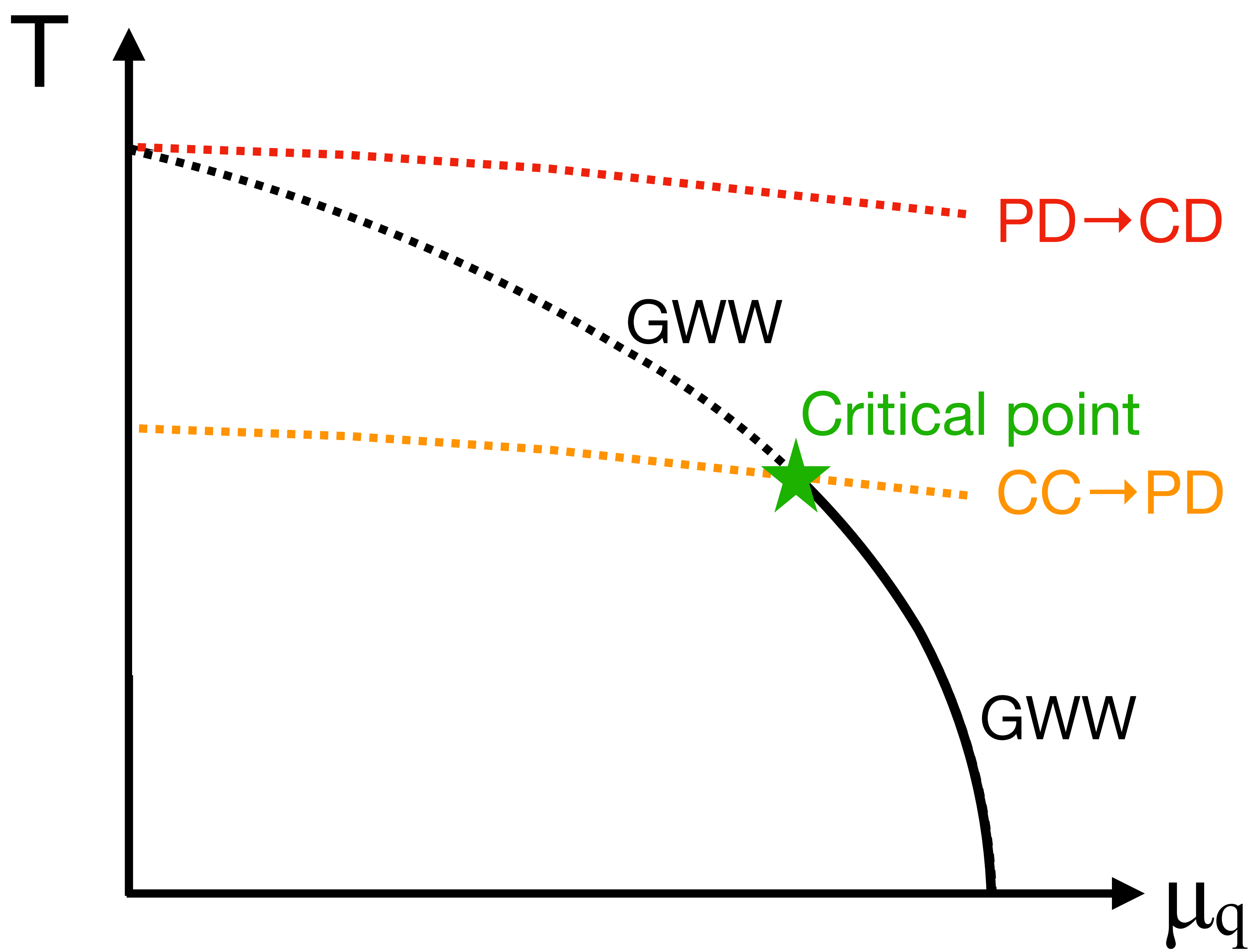
Guenther, 2010.15503

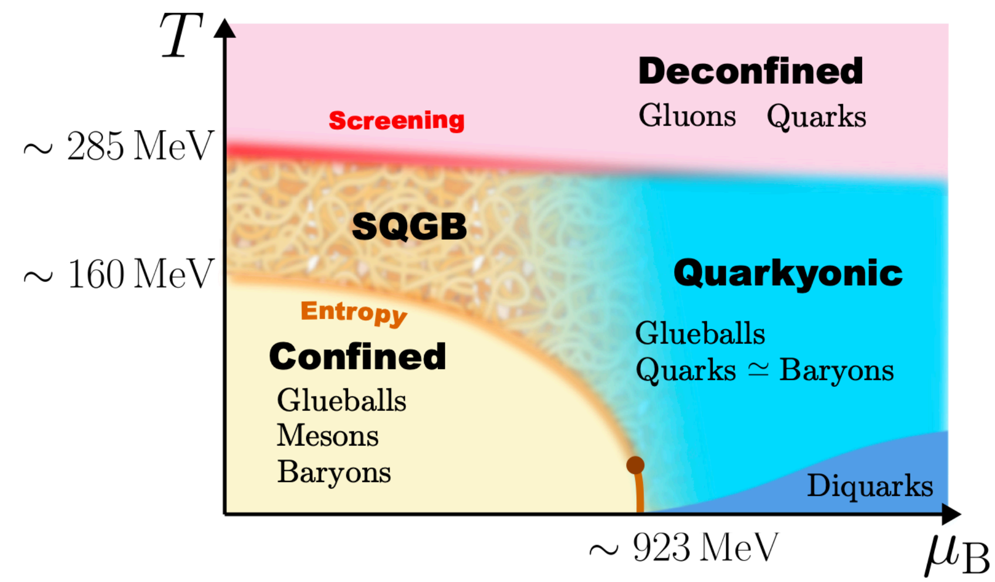
Fujimoto-Fukushima-Hidaka-McLerran
2506.00237



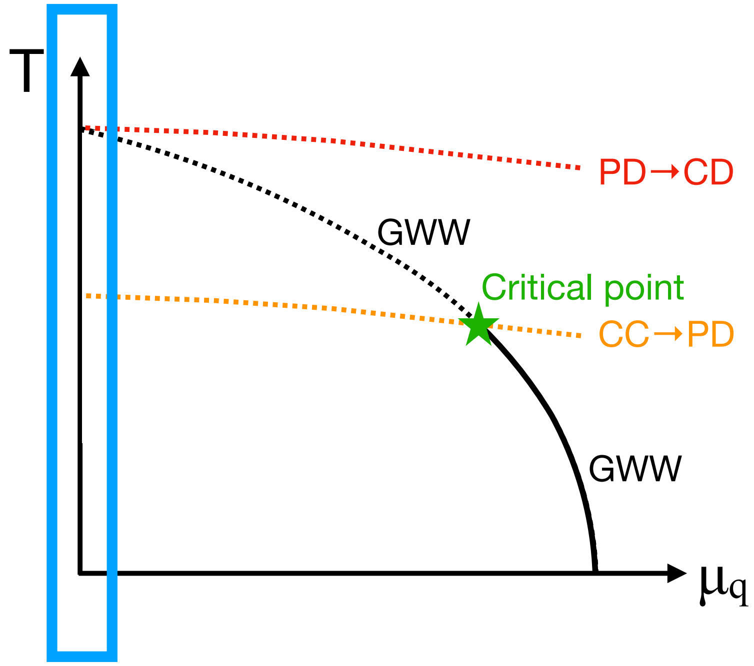
- We saw <u>three phases</u> at finite T, zero μ.
- Fujimoto, Fukushima, Hidaka, and McLerran considered finite chemical potential and conjectured three phases. (2506.00237)
- We study Veneziano large-N limit (N_f/N_c fix) and that there are <u>four phases</u>.

A natural scenario

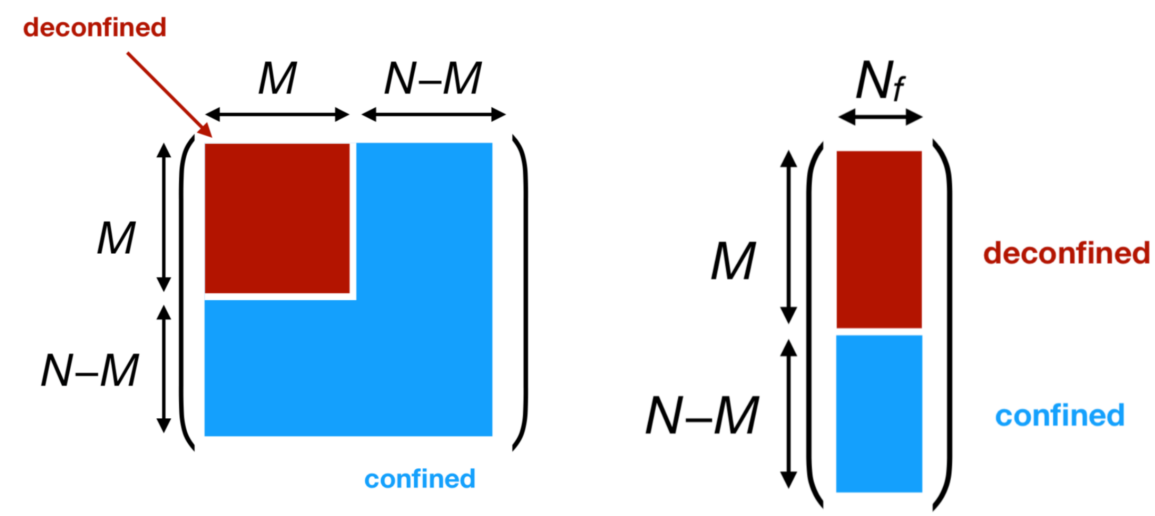


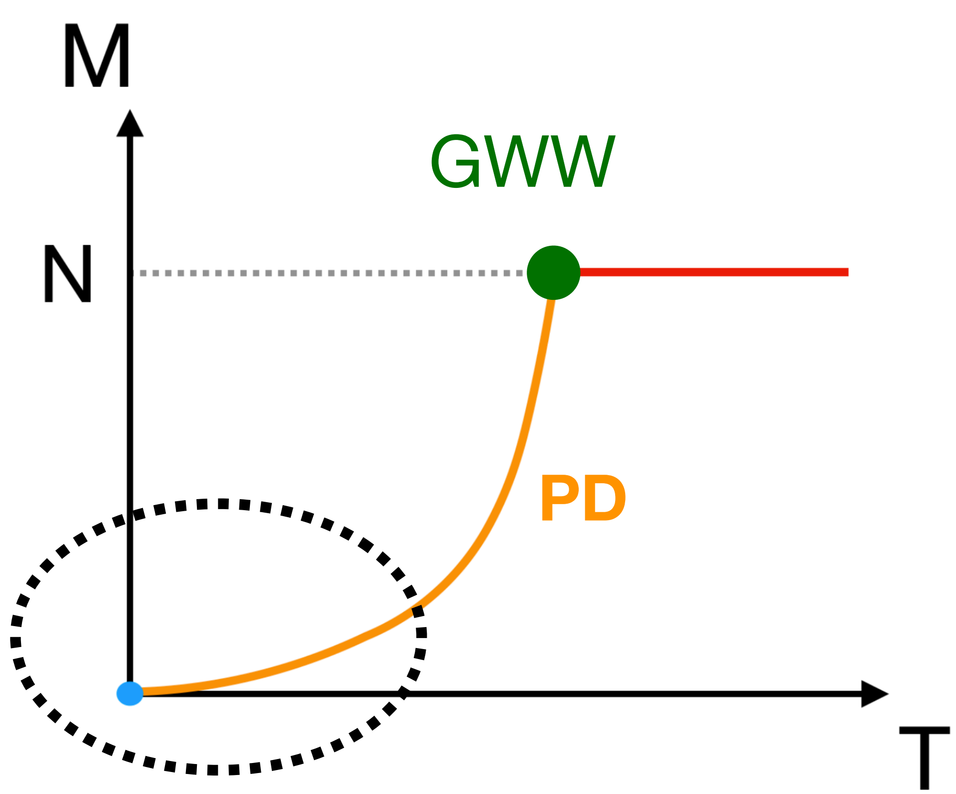


Fujimoto-Fukushima-Hidaka-McLerran
2506.00237

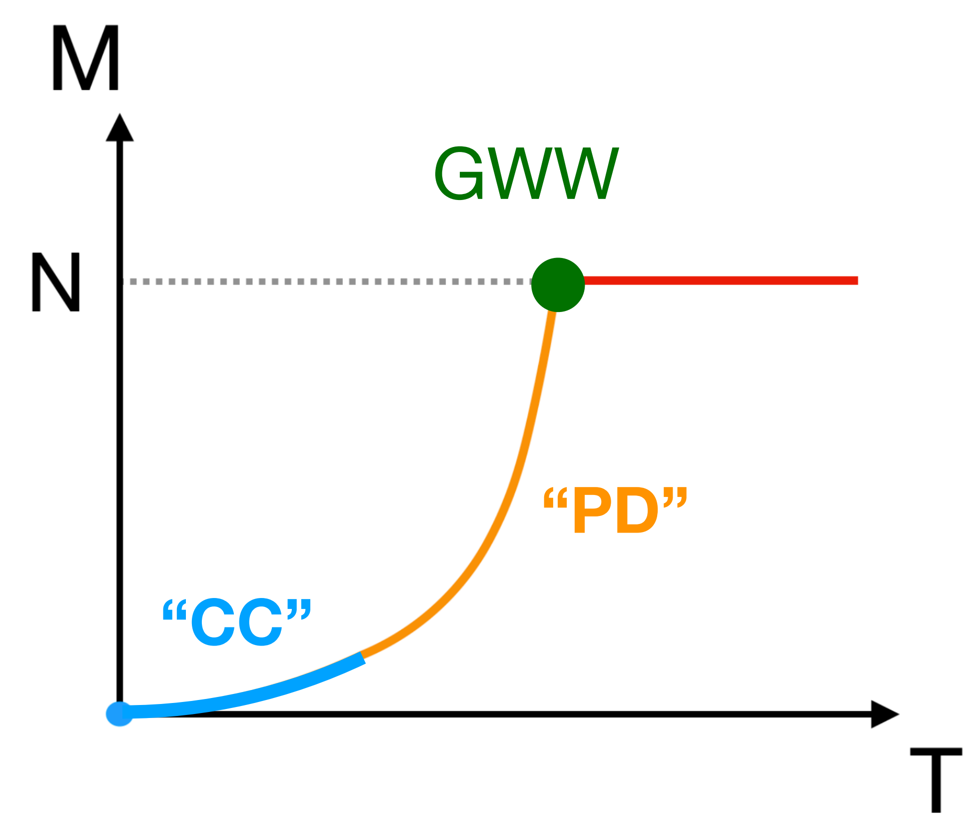


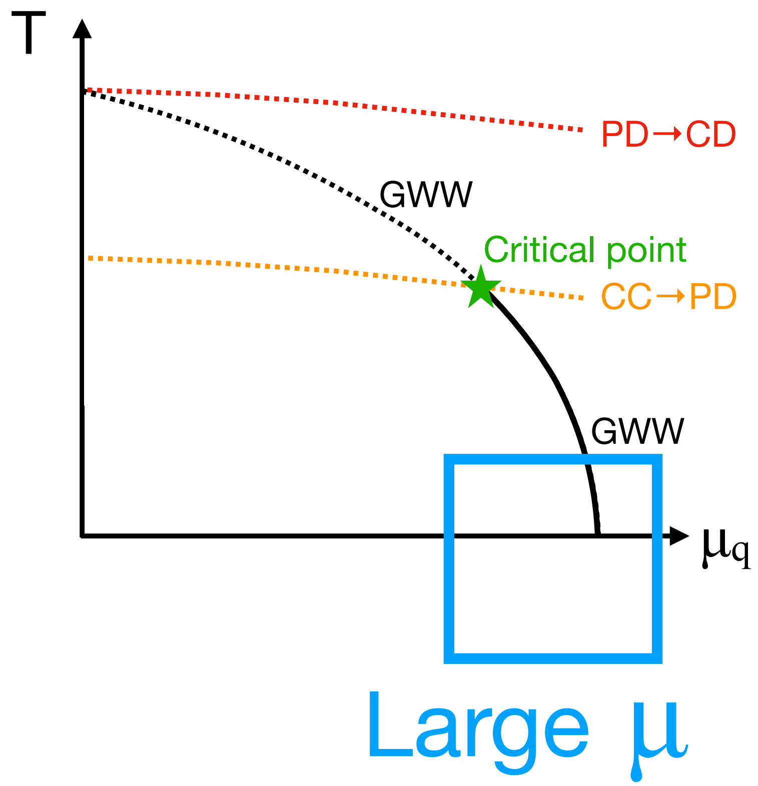
$\mu = 0$





(Almost) Completely Confined





$$\eta \equiv \frac{N_f}{N_c}$$

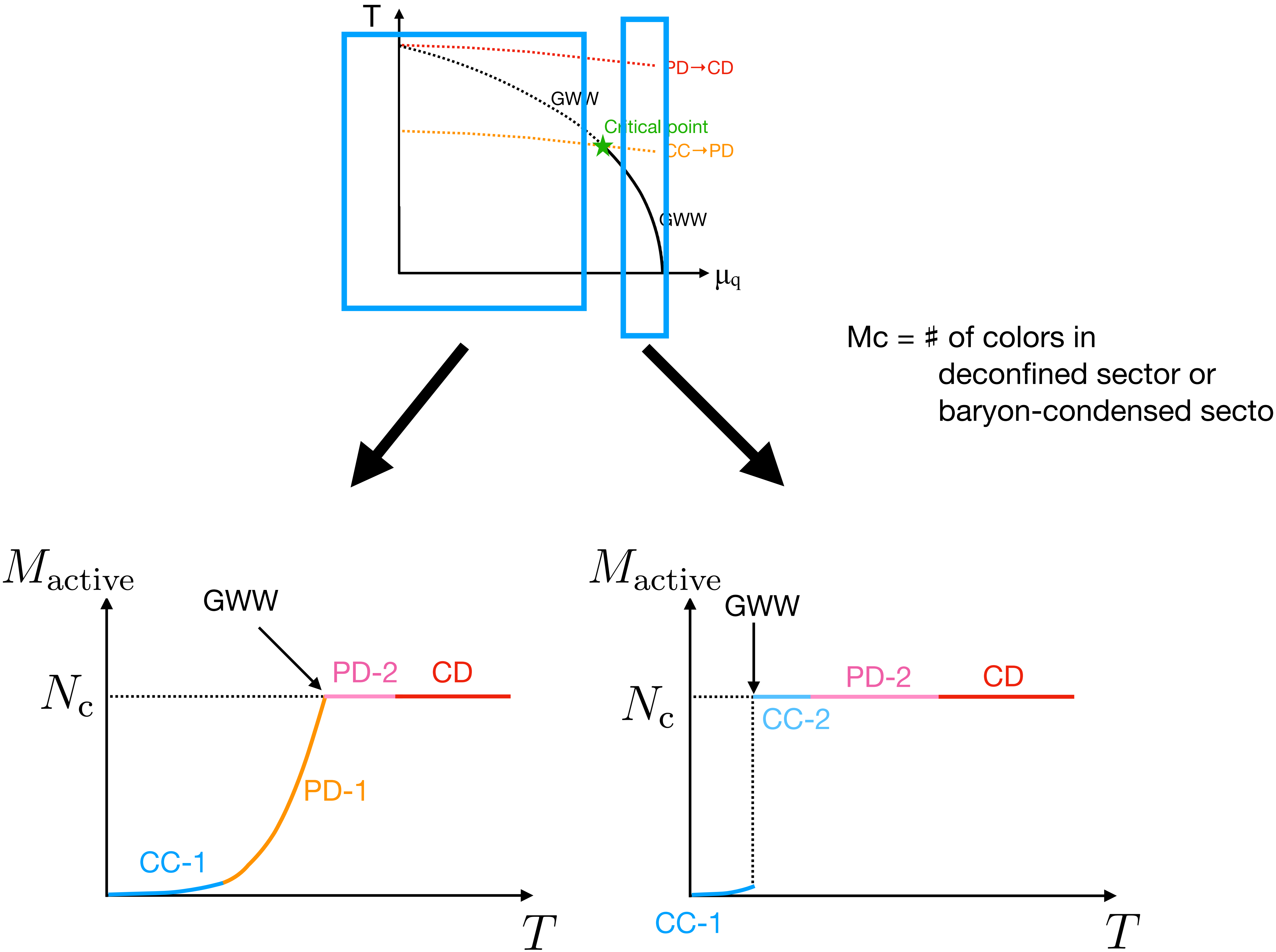
Baryon condensation @

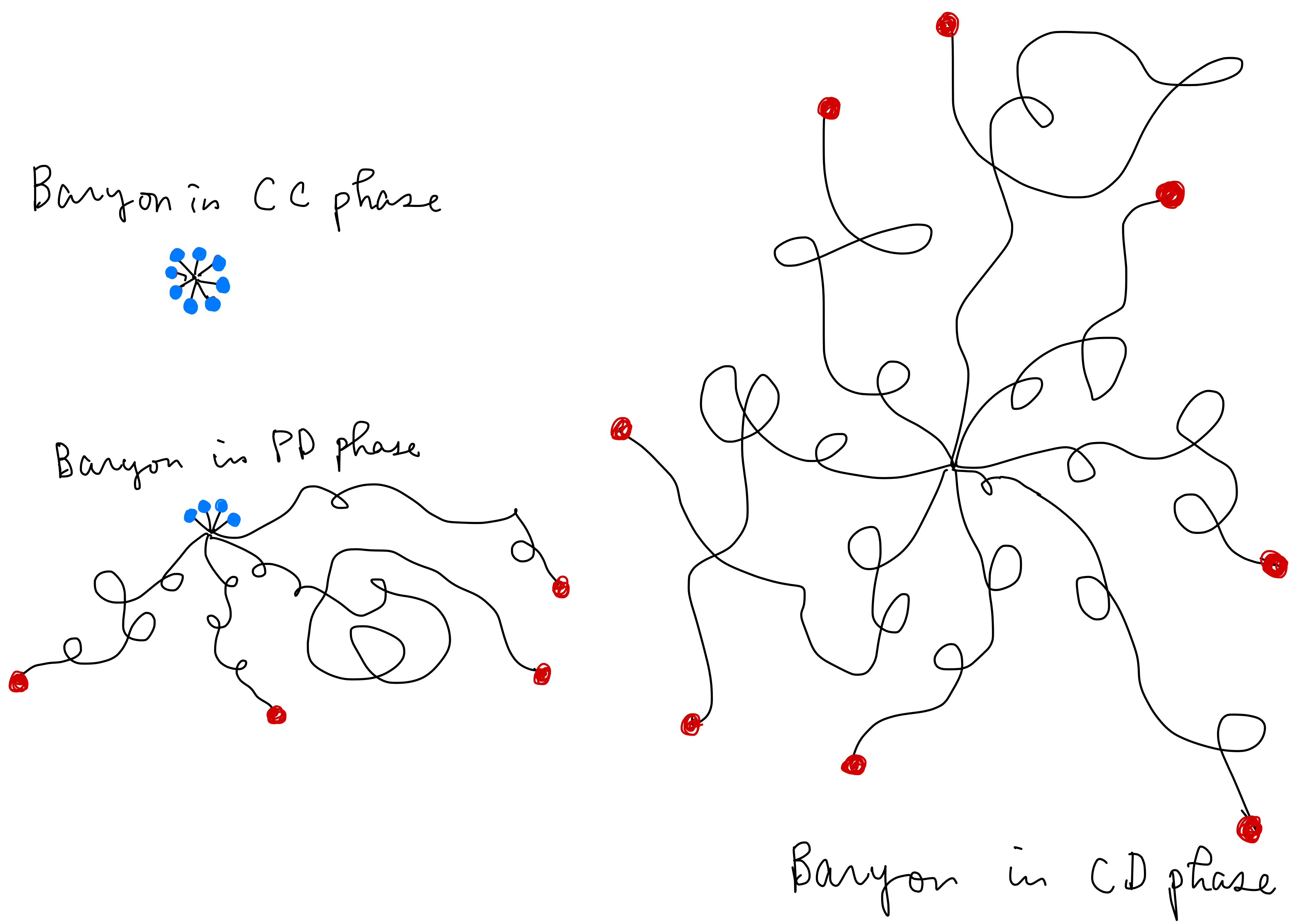
$$T_B = \frac{m - \mu_q}{(1+2\eta)\log(1+2\eta) - 2\eta\log(2\eta)}$$

(Hidaka, McLerran, Pisarski, 2008)

- N_c colors are split into $2N_f$ clusters (flavor and spin up/down)
  → various kinds of baryons
- Baryon condensation can cause GWW-ish transition.

String condensation is not the only way to introduce redundancy


T
PD→CD
GWW
Critical point
CC→PD
GWW
μ_q
Mc = # of colors in
deconfined sector or
baryon-condensed sector
$M_{\rm active}$
GWW
$N_{\rm c}$
PD-2
CD
PD-1
CC-1
T
$M_{\rm active}$
GWW
$N_{\rm c}$
PD-2
CD
CC-2
CC-1
T



Speculation

Summary

- The amount of gauge redundancy has physical consequences
- Deconfinement = String condensation
- Offset of Polyakov line phase distribution
  → size of deconfined sector ~ string length (at $\mu=0$)
- String condensation + baryon condensation
  → 4 phases at fixed μ