Semiclassics for QCD vacuum via T^2 compactification

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Buenas Ideas on the QCD Phase Diagram @ YITP, KyotoU.

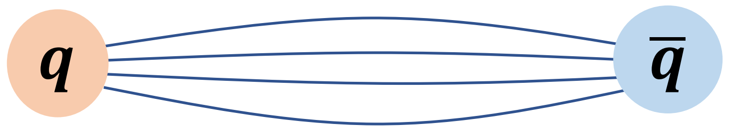
26–29 May 2026

Based on
JHEP **08** (2024) 001 [arXiv:2402.04320] with Yuya Tanizaki (YITP, Kyoto U.)

Introduction: confinement and θ angle

“Confinement implies multi-branch structure of θ dependence“

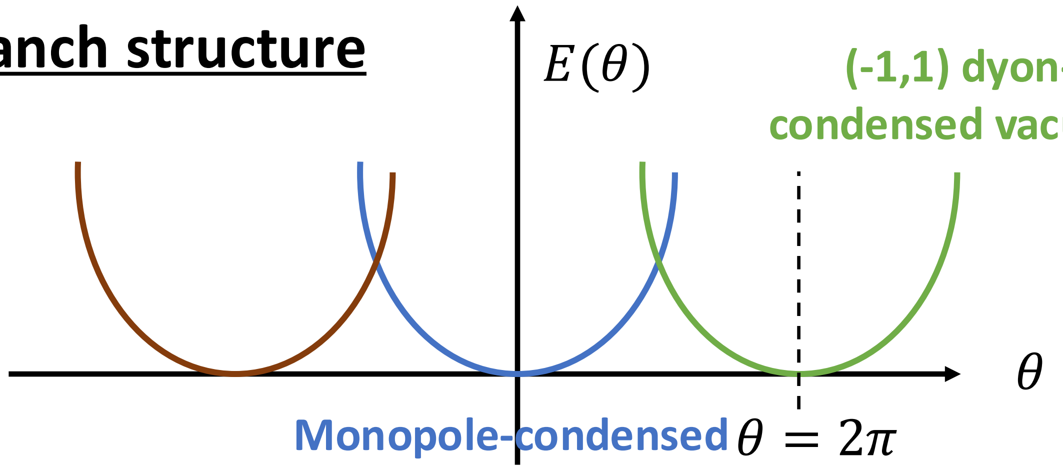
(monopole condensation) ➡



Witten effect: monopole acquires electric charge $\theta/2\pi$ by increasing θ

⬇

<u>Multi-branch structure</u>



cf.) Large-N [Witten ‘80]; anomaly [Gaiotto-Kapustin-Komargodski-Seiberg ‘17]

Introduction: chiral Lagrangian

- **Low-energy effective theory of QCD: $SU(N_f)$ Chiral Lagrangian**

Light pseudoscalar mesons: Nambu-Goldstone bosons of (approximate) $SU(N_f)_{\mathrm{chiral}}$

$$\Rightarrow S[U] = \int f_\pi^2 \, |dU|^2 - \Lambda^3 \mathrm{tr}\,(MU) + c.c.$$

- **Chiral Lagrangian with η'**

Often, one includes η' by considering $U(N_f)$ chiral Lagrangian and adds the instanton-induced η' mass term (Kobayashi-Maskawa-'t Hooft vertex).

$$\Rightarrow S[U] = \int f_\pi^2 \, |dU|^2 - \Lambda^3 \mathrm{tr}\,(MU) - \Delta \, \mathrm{e}^{-\mathrm{i}\theta} \det(U) + c.c.$$

We know, at large N, this should be $\log \det(U)$-vertex

Problem: where is the YM vacuum label?

cf.) Flavor-symmetric QCD has discrete anomaly at $\theta = \pi$ when $\gcd(N, N_f) \neq 1$ [Gaiotto-Komargodski-Seiberg '17].

Short summary

Our suggestion:
Extend the periodicity of η’ by N (to include YM vacuum label)
and use fractional-instanton-vertex $(\det U)^{1/N}$

- **We obtain this in “a tractable confinement (a limit of QCD)”.**

QCD / SU(N) Yang-Mills
(strongly coupled, hard problem)

→

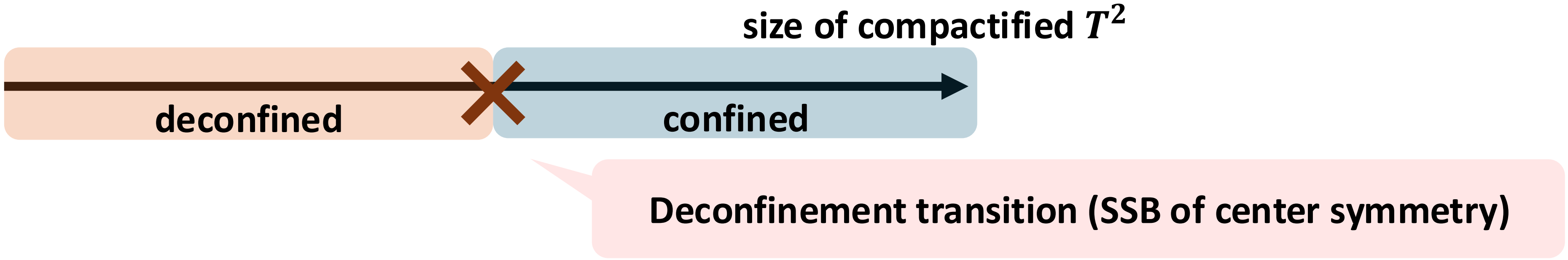
Deformed theory
(weakly coupled, easy problem)

We want to know

Confinement by gas of fractional instantons

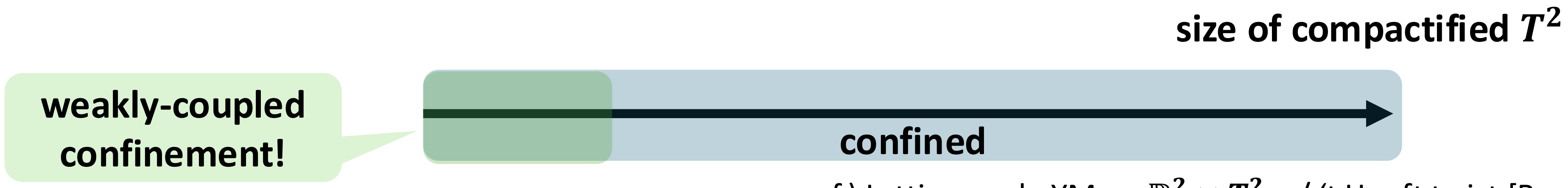
Adiabatic continuity

- In this talk, we consider T^2 compactification to have weakly-coupled theory
- With the naïve compactification, there is a deconfinement transition somewhere



- **Nice trick: inserting 't Hooft flux,** which stabilizes the center symmetry / avoids deconfinement ⇒ We expect the **adiabatic continuity** in this setup.

  in a similar spirit as twisted Eguchi-Kawai [González-Arroyo, Okawa '83--]

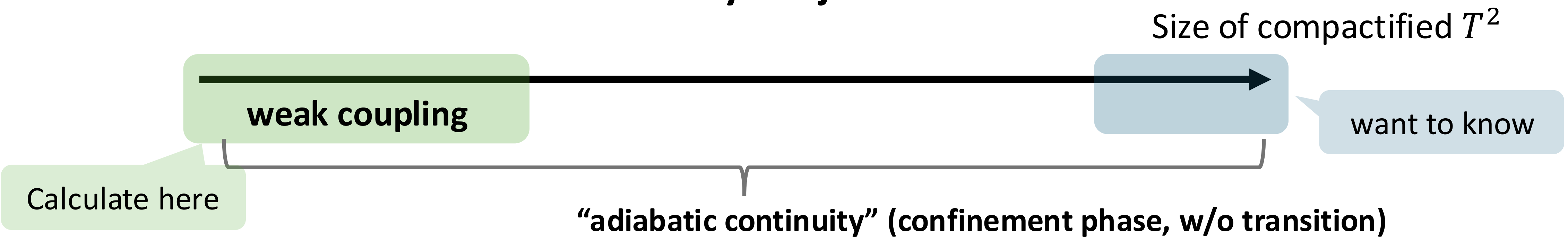


cf.) Lattice work: YM on $\mathbb{R}^2 \times T^2$ w/ 't Hooft twist [Bergner–González-Arroyo–Soler '25]

Semiclassics via T^2 compactification

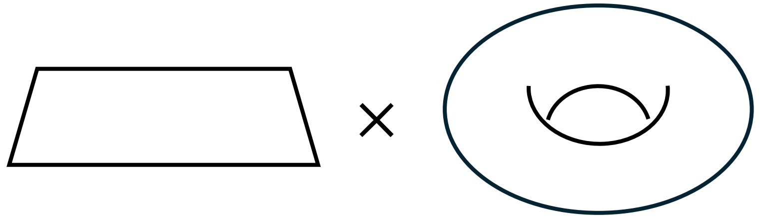
Today's talk: We investigate SU(N) YM/QCD vacuum structure through semiclassical analysis on $\mathbb{R}^2 \times T^2$ with **'t Hooft flux** (+ baryon magnetic flux for QCD)**.**

Main ansatz: adiabatic continuity conjecture



✅ Empirically, this method successfully gives a reasonable picture for confining vacuum in SU(N) YM, SU(N) N=1 SYM, QCD(F), QCD(Sym), QCD(AS), QCD(BF) [Tanizaki-Ünsal '22 '23][Tanizaki-YH-Watanabe '23 '24]. (cf. [Yamazaki-Yonekura '17])
This work: expanding analysis for QCD(F).

Contents

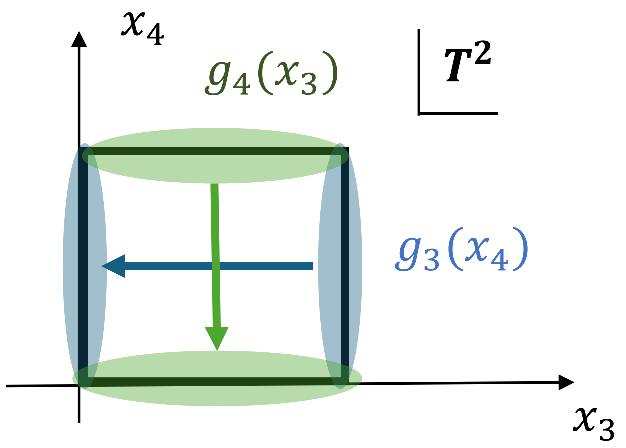


Plan:
Consider SU(N) Yang-Mills theory on $\mathbb{R}^2 \times T^2$ with ‘t Hooft flux;
Study physics at small T^2, and predict the original theory on $\mathbb{R}^4$ [Tanizaki-Ünsal ’22]

SU(N) YM on $\mathbb{R}^2 \times T^2$ with 't Hooft flux (1)

[Tanizaki-Ünsal '22, ……] (cf. [Yamazaki-Yonekura '17])

- **'t Hooft flux for T^2** **(or $\mathbb{Z}_N^{[1]}$ background)**



A unit 't Hooft flux ⇔ choose $g_3(0)g_4(L)g_3^\dagger(L)g_4^\dagger(0) = e^{\frac{2\pi \mathrm{i}}{N}}$.

($g_3(x_4), g_4(x_3)$: transition functions on T^2)

$$\begin{cases} a(\vec{x}, x_3 + L, x_4) = g_3^\dagger a g_3 - \mathrm{i}\, g_3^\dagger \mathrm{d} g_3 \\ a(\vec{x}, x_3, x_4 + L) = g_4^\dagger a g_4 - \mathrm{i}\, g_4^\dagger \mathrm{d} g_4 \end{cases}$$

Up to gauge, we can take $g_3 = S,\ \ g_4 = C$ (shift and clock matrices of $SU(N)$).

⇒ In this gauge, inserting 't Hooft flux ⇔ twisted boundary condition:

$$\begin{cases} a(\vec{x}, x_3 + L, x_4) = S^\dagger\, a\, S \\ a(\vec{x}, x_3, x_4 + L) = C^\dagger\, a\, C \end{cases}$$

e.g.) $N = 3$

$$S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},\ C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{2\pi i}{3}} & 0 \\ 0 & 0 & e^{\frac{4\pi i}{3}} \end{pmatrix}$$

SU(N) YM on $\mathbb{R}^2 \times T^2$ with 't Hooft flux (2)

- **Consequences from 't Hooft-twisted compactification**

✓**Center symmetry is kept at small T^2**

Classically, $P_3 = S$ and $P_4 = C \Rightarrow \langle \mathrm{tr}\, P_3 \rangle = \langle \mathrm{tr}\, P_4 \rangle = 0$.

✓**Perturbatively gapped gluons:**

$$\begin{cases} a(\vec{x}, x_3 + L, x_4) = S^\dagger\, a\, S \\ a(\vec{x}, x_3, x_4 + L) = C^\dagger\, a\, C \end{cases}$$

∼**adjoint higgsing by Polyakov loops P_3, P_4:** $SU(N) \to \mathbb{Z}_N$

⇒ no zeromode; O(1/NL) KK mass

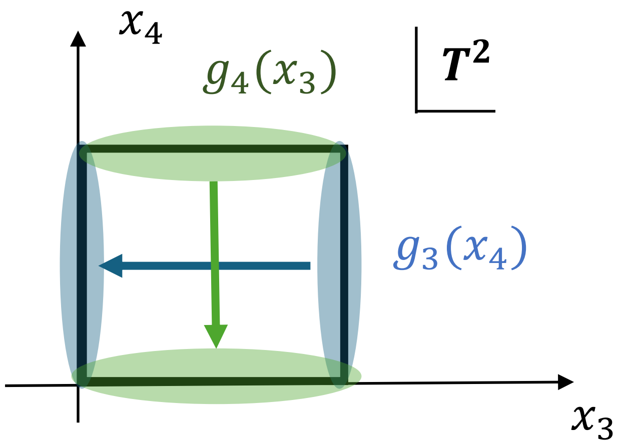
For confinement on $\mathbb{R}^2$:

✓**Numerical evidence for $^\exists$center vortex/fractional instanton with $S = \frac{8\pi^2}{Ng^2}$, $Q_{top} = 1/N$ as a "local solution"** (scale ∼ $\mathrm{Size}(T^2)$)

[Gonzalez-Arroyo–Montero '98, Montero '99 '00]

(cf. [García Pérez–Gonzalez-Arroyo–Soderberg '90; Itou '18] for $\mathbb{R} \times T^3$)

Note) fractional topological charge: it cannot exist alone if the boundary condition for $\mathbb{R}^2$ is regular



$g_3 = S,\ g_4 = C$

e.g.) $N = 3$

$$S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

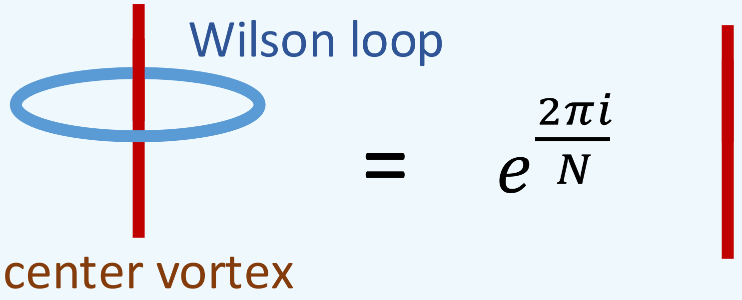
$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{2\pi i}{3}} & 0 \\ 0 & 0 & e^{\frac{4\pi i}{3}} \end{pmatrix}$$

Remark: center vortex?

Center vortex (in general context)

['t Hooft '78, …]

Co-dim-2 object carrying "magnetic flux of center element":

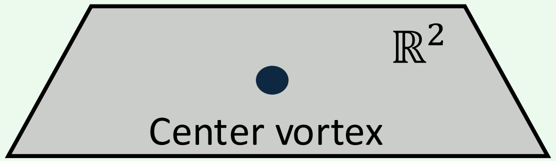


(expected to play an important role in quark confinement)

Center vortex (we consider here)

In addition, it is a 1/N fractional instanton satisfying the BPS bound

$$S = \frac{8\pi^2}{Ng^2}, \qquad Q_{top} = 1/N$$



In the asymmetric limit of T^2 ($L_3 \gg L_4$),
the center-vortex configuration can be constructed
from BPS/KK monopole (KvBLLY caloron) [YH-Tanizaki '24]

Semiclassics on $\mathbb{R}^2 \times T^2$ in $SU(N)$ YM [Tanizaki-Ünsal '22]

- **Dilute gas of center vortices**

The center-vortex and anti-center-vortex vertices are:

$$K e^{-\frac{8\pi^2}{N g^2} + i\,\theta/N}, \qquad K e^{-\frac{8\pi^2}{N g^2} - i\,\theta/N}$$

with a dimensionful constant K.

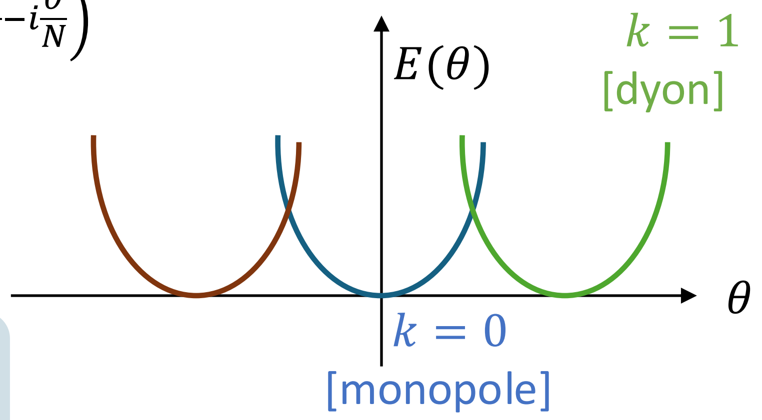
For calculating partition function, we compactify $\mathbb{R}^2$ without 't Hooft flux. ⇒ total topological charge is constrained $Q_{top} \in \mathbb{Z}$

Then, the dilute gas approximation yields, (only configurations with $Q_{top} \in \mathbb{Z}$ are admitted)

$$Z_{YM} = \sum_{n,\bar{n}\geq 0} \frac{1}{n!\,\bar{n}!} \delta_{n-\bar{n}\in N\mathbb{Z}} \left(V K e^{-\frac{8\pi^2}{N g^2} + i\frac{\theta}{N}} \right)^{n} \left(V K e^{-\frac{8\pi^2}{N g^2} - i\frac{\theta}{N}} \right)^{\bar{n}}$$
$$= \sum_{k\in\mathbb{Z}_N} \exp\left[-V \left(-2 K e^{-\frac{8\pi^2}{N g^2}} \cos\left(\frac{\theta - 2\pi k}{N} \right) \right) \right]$$

N semiclassical vacua

Energy density of k-th vacuum
→multibranch structure!



✅ One can also derive area-law falloff of the Wilson loop from the dilute gas of center vortices.

Contents

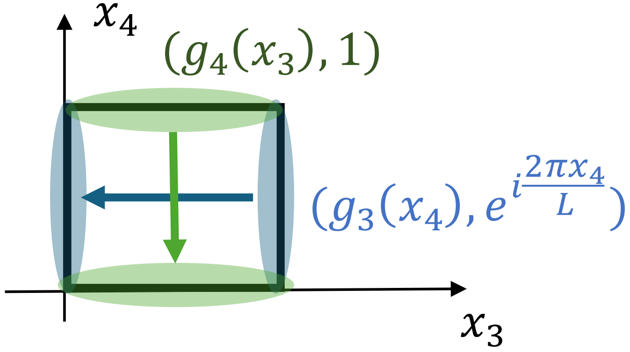
Plan:
Consider similar T^2 compactification for QCD, and study small-T^2 physics
New insights on η’ meson: periodicity extension / η’ mass from fractional instanton
[Tanizaki-Ünsal ’22; YH-Tanizaki ‘24]

Setup for QCD [Tanizaki-Ünsal '22]

- In the presence of fundamental quarks, it is impossible to insert 't Hooft flux alone ($g_3(0)g_4(L)g_3^\dagger(L)g_4^\dagger(0) = e^{\frac{2\pi i}{N}}$ leads to an inconsistency).
- To avoid this problem, we also introduce **baryon magnetic flux** simultaneously: $\int_{T^2} dA_B = 2\pi$. (e.g., we can take $A_B = \frac{2\pi}{L^2} x_3 dx_4$)

Boundary conditions for quarks (in the gauge $g_3 = S,\ g_4 = C$):

$$\begin{cases} \psi(\vec{x}, x_3 + L, x_4) = e^{i\frac{2\pi x_4}{NL}} S^\dagger \psi(\vec{x}, x_3, x_4) \\ \psi(\vec{x}, x_3, x_4 + L) = C^\dagger \psi(\vec{x}, x_3, x_4) \end{cases}$$



- At small T^2 , there is one 2d Dirac "low-energy mode" (⇔without KK mass) per flavor.

(obtained by solving zeromode equation)

Index theorem "$N \times \int_{T^2} dA_q = 1$" $(U(1)_B = U(1)_q/\mathbb{Z}_N)$

Constructing 2d effective theory

$N_f = 1$ case:

- Low-energy mode: one 2d Dirac fermion (⇔ compact scalar φ)
- Center-vortex vertex: $Ke^{-\frac{8\pi^2}{Ng^2}+i\,\theta/N}$ “$e^{-i\,\varphi/N}$” from $U(1)_{\mathrm{chiral}}$ spurious symmetry
- Dilute gas approximation

Invariance under
$\theta \to \theta + \alpha, \varphi \to \varphi + \alpha$

$$\longrightarrow \quad S[\varphi] = \int \frac{1}{8\pi}|d\varphi|^2 - m\mu\cos\varphi \; - 2Ke^{-\frac{8\pi^2}{Ng^2}}\cos\left(\frac{\varphi-\theta-2\,\pi k}{N}\right)$$

$\boldsymbol{\varphi}$ “eats” the vacuum label $\boldsymbol{k} \in \mathbb{Z}_N$ and extends its periodicity to $\boldsymbol{\varphi \sim \varphi + 2\pi N}$.

residual gauge $\boldsymbol{SU(N) \to \mathbb{Z}_N}$

$N_f \geq 2$ case: the non-abelian bosonization gives the 2d analog of $U(N_f)$ chiral Lagrangian with $\eta' \sim \eta' + 2\pi N$ & $(\det U)^{1/N}$-type $\boldsymbol{\eta}'$ mass.

2d version of chiral Lagrangian

- For $N_f > 1$, we use the non-Abelian bosonization: looks like **chiral Lagrangian with η'!**

[$U \in U(N_f)$ with $2\pi N$-periodic (det U)]

$$S[U] = \int \frac{1}{8\pi}|dU|^2 - m\mu \,\mathrm{tr}\,(U) \; - K \mathrm{e}^{-\frac{8\pi^2}{Ng^2}} \mathrm{e}^{-i\theta/N} (\det U)^{1/N} + c.c. + S_{WZW}^{3d}[U]$$

quark-mass deformation (if present)

Center-vortex-induced η' mass term "finite-N version of log-det vertex"

Up to gapped **η'**, this 2d effective theory

= T^2 compactification with $U(1)_B$ flux of 4d $SU(N_f)$ chiral Lagrangian

Coupling to $U(1)_B$ background

$$\int_{M_3 \times T^2} \mathrm{d}A_B \wedge \left(\frac{1}{24\pi^2} \,\mathrm{tr}\,(U^{-1}\mathrm{d}U)^3 \right) \Rightarrow \int_{M_3} \left(\frac{1}{12\pi} \,\mathrm{tr}\,(U^{-1}\mathrm{d}U)^3 \right) = S_{WZW}^{3d}[U]$$

Results

- **2d effective theory on $\mathbb{R}^2$**

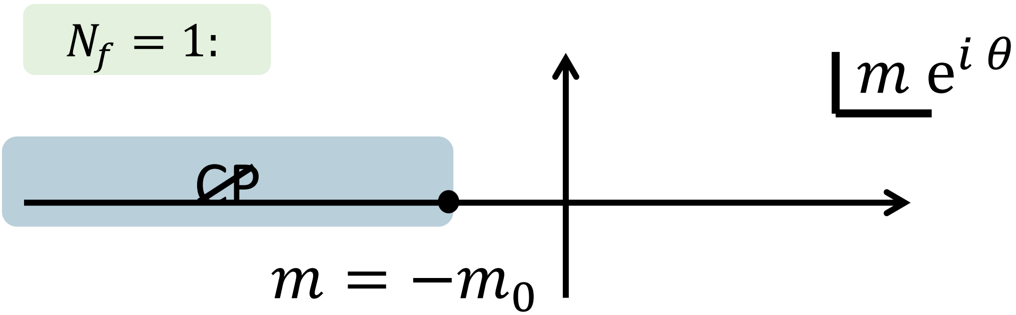
  = 2d analog of chiral Lagrangian + periodicity-extended $\boldsymbol{\eta'}$

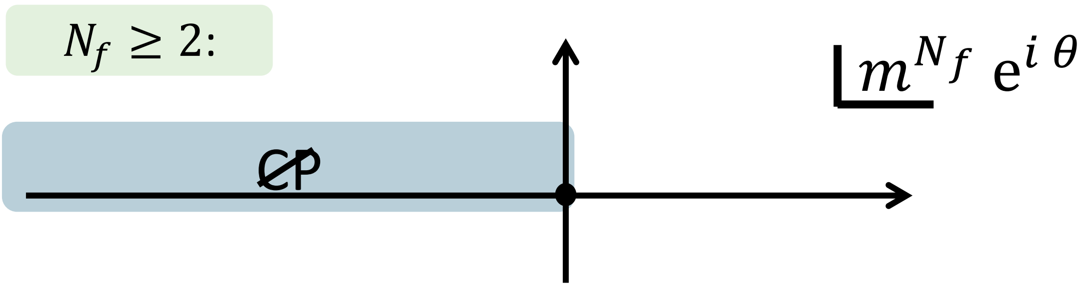
  + corresponding $\boldsymbol{\eta'}$ mass term $(\det U)^{1/N}$

$\eta' \sim \eta' + 2\pi$
$\Rightarrow \eta' \sim \eta' + 2\pi N$

finite-N version of log-det vertex

- This 2d effective theory explains the expected vacuum structure of QCD (phase diagram on $m^{N_f} \mathrm{e}^{i\theta}$):

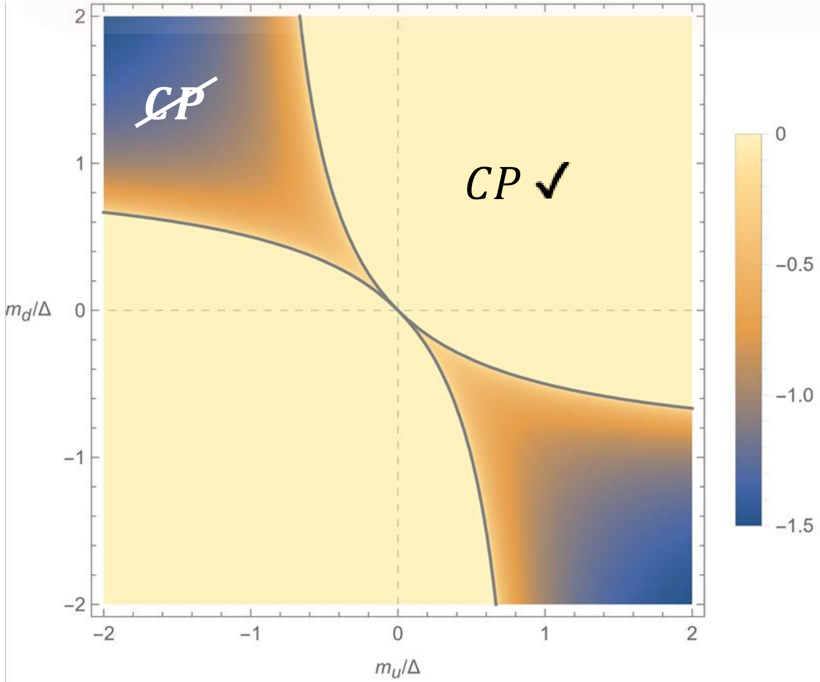




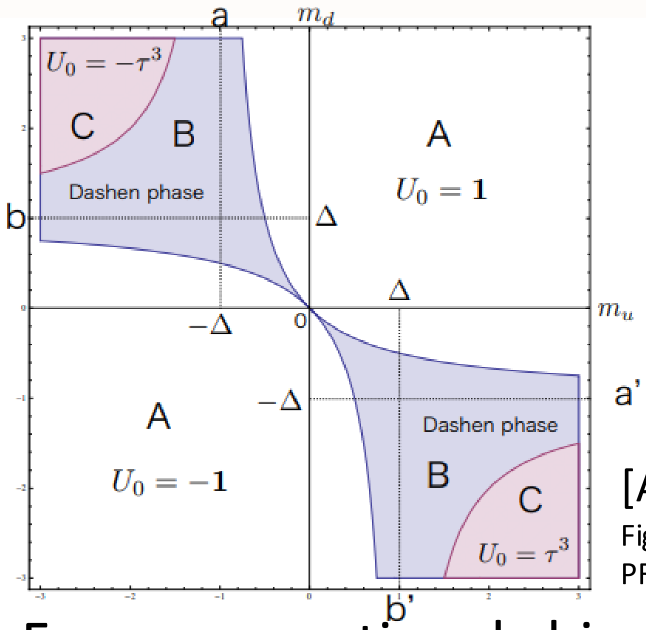
- **$\boldsymbol{\eta'}$ extends its periodicity by absorbing the $\mathbb{Z}_N$ vacuum label;** also for 4d chiral Lagrangian, this prescription improves the global aspects.

Application: Dashen phase on (m_u, m_d) plane

Phase diagram of (1+1)-flavor QCD on (m_u, m_d) plane:
The conventional U(2) chiral Lagrangian with det-type η mass has an artificial CP-restored phase ("phase C"). The periodicity extension of η eliminates the artificial phase.



with $\eta \sim \eta + 2\pi N$ & $(\det U)^{1/N}$ mass



[Aoki-Creutz '14]
Fig. 1 of S. Aoki and M. Creutz, PRL **112** 141603 (2014)

From conventional chiral Lagrangian with η (with det-type mass term)

Summary

describing a confining vacuum by dilute gas of **center vortices** [Tanizaki-Ünsal '22]

We study QCD through semiclassics on $\mathbb{R}^2 \times T^2$ with 't Hooft flux & $U(1)_B$ magnetic flux

Our results:

- **2d effective theory on $\mathbb{R}^2$**

  = 2d analog of chiral Lagrangian + periodicity-extended η'

  + corresponding η' mass term $(\det U)^{1/N}$

$\eta' \sim \eta' + 2\pi$
$\Rightarrow \eta' \sim \eta' + 2\pi N$

Center-vortex induced mass

- This 2d effective theory explains the expected vacuum structure of QCD (phase diagram on $m^{N_f} e^{i\theta}$).
- **The periodicity extension of η' = inclusion of YM vacuum label**

Also for 4d chiral Lagrangian with η', the periodicity extension improves global aspects (particularly, smooth connection to quenched limit; reproducing the discrete anomaly).

Backups

Technicality: $\mathbb{Z}_N$ gauging and vacuum label

- Problem: Center-vortex vertex: $K e^{-\frac{8\pi^2}{Ng^2}+i\,\theta/N}$ "$e^{-i\,\varphi/N}$" looks ill-defined/non-genuine.
- Key: **residual** $\mathbb{Z}_N$ **gauge** after adjoint higgsing by Polyakov loops： $SU(N) \to \mathbb{Z}_N$.
- The residual $\mathbb{Z}_N$ gauge is vector-like to fermion ψ. It couples to φ magnetically

$$\frac{i}{2\,\pi}\int a_{\mathbb{Z}_N} \wedge d\varphi$$

(#fermions) = (#kinks).

Integrating out $a_{\mathbb{Z}_N} \Rightarrow$ constraint $\int d\varphi \in 2\pi N\,\mathbb{Z}$

$\Rightarrow$ It is possible to regard $\varphi \in \mathbb{R}/2\pi N\mathbb{Z}$.

$e^{-i\,\varphi/N}$ becomes well-defined.

- In the lift from 2π-periodic field to $2\pi N$-periodic field, there is $\mathbb{Z}_N$ ambiguity: $\varphi \to \varphi + 2\pi k$. This 1-to-N correspondence absorbs the vacuum label k. In summary,

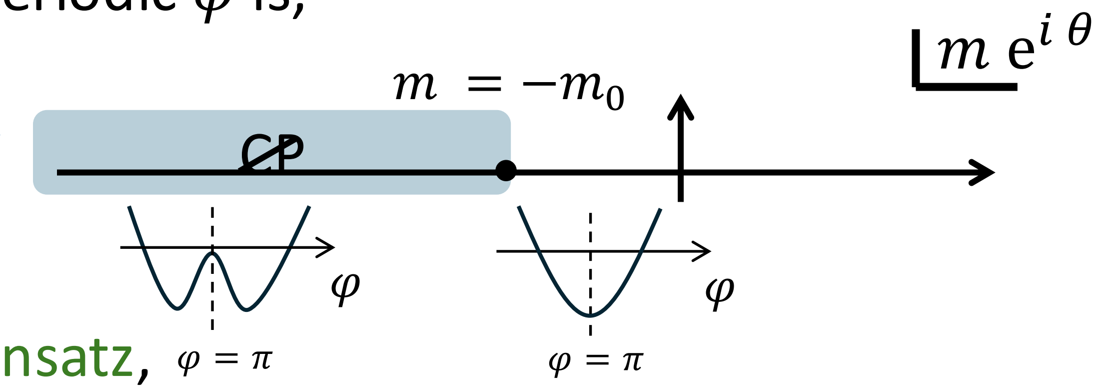
$$\int D a_{\mathbb{Z}_N} \sum_{k\in\mathbb{Z}_N} \int_{\varphi\sim\varphi+2\pi} D\varphi \;\ldots \;\Rightarrow\; \int_{\varphi\sim\varphi+2\pi N} D\varphi \;\ldots$$

Vacuum structure from 2d effective theory

The 2d effective theory explains the vacuum structure, just by finding potential minima:

- **$N_f = 1$ case**: the effective potential for $2\pi N$-periodic φ is,

$$V[\varphi] = -m\mu \cos\varphi \;-\; 2K e^{-\frac{8\pi^2}{Ng^2}} \cos\left(\frac{\varphi-\theta}{N}\right)$$



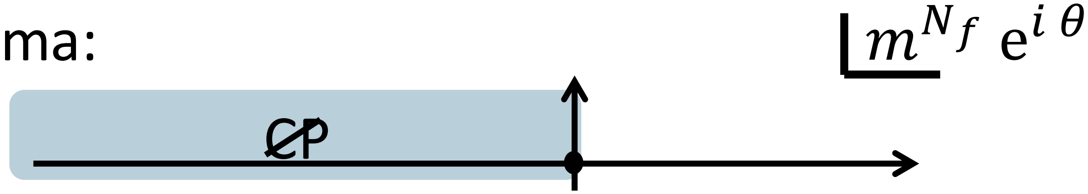
- **$N_f \geq 2$ case**: we take the $SU(N_f)$ symmetric ansatz,

$U = e^{i\,\varphi} 1$ with $(\log\det U) = N_f\varphi + 2\pi k$ $(-\pi < \varphi \leq \pi, k \in \mathbb{Z}_N)$

$$\Rightarrow\; V[\varphi] = -N_f m\mu\cos\varphi \;-\; 2Ke^{-\frac{8\pi^2}{Ng^2}}\cos\left(\frac{N_f\varphi + 2\pi k - \theta}{N}\right)$$

At $\theta = \pi$, this potential has two degenerate minima:

$(\varphi = \varphi_*, k = 0)$ and $(\varphi = -\varphi_*, k = 1)$



Discrete anomaly

Baryon-color-flavor anomaly:
Flavor-symmetric QCD with N_f quarks at $\theta = \pi$ has mixed anomaly between $\frac{SU(N_f)\times U(1)_q}{\mathbb{Z}_N \times \mathbb{Z}_{N_f}}$ and CP if $\gcd(N, N_f) \neq 1$. [Gaiotto-Komargodski-Seiberg '17]

- For $\gcd(N, N_f) = 1$, the variables (k, φ) in the $SU(N_f)$ symmetric ansatz can be combined into single $2\pi N$-periodic one φ: $N_f \varphi + 2\pi k \Rightarrow N_f \varphi \ (\mathrm{mod}\ 2\pi N)$. Like the mass deformation in $N_f = 1$ case, a suitable symmetric deformation can single out a unique gapped vacuum (the absence of anomaly).
- For $\gcd(N, N_f) \neq 1$, the $\mathbb{Z}_{\gcd(N, N_f)}$ discrete label cannot be absorbed. (Intuitively, quark fluctuation only bridges k-th vacuum and $(k + N_f)$-th vacuum, so it cannot split the degeneracy of CP-broken vacua: $k = 0$ and $k = 1$.)
- 4d chiral Lagrangian with periodicity-extended η' reproduces this discrete anomaly.

(A more essential point is that the coupling $\int \eta' dA_B \wedge dA_B$ becomes well-defined thanks to the periodicity extension.)

Digression: 2d center vortex/fractional instanton

The **2d center vortex** can be understood as **BPS/KK monopole** in 3d semiclassics (w/ center-stabilizing deformation [Ünsal-Yaffe '08]) [YH-Tanizaki '24] (cf. [Güvendik-Schäfer-Ünsal; Wandler '24])

2d center-vortex

3d BPS/KK monopole in $(\mathbb{R}^2 \times S^1) \times S^1_{\mathrm{small}}$
(with 't Hooft flux and center-stabilizing deformation for S^1_{small})