Spectral properties of QCD at finite temperature

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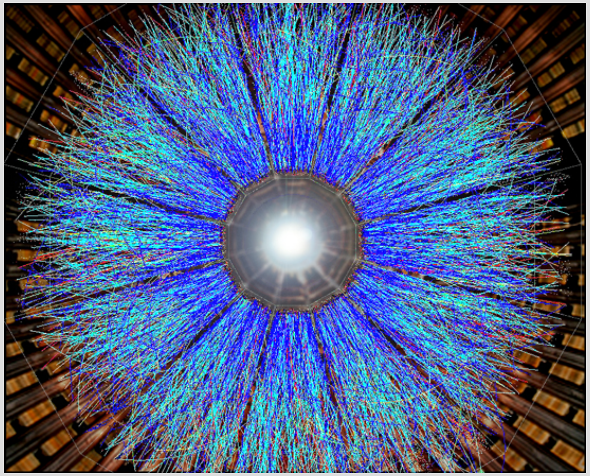
Outline

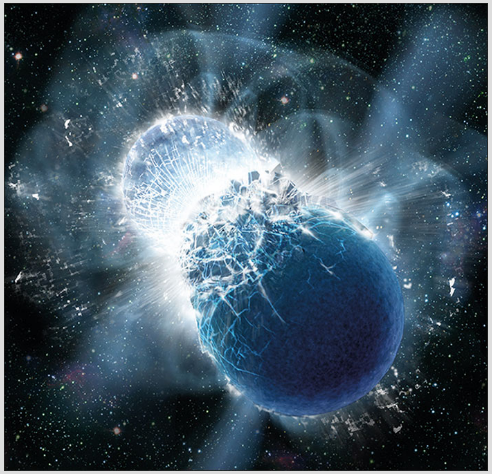
1. QFT beyond the vacuum
2. Thermoparticles
3. Hadrons at high temperature
4. Spontaneous symmetry breaking when $T > 0$
5. Finite-temperature perturbation theory

1. QFT beyond the vacuum

- To describe physical phenomena in "extreme environments" one must understand how QFT applies to systems that are hot, dense, or both



[Brookhaven National Lab]



[Skyworks Digital Inc.]

- Therefore need to establish how the inclusion of temperature $T = 1/\beta$ or density ($|\mu| > 0$) modifies the standard QFT assumptions, and what effect this has on the correlation functions $<\varphi(x_1)...\varphi(x_n)>_{\beta,\mu}$

$\rightarrow$ Here we stick to the case $T > 0$ and vanishing density

1. QFT beyond the vacuum

- As soon as $T > 0$ there are some immediate implications:

  - **Lorentz invariance** ✘ $\rightarrow$ but can retain rotational invariance
  - **Spectral condition ($H > 0$)** ✘ $\rightarrow$ replaced by KMS condition
  - **Field locality ($[\varphi(x),\varphi(y)]=0,\ (x\text{-}y)^2<0$)** ✓ $\rightarrow$ this is important!

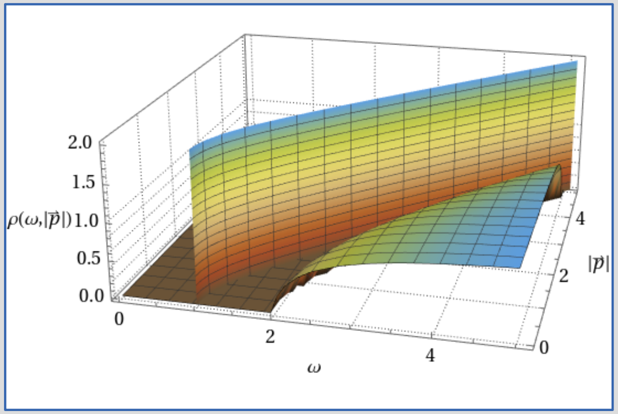
- Correlation functions of particular importance are "spectral functions"

$$\rho(\omega,\vec{p}) = \int d^4x\, e^{i(\omega x_0 - \vec{p}\cdot\vec{x})} \langle [\phi(x),\phi(0)] \rangle$$

When $T = 0$:

Delta peak $\rightarrow$ stable particle component

Continuum onset $\rightarrow$ multi-particle states



What changes when $T > 0$?

2. Thermoparticles

- Taking into account the additional constraints for $T > 0$ the spectral function has the representation*

$$\boxed{\rho(\omega,\vec{p}) = \int_0^\infty ds \int \frac{d^3\vec{u}}{(2\pi)^2}\, \epsilon(\omega)\, \delta\big(\omega^2 - (\vec{p}-\vec{u})^2 - s\big)\, \widetilde{D}_\beta(\vec{u},s)}$$

"**Thermal spectral density**"

This is the $T > 0$ generalisation of the textbook *Källén-Lehmann* representation!

$$\boxed{\rho(\omega,\vec{p}) = 2\pi\epsilon(\omega)\int_0^\infty ds\, \delta(p^2 - s)\, \varrho(s)}$$

- $T > 0$ effects amount to understanding: $\rho(s) \rightarrow \widetilde{D}_\beta(\vec{u},s)$, which tell us about the possible excitations that can exist in a thermal medium

  → $\vec{u}$ dependence: deviations away from the mass-shell $p^2 = s$

  → s dependence: energy ω thresholds, just like for $T = 0$

- There is theoretical evidence* that $\widetilde{D}_\beta(\vec{u},s)$ contains ***discrete*** components

$$\boxed{\widetilde{D}_{m,\beta}(\vec{u})\, \delta(s - m^2)}$$

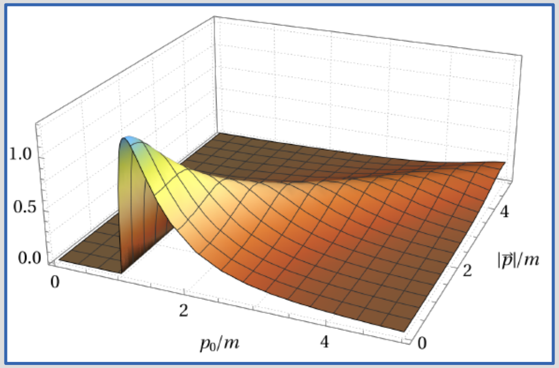
"**Thermoparticle**"

→ $T \rightarrow 0$ implies: $\widetilde{D}_{m,\beta}(\vec{u}) \rightarrow \delta^3(\vec{u})$, and vacuum particle $\delta(p^2 - m^2)$ recovered

* See: *J. Bros and D Buchholz, Z. Phys. C* 55 (1992), *Ann. Inst. H.Poincare Phys.Theor.* 64 (1996)

- Non-trivial "Damping factor" $\widetilde{D}_{m,\beta}(\vec{u})$ results in thermally-broadened peak in the spectral function $\rightarrow$ *parametrises collisional broadening effects*

Thermoparticles have distinctive characteristics



- Spectral function has a threshold at $\omega = m$

$$\boxed{\rho_{\mathrm{TP}}(\omega,\vec{p}) = \theta(\omega^2 - m^2)\varrho(\omega,\vec{p})}$$

- Two-point function has factorised form

$$\boxed{\mathcal{W}_{\mathrm{TP}}(x_0,\vec{x}) = D_{m,\beta}(\vec{x})\mathcal{W}_{(0)}(x_0,\vec{x})}$$

$\rightarrow$ They dominate the large-time behaviour of $\langle\phi(x_0,\vec{x})\phi(0)\rangle$

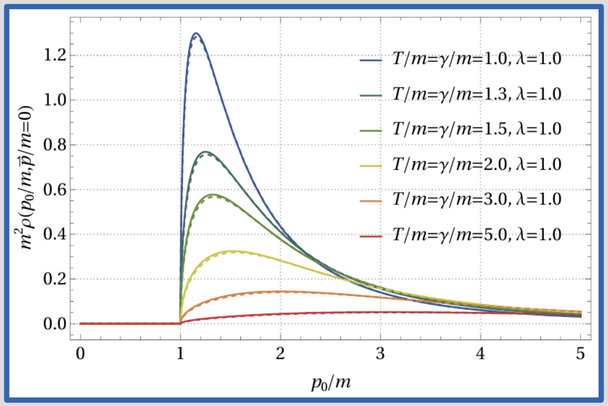
$\rightarrow$ Along direction $\vec{x} = \vec{v}\, x_0$ they behave like $D_{m,\beta}(\vec{v}x_0)\,|x_0|^{-3/2}$ hence damping depends on velocity relative to the medium [Bros, Ann. H. Poincare 4, (2003)]

- Singularities of propagators can be broader than simple poles, e.g. when $D_{m,\beta}(\vec{x}) = e^{-\gamma|x|}$

$$\boxed{\widetilde{G}_{\mathrm{TP}}(k_0,\vec{p}) = -\frac{1}{k_0^2 - |\vec{p}|^2 - m^2 - 2\gamma\sqrt{m^2-k_0^2} - \gamma^2}}$$

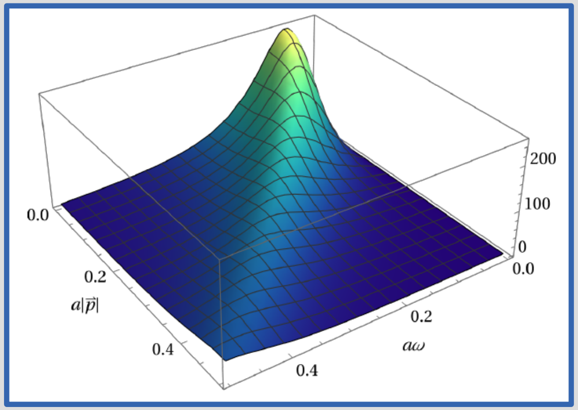
- There is mounting evidence that thermoparticles play a central role in determining the characteristics of QFTs at finite temperature

$T > 0$ perturbation theory

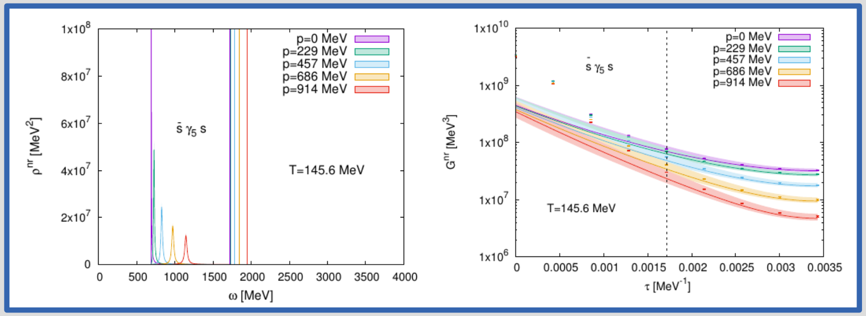


Thermoparticles

Spontaneous symmetry breaking when $T > 0$

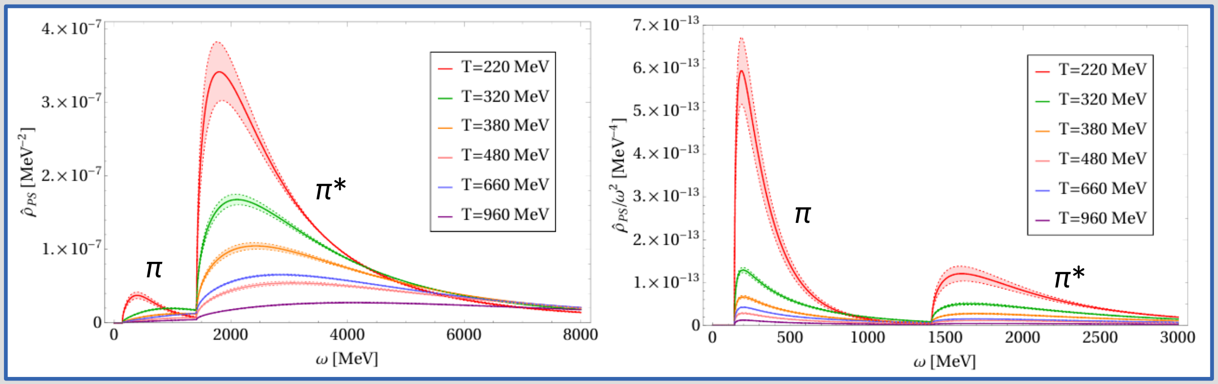


Hadrons at high temperature



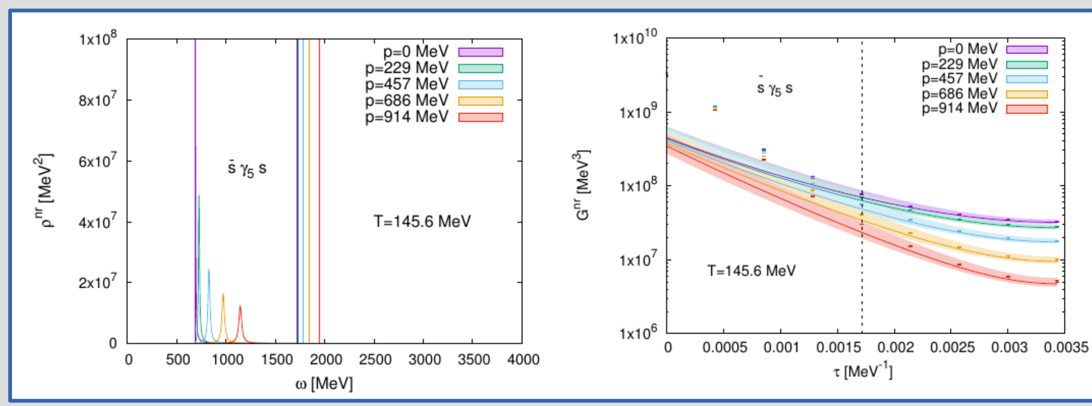
3. Hadrons at high temperature

- Evidence for low-energy thermoparticle excitations in QCD $\to$ extraction of spectral function for pseudo-scalar meson operator $\boxed{\mathcal{O}^a_{\mathrm{PS}} = \overline{\psi}\gamma_5\frac{\tau^a}{2}\psi}$ in several different quark channels



Light-light pseudo-scalar meson (pion) channel [P.L., O. Philipsen, *JHEP* 10, 161 (2022)]

Light-strange (kaon) and strange-strange (eta) pseudo-scalar meson channels [D. Bala, O. Kaczmarek, P. L., O. Philipsen, and T. Ueding, *JHEP* 05, 332 (2024)]



Data in *all* channels consistent with a thermoparticle-type ground state: suggests light pseudo-scalar mesons (pions, kaons,..) still have a bound-state-like structure, even at high T

3. Hadrons at high temperature

<u>Goal</u>: Extract information about the finite T spectral function $\rho_\Gamma(\omega,\boldsymbol{p})$ from data of *Euclidean* correlator $C_\Gamma(\tau,\vec{x}) = \langle O_\Gamma(\tau,\vec{x})\, O_\Gamma(0,\vec{0})\rangle_T$ O_Γ = scalar operator

- <u>Standard approach</u>: extract $\rho_\Gamma(\omega,\boldsymbol{p})$ from temporal correlator $\widetilde{C}_\Gamma(\tau,\boldsymbol{p})$

$$\boxed{\widetilde{C}_\Gamma(\tau,\vec{p}) = \int_0^\infty \frac{d\omega}{2\pi} \frac{\cosh\left[\left(\frac{\beta}{2}-|\tau|\right)\omega\right]}{\sinh\left(\frac{\beta}{2}\omega\right)} \rho_\Gamma(\omega,\vec{p})}$$

$\rightarrow$ *Problem is ill-conditioned, need more information!*

- Instead, one can use the ***spatial*** correlator, where one integrates $C_\Gamma(\tau,\boldsymbol{x})$ over $\{\tau,x,y\}$ and fixes a spatial direction z

$$\boxed{C_\Gamma(z) = \int_{-\infty}^\infty \frac{dp_z}{2\pi} e^{ip_z z} \int_0^\infty \frac{d\omega}{\pi\omega}\, \rho_\Gamma(\omega, p_x = p_y = 0, p_z)}$$

- It turns out that *if* thermoparticles exist, then they will give a distinct contribution to $C(z)$

$$\boxed{C(z) \approx \frac{1}{2}\int_{|z|}^\infty dR\; e^{-mR} D_{m,\beta}(R)}$$

[P.L., *PRD* 106 (2022); P.L., O. Philipsen, *JHEP* 10, 161 (2022)]

$\rightarrow$ This component can be extracted *directly* from data

3. Hadrons at high temperature

- In particular, one can extract the damping factors $D_\beta(\boldsymbol{x})$ and their temperature dependence from $C(z)$, and then use this to determine the spectral contribution $\rho_{\mathrm{TP}}(\omega,\boldsymbol{p})$

$C(z)$ data $\longrightarrow$ $D_\beta(\boldsymbol{x})$

$+$

$$\boxed{\rho(\omega,\vec{p}) = \int_0^\infty ds \int \frac{d^3\vec{u}}{(2\pi)^2}\, \epsilon(\omega)\, \delta\big(\omega^2 - (\vec{p}-\vec{u})^2 - s\big)\, \widetilde{D}_\beta(\vec{u},s)}$$

$\Big\}\ \rho_{\mathrm{TP}}(\omega,\boldsymbol{p})$

- Can use temporal correlator $\widetilde{C}_\Gamma(\tau,\boldsymbol{p})$ data to check whether extracted $\rho_{\mathrm{TP}}(\omega,\boldsymbol{p})$ components are consistent $\longrightarrow$ a *highly non-trivial test!*

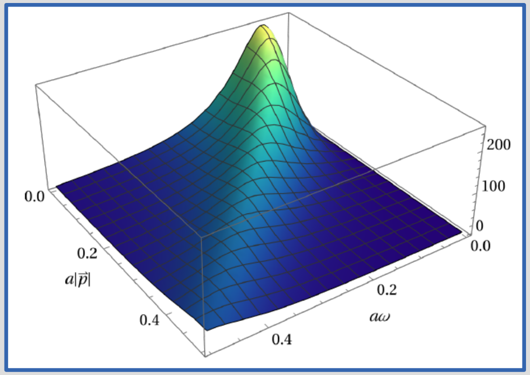
- Test was performed for $\boldsymbol{p}=0$ in [P.L., Philipsen, 2022] (light-light channel), and generalised to $|\boldsymbol{p}|>0$ in [Bala, Kaczmarek, P.L., Philipsen, Ueding, 2024] (light-strange and strange-strange channels)

4. Spontaneous symmetry breaking when $T > 0$

- Thermoparticles play an essential role in the generalisation of spontaneous symmetry breaking to finite temperature [Bros, Buchholz, *PRD* 58 (1998)]

- Evidence [PL, O. Philipsen, 2501.17120, 2507.14348] that $T > 0$ Goldstone mode in U(1) complex scalar theory is a ***massless*** thermoparticle, and persists for $T > T_c$

$$\boxed{D_\beta^G(\vec{x}, s) = D_\beta^G(\vec{x})\delta(s)}$$



$\rightarrow$ *The damping factor determines the phase structure, i.e. value of order parameter q*

"Strong dissipation", $D^G(\boldsymbol{x}) \rightarrow 0$ for $|\boldsymbol{x}| \rightarrow \infty$, implies $q = 0$, **Restored phase**

$T = T_c$

"Weak dissipation", $D^G(\boldsymbol{x}) > 0$ for $|\boldsymbol{x}| \rightarrow \infty$, implies $q > 0$, **Broken phase**

$T = 0$ $\quad$ $D^G(\boldsymbol{x}) = \text{const}$, i.e. no thermal effects, hence $p^2 = 0$ mode

This captures the physics! $\rightarrow$ sufficiently strong thermal effects destroy the long-range order, which leads to symmetry restoration

4. Spontaneous symmetry breaking when $T > 0$

Goldstone's theorem $\rightarrow$ *What happens when a continuous global symmetry of a QFT is spontaneously broken?*

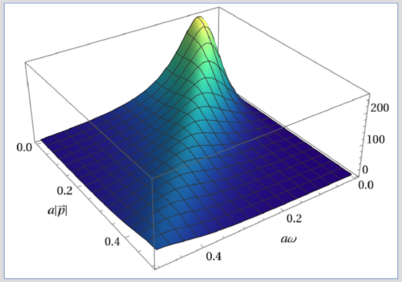
- If j^μ is the conserved current associated with the symmetry, and A is some local field whose transformation δA under the symmetry has a non-trivial expectation value: $<\delta A> = \lim_{R\to\infty} <[Q_R, A]> = q \neq 0$, then*

  $\rightarrow$ When $T = 0$ the Fourier transform of $<[j^0(x), A(0)]>$ contains a $\delta(p^2)$ component, i.e. a massless Goldstone boson

$T > 0$ case: using spectral representation for $<[j^0(x), A(0)]>_\beta$ the SSB condition $<\delta A>_\beta \neq 0$ implies [Bros, Buchholz, 1998] that the Fourier transform contains a massless **thermoparticle** $\boxed{D^G_\beta(\vec{x}, s) = D^G_\beta(\vec{x})\delta(s)}$

$\rightarrow$ This a **thermal Goldstone boson**: in the $T \rightarrow 0$ limit, $D^G(\boldsymbol{x}) \rightarrow$ const, and one recovers $\delta(p^2)$

$\rightarrow$ Non-trivial damping factor $D^G(\boldsymbol{x})$ causes broadening of the $p^2 = 0$ Goldstone peak



* See: [F. Strocchi, Symmetry Breaking, Lect. Notes Phys. 732 (2008)]

4. Spontaneous symmetry breaking when $T > 0$

- Now we know what the signatures of thermal Goldstone modes are, one can look for them in lattice data

- Consider a simple model with SSB: U(1) complex scalar field theory

$$\boxed{\mathcal{L} = \frac{1}{2}\partial_\mu \phi^\dagger \partial^\mu \phi - \frac{1}{2}m^2 \phi^\dagger \phi - \frac{\lambda}{4!}(\phi^\dagger \phi)^2}$$

- In the broken phase at $T = 0$ the model contains a massless Goldstone boson and a resonance-like mode

  $\rightarrow$ Model expected* to undergo a second-order phase transition: for $T > T_c$ the U(1) symmetry is restored, and $|v|^2 = <\Phi><\Phi^\dagger> = 0$

- Investigate theory on a $L_\tau \times L^3$ lattice ($L_\tau = a\,N_\tau$, $L = a\,N_s$) with action

$$\boxed{S = a^4 \sum_{x \in \Lambda_a} \left[\sum_\mu \left(\frac{1}{2} \Delta^f_\mu \phi^*(x) \Delta^f_\mu \phi(x) \right) + \frac{m_0^2}{2} \phi^*(x)\phi(x) + \frac{g_0}{4!} \left(\phi^*(x)\phi(x) \right)^2 \right]}$$

$\rightarrow$ Avoid triviality by keeping a>0 fixed, hence $T = (aN_\tau)^{-1}$ is varied in discrete steps

* See e.g. [J. I. Kapusta and C. Gale, *Finite-temperature Field Theory*]

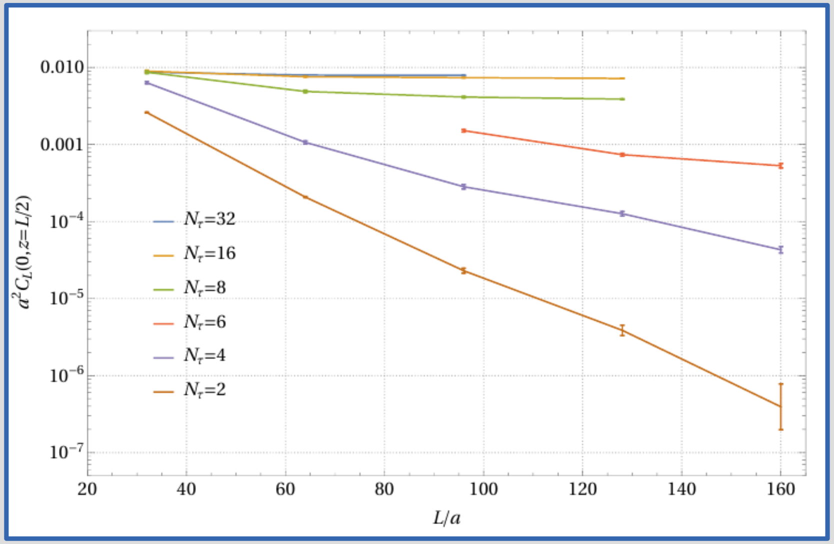
4. Spontaneous symmetry breaking when $T > 0$

- SSB does not occur in a finite spatial volume $V = L^3$, i.e. there is no notion of a "vev" on the lattice

  $\rightarrow$ Need to perform $L \rightarrow \infty$ extrapolation of lattice results!

- For the finite-volume correlator $C_L(\tau, \boldsymbol{x})$: $\boxed{\lim_{L\rightarrow\infty} C_L(\tau,\vec{x}) \xrightarrow{|\vec{x}|\rightarrow\infty} |v|^2}$

- Based on this property there are different approaches* for extracting $|v|^2$, here we use: $\boxed{|v|^2 = \lim_{L\rightarrow\infty} C_L(0,|\vec{x}| = L/2)}$



- $N_\tau = 8, 16, 32$, non-zero for large L $\rightarrow$ **U(1)** ***broken***
- $N_\tau = 2, 4, 6$, vanishing for large L $\rightarrow$ **U(1)** ***restored***
- Infinite-volume extrapolation requires a parametrisation for $C_L(\tau{=}0,\, |\boldsymbol{x}| = z)$
- Use finite-volume version of $\boxed{C^G(0,\vec{x}) = \frac{\coth\left(\frac{\pi|\vec{x}|}{\beta}\right)}{4\pi\beta|\vec{x}|} D^G_\beta(\vec{x})}$

* See e.g. [H. Neuberger, *PRL* 60,(1988).]

<u>Broken phase</u> ($N_\tau = 8, 16, 32$)

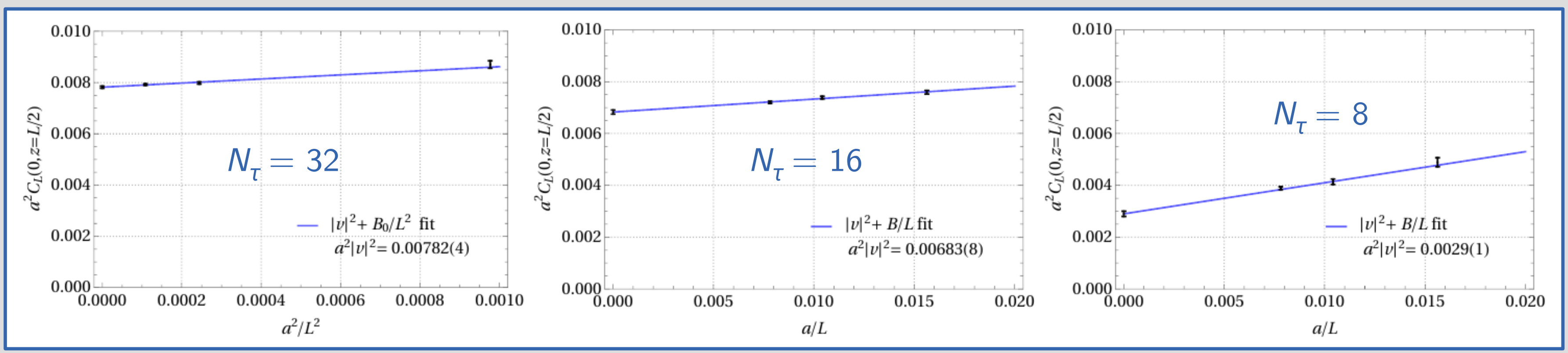
- In this case we assume $D^G(\boldsymbol{x}) \sim$ const and fit the ansatz:

$$C_L(0,z) = c_L + b_L \left[\frac{\coth\left(\frac{\pi z}{\beta}\right)}{z} + \{z \to (L-z)\} \right]$$

Non-zero if in broken phase

Finite *L* symmetrisation

- Functional form provides very good description of the data for each volume considered ($L/a = 32, 64, 96, 128$). Use this to do $L \to \infty$ extrapolation:



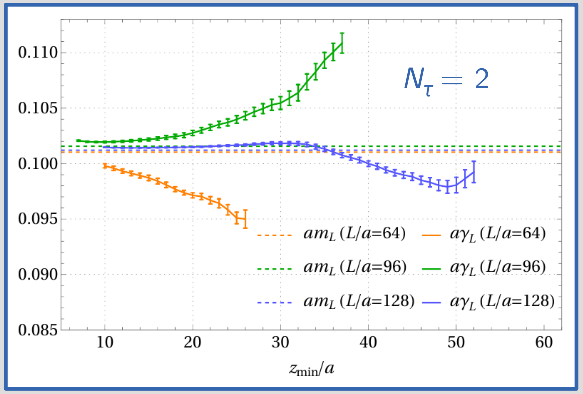
4. Spontaneous symmetry breaking when $T > 0$

Restored phase ($N_\tau = 2, 4, 6$)

- As outlined previously, the Goldstone mode can still persist in this regime, and would have the properties of a massless thermoparticle
- What is the structure of the damping factor $D^G(\boldsymbol{x})$? Use **spatial** correlator:

$$\boxed{C_L(z) = d_L\left[e^{-m_L z} + \{z \to (L-z)\}\right]} \rightarrow \boxed{D^G_\beta(\vec{x}) = \alpha\, e^{-\gamma|\vec{x}|}} \rightarrow \boxed{C^G(0,\vec{x}) = \frac{\coth\left(\frac{\pi|\vec{x}|}{\beta}\right)}{4\pi\beta|\vec{x}|}\alpha\, e^{-\gamma|\vec{x}|}}$$

- Spatial correlator fits provide excellent description of data over full range $[0, L/2]$ for each of the (large) volumes considered ($L/a = 64, 96, 128, 160$)
- Test damping by fitting: $\boxed{C_L(0,z) = b_L\left[\frac{\coth\left(\frac{\pi z}{\beta}\right)}{z}e^{-\gamma_L z} + \{z \to (L-z)\}\right]}$
- Consistency of the thermoparticle hypothesis requires that the fit parameters γ_L and m_L approach one another in the infinite-volume $L \to \infty$ limit



4. Spontaneous symmetry breaking when $T > 0$

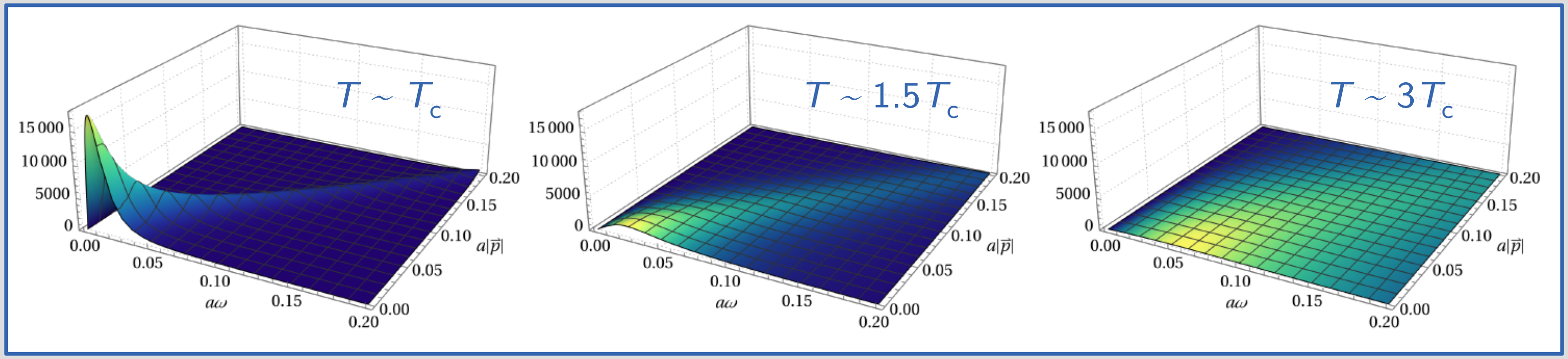
Restored phase ($N_\tau = 2, 4, 6$)

- Data at each N_τ consistent with the correlator being dominated by an exponentially-damped massless thermoparticle $\rightarrow$ **thermal Goldstone**

- One can use the extracted damping factor $D^G(\boldsymbol{x}) = a\, e^{-\gamma|\boldsymbol{x}|}$ to compute the spectral function $\rho_G(\omega,\boldsymbol{p})$ of the Goldstone mode

$$\rho_G(\omega, \vec{p}) = \frac{4\alpha\,\omega\gamma}{(\omega^2 - |\vec{p}|^2 - \gamma^2)^2 + 4\omega^2\gamma^2}$$

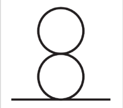
- Spectral properties are very different to the $T=0$ case $\rho_G(\omega,\boldsymbol{p}) \sim \delta(p^2)$



- Broadened peak structure around $p^2 = 0$ increases with temperature

  $\rightarrow$ Represents the increasingly strong thermal effects of the medium!

5. Finite-temperature perturbation theory

- Perturbation theory has several issues related to the infrared structure of certain classes of diagrams when $T > 0$, e.g. the $m \to 0$ divergence of 

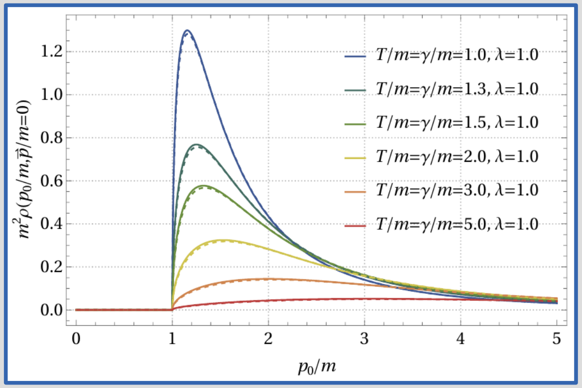
- This seems to be a realisation of a more fundamental constraint: non-trivial scattering at $T > 0$ is incompatible with real dispersion laws (*NRT theorem*, Commun. *Math. Phys.* 92, 247 (1983))

- Now both analytic (Weldon, *PRD* 65 (2002)) and numerical (PL, Philipsen, 2405.02009) evidence for this, even for massive theories, and at small T

- A resolution of these issues is to use propagators which don't have real dispersion laws $\to$ *thermoparticles* are a natural candidate

- **Thermoparticle perturbation theory** gives precise predictions consistent with lattice Φ^4 theory data [Ali, PL, Philipsen, (2026) paper out soon!]

  $\to$ Framework can be generalised to any QFT, so potentially many applications



- The Gell-Mann-Low (GML) relation is the basis of perturbation theory

When $T=0$:

$$\langle\Omega|T\{\phi(x_1)\phi(x_2)\cdots\phi(x_n)\}|\Omega\rangle = \frac{\langle\Omega_0|T\left\{\phi_0(x_1)\phi_0(x_2)\cdots\phi_0(x_n)e^{i\int d^4z\,\mathcal{L}_I[\phi_0(z)]}\right\}|\Omega_0\rangle}{\langle\Omega_0|T\left\{e^{i\int d^4z\,\mathcal{L}_I[\phi_0(z)]}\right\}|\Omega_0\rangle}$$

Fully interacting fields

Asymptotic fields

- The asymptotic (large-time) fields Φ_0 are non interacting. Expanding the exponential in the coupling, and applying Wick's theorem, all terms (diagrams) involve products or convolutions of the free Φ_0 propagator $\frac{i}{p^2-m^2+i0^+}$
- Perturbation theory for $T>0$ requires a **thermal** generalisation of the GML relation $\longrightarrow$ But NRT theorem rules out that the asymptotic fields are free or have real dispersion relations, hence propagators cannot have real poles

Standard $T>0$ relation:

$$\langle\Omega_\beta|T\{\phi(x_1)\phi(x_2)\cdots\phi(x_n)\}|\Omega_\beta\rangle = \frac{\mathrm{Tr}\left(e^{-\beta H}\,T\left\{\phi_0(x_1)\phi_0(x_2)\cdots\phi_0(x_n)e^{i\int_C\mathcal{L}_I[\phi_0]}\right\}\right)}{\mathrm{Tr}\left(e^{-\beta H}\,T\left\{e^{i\int_C\mathcal{L}_I[\phi_0]}\right\}\right)}$$

Assumes **free** asymptotic fields

5. Finite-temperature perturbation theory

- Due to their large-time dominance, thermoparticles are natural candidates for describing scattering states when $T > 0$
- Since scattering states fix the propagators of the perturbative expansion, use thermoparticle propagators instead! $\rightarrow$ "Thermoparticle perturbation theory"
- **What form do these propagators have?** $\rightarrow$ From Φ^4 theory data there are strong indications [PL, O. Philipsen, 2405.02009] that these excitations are present, and that their damping factor has the form: $D_\beta(\boldsymbol{x}) = e^{-\gamma|\boldsymbol{x}|}$

$$\boxed{G(\omega_n, \vec{p}) = \frac{1}{\omega_n^2 + |\vec{p}|^2 + m^2 + \gamma^2 + 2\gamma\sqrt{m^2 + \omega_n^2}}}$$

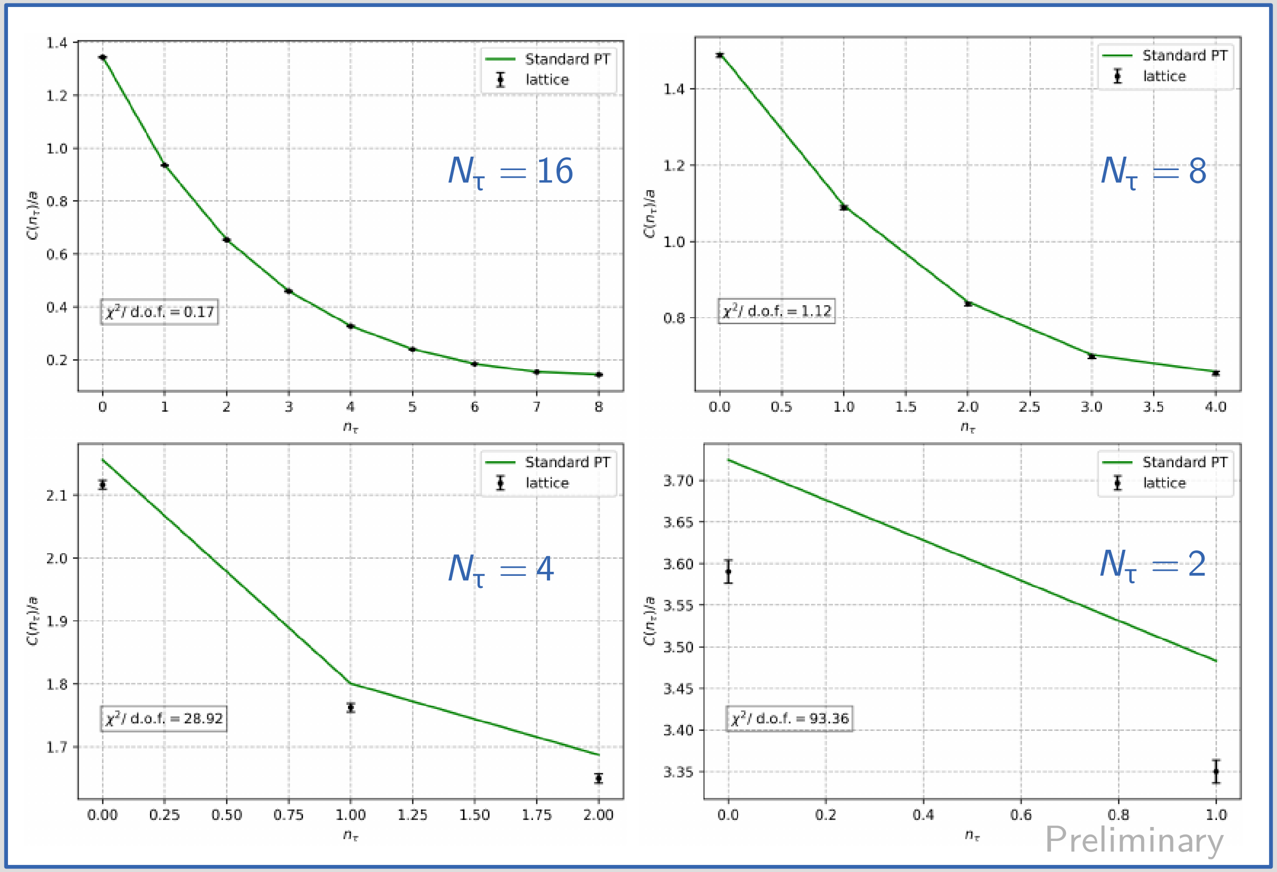
$\rightarrow$ Generalises the free field (γ=0) result. Width γ>0 cuts off IR divergences!

<u>In principle</u>: the form of the thermoparticle propagators, including γ, are *uniquely* fixed by the dynamical equations of the theory [Bros, Buchholz, 2002]
<u>In practice</u>: we determine γ by adjusting it such that 2-loop lattice perturbation theory results are consistent with the spatial correlator data $C(z)$

- Perform procedure for different values of N_τ and different lattice parameters (am_0, g_0), and then use the extracted $\gamma(am_0, g_0, N_\tau)$ to make predictions

  Test: Compute 2-loop lattice PT predictions of $C(\tau)$ with extracted γ and compare with the data $\rightarrow$ *a highly non-trivial consistency check!*

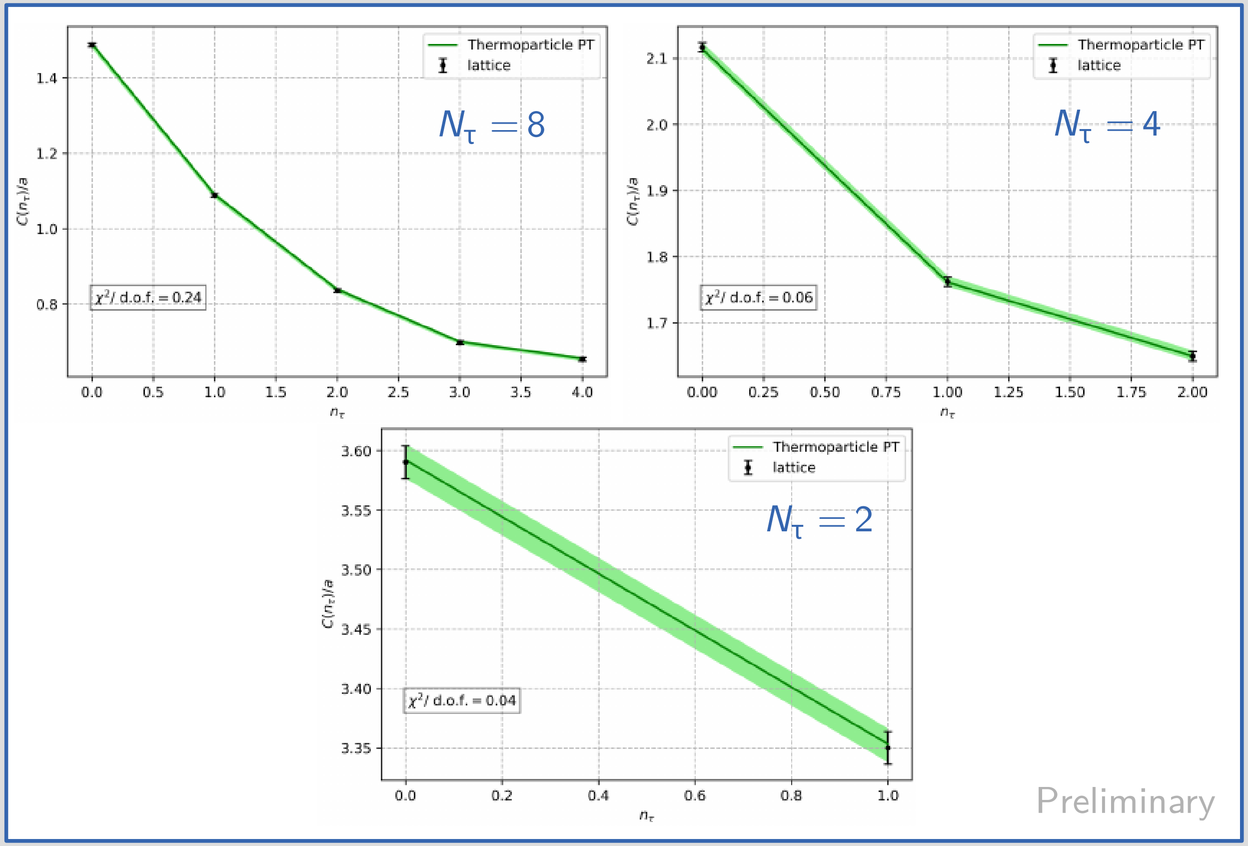
C(τ) predictions of **standard perturbation theory** (γ=0):



$\rightarrow$ **Predictions increasingly inconsistent with the data for higher T (lower N_τ)**

5. Finite-temperature perturbation theory

- Now compute two-loop $C(\tau)$ predictions using **thermoparticle perturbation theory**, with the $\gamma(am_0, g_0, N_\tau)$ values determined from $C(z)$ data



- Predictions are highly consistent with the data at **all** temperatures (lattice errors $<1\%$)

Evidence that thermoparticles are the correct basis for the $T>0$ perturbative expansion

Summary & outlook

- One can use the non-perturbative constraints imposed by causality to gain new insights $\rightarrow$ $\rho(\omega, \boldsymbol{p})$ have spectral representations. This narrows down the potential excitations that can exist $\rightarrow$ **"Thermoparticles"**

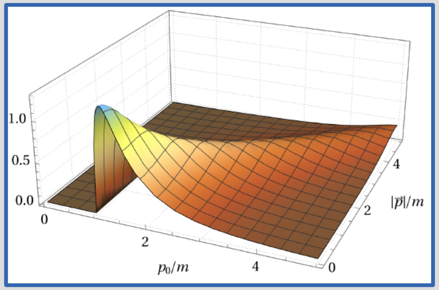
- Thermoparticles play a central role in QFTs at finite temperature:

(i) Light mesons appear to have a thermoparticle-like structure at high temperature, i.e. bound-state properties are still important

(ii) Thermal Goldstone modes behave like massless thermoparticles, and persist when $T > T_c$ $\rightarrow$ *Phase structure is determined by their damping*

(iii) Inconsistencies of standard $T > 0$ perturbation theory are caused by using the wrong choice of scattering states, i.e. free particles.
$\rightarrow$ *Thermoparticles fix these problems and lead to precise predictions*

So far, just considered $T > 0$ and $\mu = 0$.
What happens in systems with non-vanishing density? $\rightarrow$ *Interesting implications!*



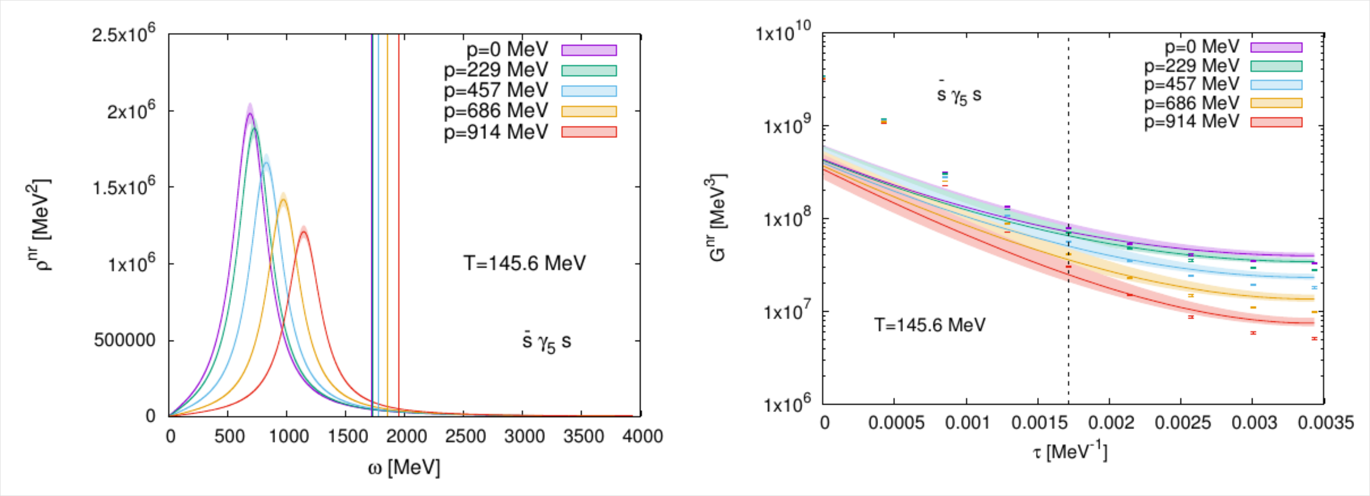
Backup: Hadrons at high temperature

- The robustness of the thermoparticle hypothesis can also be tested by comparing with different causal models, e.g. a Breit Wigner

$$\boxed{\rho_{\mathrm{BW}}(\omega,\vec{p}) = \frac{4\omega\Gamma}{(\omega^2-|\vec{p}|^2-m^2-\Gamma^2)^2+4\omega^2\Gamma^2},}$$

$$\boxed{C_{\mathrm{BW}}(z) = \frac{e^{-\sqrt{m^2+\Gamma^2}|z|}}{2\sqrt{m^2+\Gamma^2}}.}$$

- Same procedure as with the thermoparticle case: (i) extract the width parameter Γ and coefficient from the spatial lattice data (ii) use this to predict the corresponding temporal correlator



→ Data is *not* consistent with a Breit-Wigner-type ground state!

Backup: SSB when $T > 0$

- In order to define SSB rigorously one needs to define a regularised charged operator Q_R, and this only converges for $R \to \infty$ within commutator $[Q_R, A]$

$$\boxed{Q_{\delta R} = \int d^4x\, \alpha_\delta(t) g_R(\vec{x}) j_0(t, \vec{x})}$$

(i) $g_R(\boldsymbol{x})=1$ for $|\boldsymbol{x}| \leq R$, $g_R(\boldsymbol{x})=0$ for $|\boldsymbol{x}| > R$

(ii) $\alpha_\delta(t)$ has compact support, and $\alpha_\delta(t) \to \delta(t)$

- The condition: $\lim_{R\to\infty} <[Q_R, A]>=q$ is then ***always*** well-defined, and $q=0$ is a necessary and sufficient condition for the existence of a charge operator Q, defined by: $<u, Qv> := \lim_{R\to\infty} <u, Q_R v>$

$\rightarrow$ If this is non-vanishing for *any* A then no such charge exists, i.e. SSB!

- SSB condition: $\boxed{\lim_{\delta\to 0}\lim_{R\to\infty}\int_0^\infty ds \int \frac{d^3\vec{u}}{(2\pi)^3}\frac{d^3\vec{p}}{(2\pi)^3}\, \widetilde{D}_\beta^{(+)}(\vec{u}, s)\, \tilde{g}(\vec{p})\, \tilde{\alpha}\left(\delta R\sqrt{((\vec{p}/R)-\vec{u})^2+s}\right) = iq\tilde{\alpha}(0)}$
- In the vacuum case this implies: $\widetilde{D}^{(+)}(\boldsymbol{u},s) \to iq\,(2\pi)^3\,\delta(\boldsymbol{u})\delta(s)$
- Value of q is ***determined*** by Goldstone damping factor $D^{(+)}(\boldsymbol{x})$ for $|\boldsymbol{x}| \to \infty$

Backup: SSB when $T > 0$

- The phase of the theory is determined by the dissipative effects experienced by the thermal Goldstone mode
- Only $D^G(\boldsymbol{x}) \to 0$ for $|\boldsymbol{x}| \to \infty$ is required to ensure the symmetry is restored, but this can happen at any rate
- If the damping factor has the functional form $D^G(\boldsymbol{x}) \sim |\boldsymbol{x}|^{-\varepsilon}$ with some $\varepsilon > 0$ at large $|\boldsymbol{x}|$, the two-point function decays as a pure power-law

> **Interesting possibility**: a symmetry could be restored at high temperatures without there being a finite correlation length on either side of the phase transition

- Damping factors are fixed by the (asymptotic) dynamics of the theory [Bros, Buchholz, (2002)] $\longrightarrow$ suggests that finite-temperature phase transitions may not be entirely characterised by universality arguments

Backup: SSB when $T > 0$

- If thermal Goldstone modes are present one can look for their signatures in (Euclidean) correlation functions
- For simplicity, consider the QFT of a single complex scalar field at finite temperature, with two-point function $C(\tau,\vec{x}) = \langle \phi(\tau,\vec{x})\phi^\dagger(0)\rangle_\beta$
- If a thermal Goldstone mode is present, it follows from the thermoparticle structure, and the spectral function representation, that:

$$\boxed{C^G(0,\vec{x}) = \frac{\coth\left(\frac{\pi|\vec{x}|}{\beta}\right)}{4\pi\beta|\vec{x}|} D_\beta^G(\vec{x})}$$

- The mode dissipation is determined by the damping factor $D^G(\boldsymbol{x})$

  → For $T \to 0$ the vacuum behaviour is recovered: $\boxed{C^G(0,\vec{x}) \xrightarrow{T\to 0} \frac{\alpha_0}{4\pi^2|\vec{x}|^2}}$

- For the spatial correlator

$$\begin{aligned} C(z) &= \int dx\, dy\, d\tau\, C(\tau,\vec{x}) \\ &= \frac{1}{2}\int_0^\infty ds \int_{|z|}^\infty dR\, e^{-R\sqrt{s}} D_\beta(R,s) \end{aligned}$$

$$\Longrightarrow \quad \boxed{C^G(z) = \frac{1}{2}\int_{|z|}^\infty dR\, D_\beta^G(R)}$$

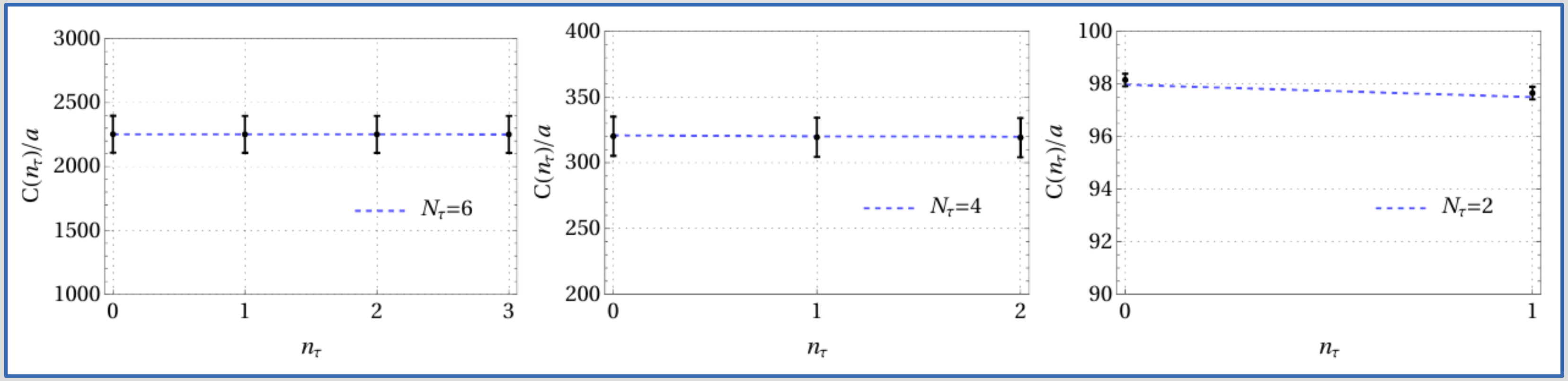
See: [PL,O. Philipsen, 2022]

Restored phase ($N_\tau = 2, 4, 6$)

- One can check the consistency of the extracted Goldstone spectral function $\rho_G(\omega,\boldsymbol{p})$ by comparing its prediction of the lattice **temporal** correlator data $C(\tau)$, since these are related via:

$$C(\tau) = \int_0^\infty \frac{d\omega}{2\pi} \frac{\cosh\left[\left(\frac{\beta}{2} - |\tau|\right)\omega\right]}{\sinh\left(\frac{\beta}{2}\omega\right)} \rho_G(\omega, \vec{p} = 0)$$

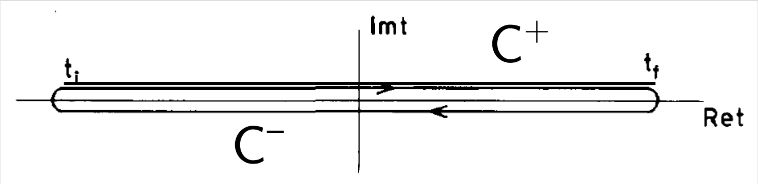
- Spectral functions are consistent with the data for all values of N_τ



Backup: Finite-temperature perturbation theory

- Perturbation theory at non-zero temperature $T = 1/\beta$ can be performed using either real or imaginary-time formulations. Both have their advantages and disadvantages (see: Kapusta & Gale, Le Bellac)

(i) **Real-time approach**: evolution along Schwinger-Keldish contour C=C$^+$∪C$^-$



Leads to "field doubling": Φ evolution along each branch expressed as separate fields Φ^+ and Φ^-

→ Perturbative calculations involve *four* propagators (++,+−,−+, − −)

$$\widetilde{\tau}^{++}_{(0)}(p) = \frac{i}{p^2 - m^2 + i\epsilon} + \frac{2\pi\delta(p^2 - m^2)}{e^{\beta|p_0|} - 1}$$

(ii) **Imaginary-time approach**: based on "Matsubara" propagators $G(\omega_n, p)$

$$G(\omega_n, \vec{p}) = \frac{1}{\omega_n^2 + |\vec{p}|^2 + m^2}$$

→ Energy integrals replaced with sums over discrete energies $\omega_n = 2\pi n/\beta$

→ Analytic continuation $i\,\omega_n \to p_0 + i\,\varepsilon$ taken at the end

Backup: Finite-temperature perturbation theory

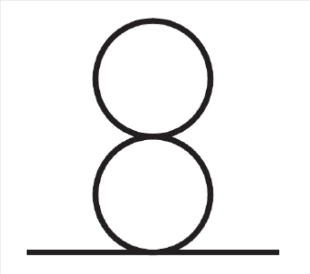
- It has long been understood that the inclusion of temperature T introduces a number of different complications:

  → In QCD, perturbation theory suffers inconsistencies above a fixed loop order (*"Lindé problem"* [*PLB* 96, 1980])

  → Perturbative series have poor convergence properties: need to reorganise the expansion (screened perturbation theory, infinite resummations, functional techniques, ...)

- All of these problems stem from the fact that $T > 0$ perturbation theory introduces a class of diagrams that are not present in the vacuum theory, and these are highly sensitive to the infrared dynamics

e.g. 2-loop order Φ^4 theory:



Includes sub-component

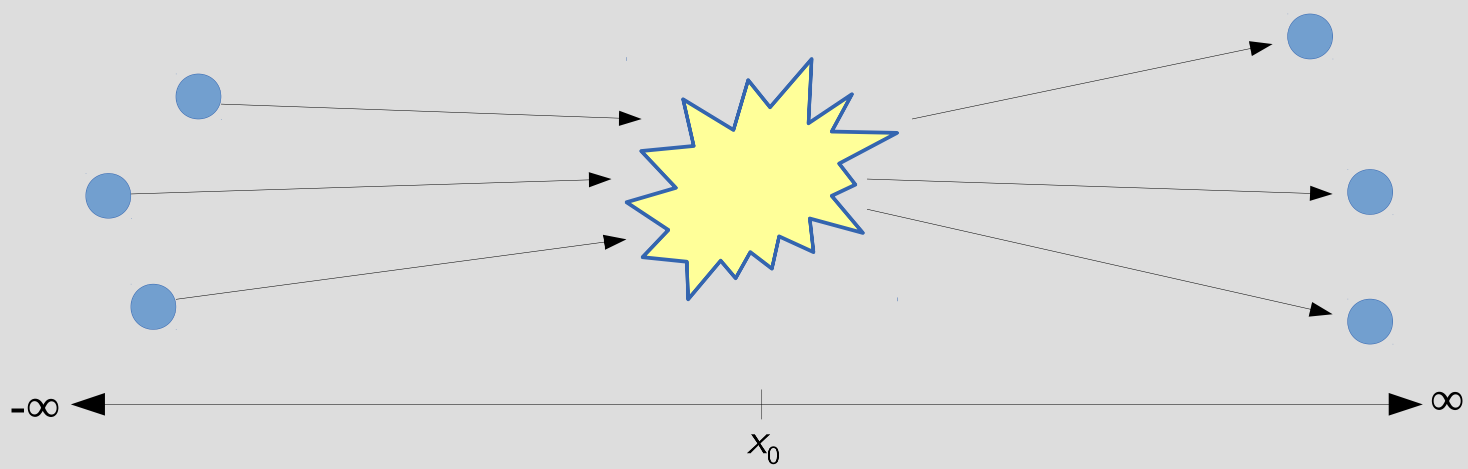
$$\boxed{T\sum_n \int \frac{d^3p}{(2\pi)^3}\frac{1}{(\omega_n^2+|\vec{p}|^2+m^2)^2}}$$

Diverges for m $\rightarrow$ 0, even after renormalisation

Backup: Finite-temperature perturbation theory

- These features are often viewed as a purely perturbative feature of finite-temperature QFT, but they are in fact a symptom of a more fundamental (non-perturbative) constraint:

"*Narnhofer-Requardt-Thirring Theorem*" [Commun. *Math. Phys.* 92, 247 (1983)]



<u>Idea</u>: construct thermal "quasi-particle" in/out scattering states with real dispersion relations $\omega = E(\boldsymbol{p}) \rightarrow$ Conclusion: *the S-matrix must be trivial!*

- So quasi-particles can exist, but only when there are no interactions...

Backup: Finite-temperature perturbation theory

Why? QFT: thermal states satisfy the KMS condition, and this gives rise to very different spectral constraints than in the vacuum case

Physics: Dissipative effects of the thermal medium are everywhere-present, so need to take these into account in the definition of scattering states!

Significant implications: *neither free field, nor quasi-particle propagators with real poles $\omega = E(\boldsymbol{p})$, can form the basis of finite-temperature perturbative expansions* [Landsman, *Ann. Phys.* 186, 141 (1988)]

Note: these constraints are not restricted to infrared regimes, but also affect systems at low temperatures, or with non-vanishing mass scales (i.e. $T/m < 1$)

$\rightarrow$ **But where do these propagator constraints come from?**

Backup: Finite-temperature perturbation theory

- **Realisation of NRT theorem constraints**: Weldon [*PRD* 65 (2002)] showed that (scalar) perturbation theory is inconsistent at some fixed loop order

  → One can classify the singularities occurring in the n-loop perturbative propagator given the singularities of the basic propagator

  → If the basic propagators have real poles $\omega = E(\boldsymbol{p})$, at some loop order the perturbative propagator will develop a branch point singularity at these poles, e.g. in Φ^4 theory this occurs at 2-loop order

$$\boxed{G_R(p) = -\frac{1}{(\omega + i\epsilon)^2 - E(\vec{p})^2}}$$

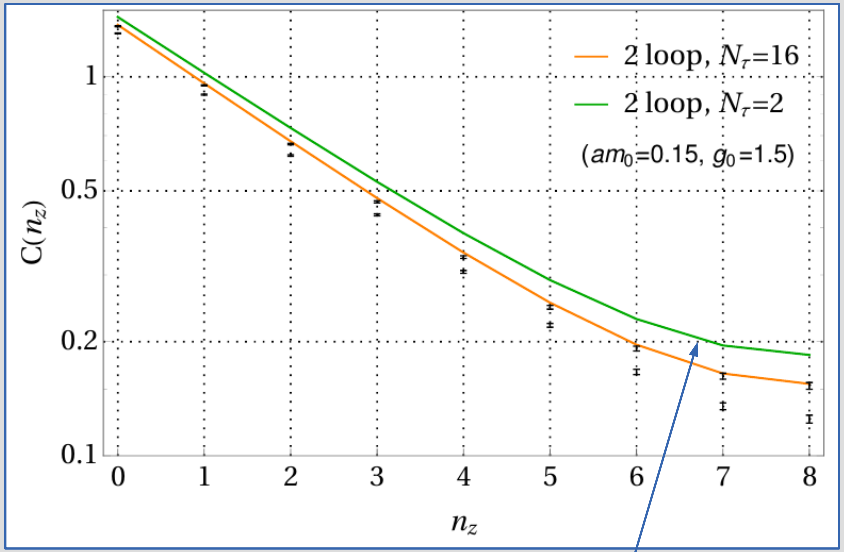
- This prohibits corrections to the perturbative thermal propagator pole from being computed, hence the standard approach breaks down

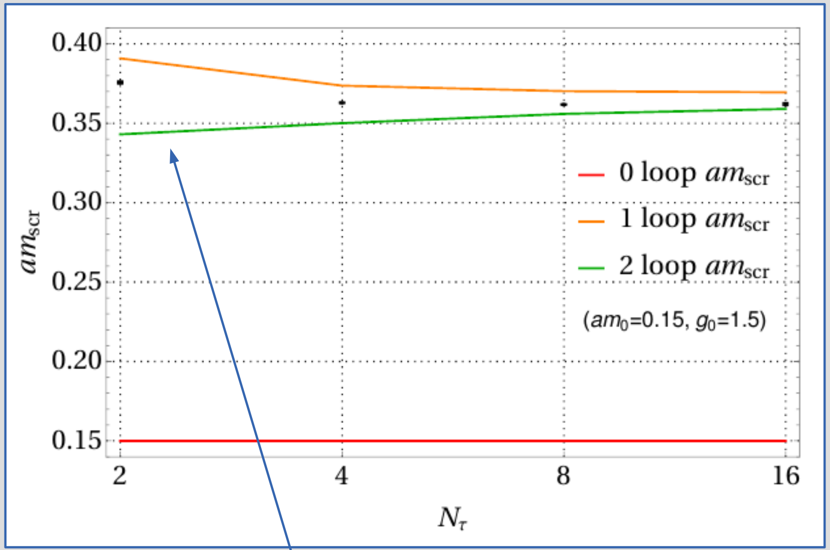
  → So far these are analytic problems. Can one find numerical evidence of this breakdown in different systems?

- How do the 2-loop LPT predictions compare with the lattice data as one varies the (lattice) temperature $T=1/(aN_\tau)$?

  1. For sufficiently small g_0 the perturbative predictions are consistent with the data for all temperatures

  2. For non-negligible g_0 the perturbative predictions for $C(z)$ increasingly deteriorate as T/m increases





For $T/m \sim 1$ the 2-loop predictions deviate drastically from the data

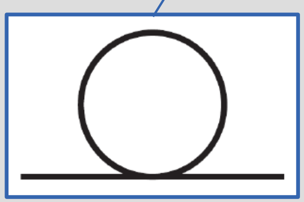
These deviations are also reflected by the screening mass m, defined: $C(z) \sim e^{-mz}$, having the *opposite* T dependence

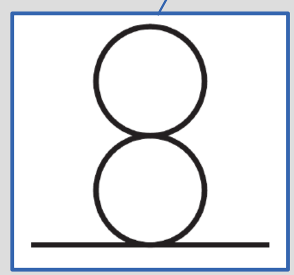
Backup: Finite-temperature perturbation theory

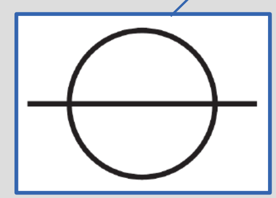
- To establish the origin of these deviations one needs to understand how the 2-loop perturbative predictions are computed

$$C(z;a,N_s,N_\tau) = \frac{1}{N_s}\sum_{k_z=0}^{N_s-1} e^{\frac{2\pi i k_z}{aN_s}z} \frac{a}{4\sin^2\left(\frac{\pi k_z}{N_s}\right) + (am_0)^2 + a^2\Pi(\omega_E = p_x = p_y = 0, p_z = \frac{2\pi k_z}{aN_s}; a, N_s, N_\tau)}.$$

$$a^2\Pi_{(2)}(p;a,N_s,N_\tau) = \frac{g_0}{2}J_1(am_0;N_s,N_\tau) - \frac{g_0^2}{4}J_1(am_0;N_s,N_\tau)J_2(am_0;N_s,N_\tau) - \frac{g_0^2}{6}I_3(p,am_0;N_s,N_\tau)$$







The functions J_n and I_3 are defined:

$$J_n(am_0;N_s,N_\tau) = \frac{1}{N_s^3 N_\tau}\sum_{p\in\mathcal{B}_a}\frac{1}{\left[\sum_\mu 4\sin^2\left(\frac{ap_\mu}{2}\right) + (am_0)^2\right]^n}, \qquad n = 1,2$$

$$I_3(p,am_0;N_s,N_\tau) = \frac{1}{(N_s^3 N_\tau)^2}\sum_{q\in\mathcal{B}_a}\sum_{r\in\mathcal{B}_a}\frac{1}{\left[\sum_\mu 4\sin^2\left(p_\mu - \frac{aq_\mu}{2} - \frac{ar_\mu}{2}\right) + (am_0)^2\right]} \times \frac{1}{\left[\sum_\nu 4\sin^2\left(\frac{aq_\nu}{2}\right) + (am_0)^2\right]}\frac{1}{\left[\sum_{\Lambda_a} 4\sin^2\left(\frac{ar_\lambda}{2}\right) + (am_0)^2\right]}$$

Backup: Finite-temperature perturbation theory

- In principle, the implications of these constraints can also be looked for by comparing perturbative predictions with lattice calculations

  Idea [PL, O. Philipsen, 2405.02009]: perform lattice simulations of basic field correlators in Φ^4 theory and compare these with the predictions of lattice perturbation theory (LPT) $\rightarrow$ see: Montvay & Münster

- In LPT one uses the discretised propagator $$G(p; a, L, L_\tau) = \frac{1}{\sum_\mu \frac{4}{a^2}\sin^2\left(\frac{ap_\mu}{2}\right) + m_0^2}$$

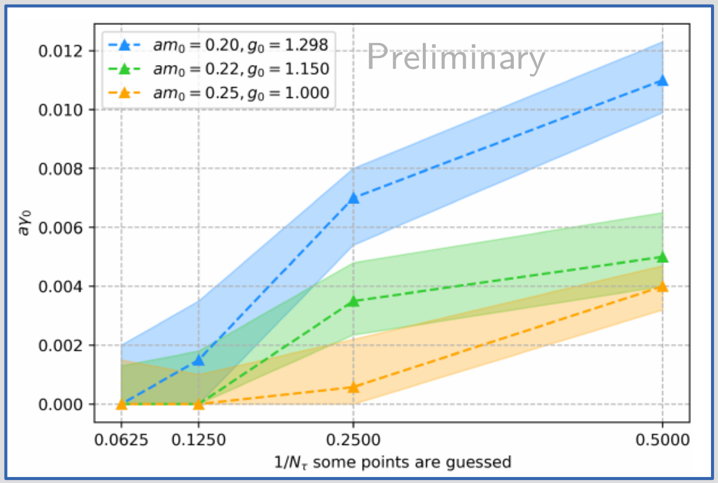
- Theory is defined on a finite $N_s^3 \times N_\tau$ lattice $\rightarrow$ internal loop integrals are replaced with sums over first Brillouin zone $\mathcal{B}_a = \{p \in \mathbb{R}^4 : -\frac{\pi}{a} < p_\mu \leq \frac{\pi}{a}\}$

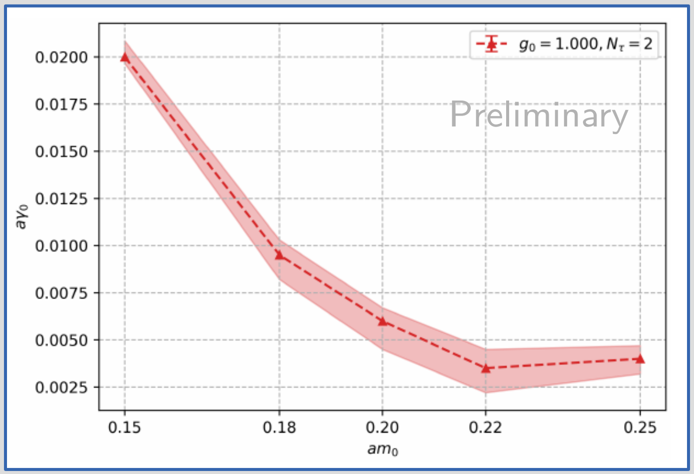
- In Φ^4 theory the lattice action is: $$S = a^4 \sum_{x\in\Lambda}\left[\frac{1}{2}\Delta_\mu^f\phi_0(x)\Delta_\mu^f\phi_0(x) + \frac{m_0^2}{2}\phi_0(x)^2 + \frac{g_0}{4!}\phi_0(x)^4\right]$$

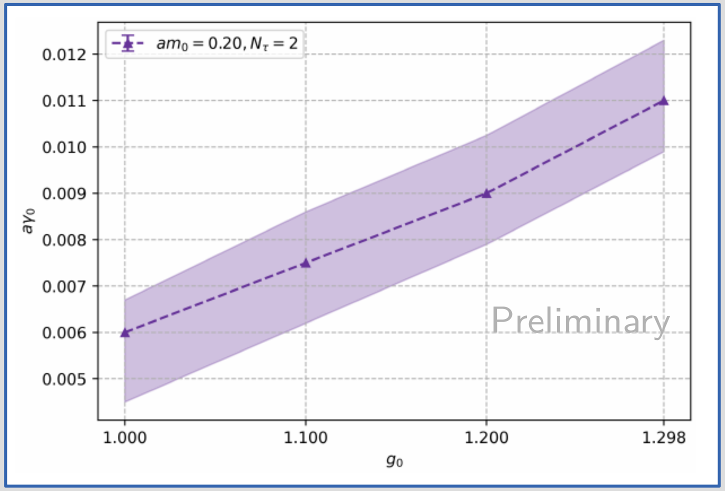
- Here we focus on perturbative calculations of the *spatial correlator*, which is defined: $$C(z; a, N_s, N_\tau) = a^3 \sum_{\tau,x,y}\langle\phi(\tau,\vec{x})\phi(0)\rangle$$

Parametric dependence of $a\gamma_0(am_0, g_0, N_\tau)$

→ By performing simulations with various different parameter sets one can determine how $a\gamma_0$ changes as one of $\{am_0, g_0, N_\tau\}$ is varied whilst keeping the others fixed







$a\gamma_0$ changes as expected:

– Larger am_0 → lower $a\gamma_0$

– Higher $1/N_\tau$ → larger $a\gamma_0$

– Lower g_0 → lower $a\gamma_0$

Backup: Finite-temperature perturbation theory

Consistency check: perform the same procedure with different propagator parametrisations, e.g. Breit-Wigner

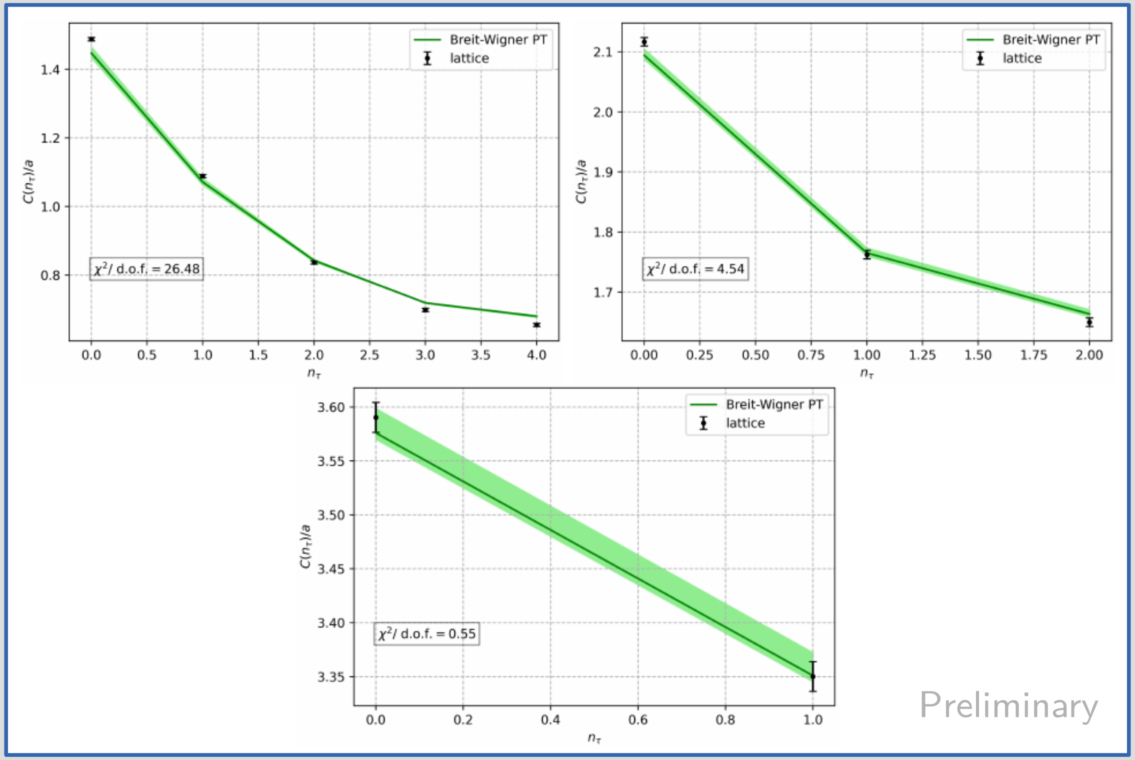
- In the Breit-Wigner case the spectral function and propagator have the form:

$$\rho_{\mathrm{BW}}(\omega, \vec{p}) = \frac{4\omega\Gamma}{(\omega^2 - |\vec{p}|^2 - m^2 - \Gamma^2)^2 + 4\omega^2\Gamma^2}$$

$$G_n(\omega_n, \vec{p}) = \frac{1}{\omega_n^2 + |\vec{p}|^2 + m^2 + \Gamma^2 + 2\Gamma|\omega_n|}$$

Extract width Γ for different (am_0, g_0, N_τ) from $C(z)$ and use this to compute $C(\tau)$

$\rightarrow$ **Poor description**!



Backup: Finite-temperature perturbation theory

Given a specific QFT, what form should the damping factors take?

Idea: thermal scattering states are defined by imposing an asymptotic field condition [Bros, Buchholz, (2002)]:

Asymptotic fields Φ_0 are assumed to satisfy dynamical equations, but only at large x_0

In Φ^4 theory

$$(\partial^2 + m^2)\phi_0(x) + \frac{\lambda}{3!}\phi_0^3(x) \xrightarrow{|x_0|\to\infty} 0$$

- Since thermoparticles dominate the large-time behaviour of correlators, they are natural candidates for describing such states. It turns out that their damping factors $\widetilde{D}_{m,\beta}(\boldsymbol{u})$ are **uniquely fixed** by the asymptotic condition
- In Φ^4 theory one finds (where κ is a thermal width):

For $g < 0$:

$$\widetilde{D}^{(-)}_{m,\beta}(\vec{u}) = \frac{2\pi^2}{\kappa^2}\delta(|\vec{u}| - \kappa),$$

$$\widetilde{G}^{(-)}_{\beta}(k_0,\vec{p}) = \frac{1}{4|\vec{p}|\kappa}\ln\left[\frac{-k_0^2 + m^2 + (|\vec{p}| + \kappa)^2}{-k_0^2 + m^2 + (|\vec{p}| - \kappa)^2}\right]$$

For $g > 0$:

$$\widetilde{D}^{(+)}_{m,\beta}(\vec{u}) = \frac{4\pi}{\kappa_0\left(|\vec{u}|^2 + \kappa^2\right)},$$

$$\widetilde{G}^{(+)}_{\beta}(k_0,\vec{p}) = \frac{i}{2|\vec{p}|\kappa_0}\ln\left[\frac{\sqrt{-k_0^2 + m^2} - i|\vec{p}| + \kappa}{\sqrt{-k_0^2 + m^2} + i|\vec{p}| + \kappa}\right]$$