Color Superconductivity Revisited

~Importance of QCD anomaly and gluons~

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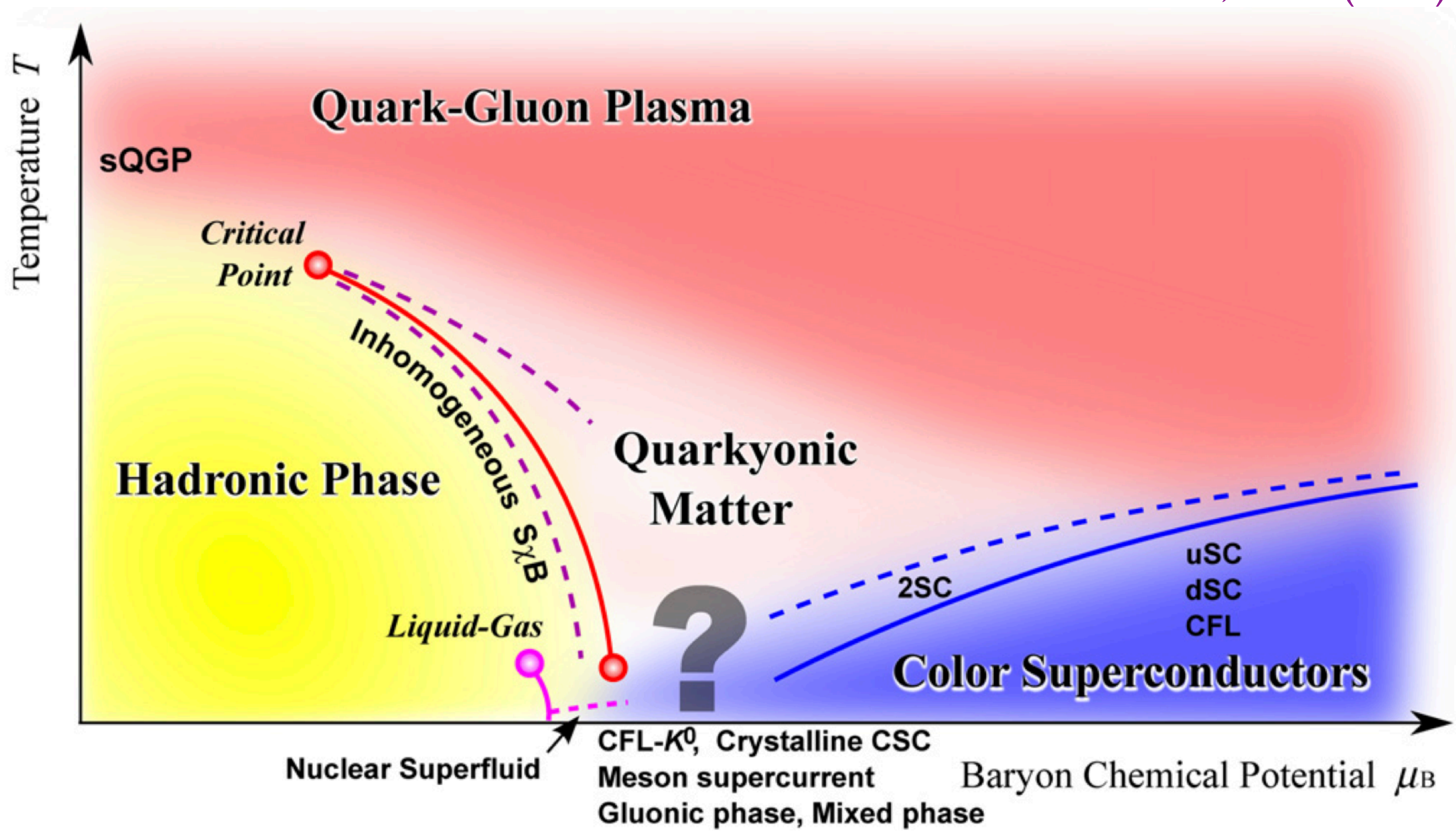
Buenas Ideas on the QCD Phase Diagram
May 26, 2026

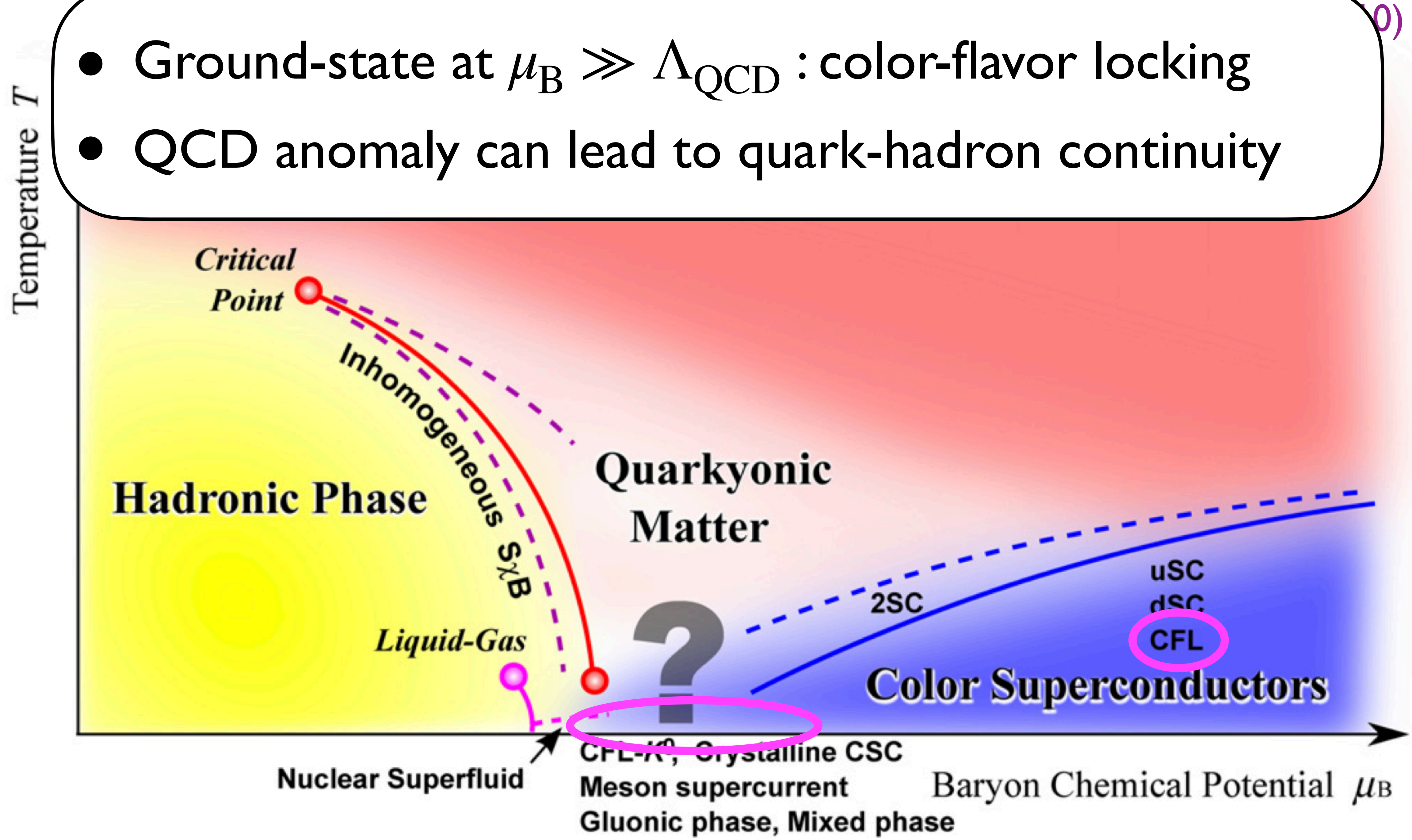
QCD phase diagram

Fukushima-Hatsuda, PPNP (2010)



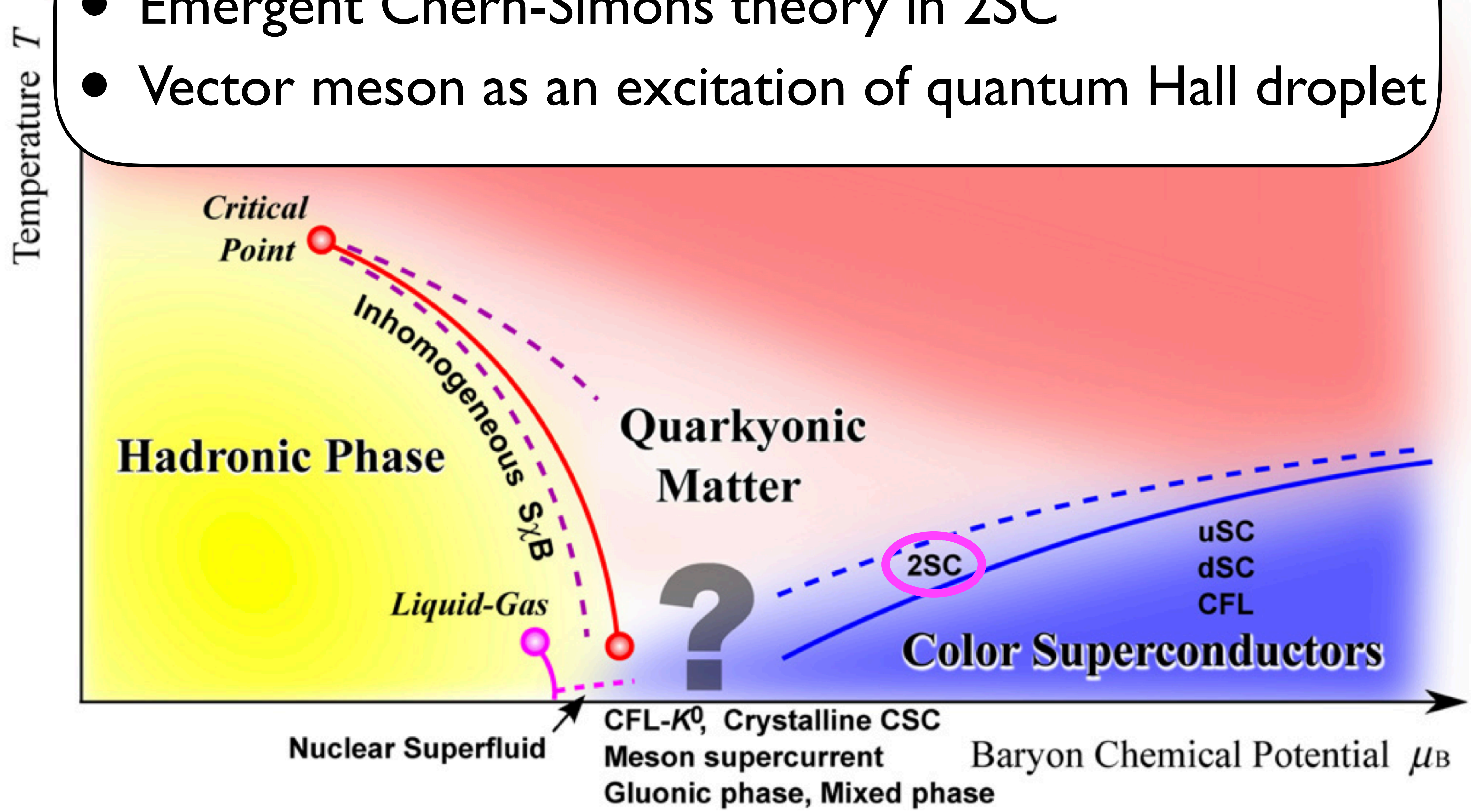
Contents

- Ground-state at $\mu_{\rm B} \gg \Lambda_{\rm QCD}$: color-flavor locking
- QCD anomaly can lead to quark-hadron continuity



Contents

- Emergent Chern-Simons theory in 2SC
- Vector meson as an excitation of quantum Hall droplet



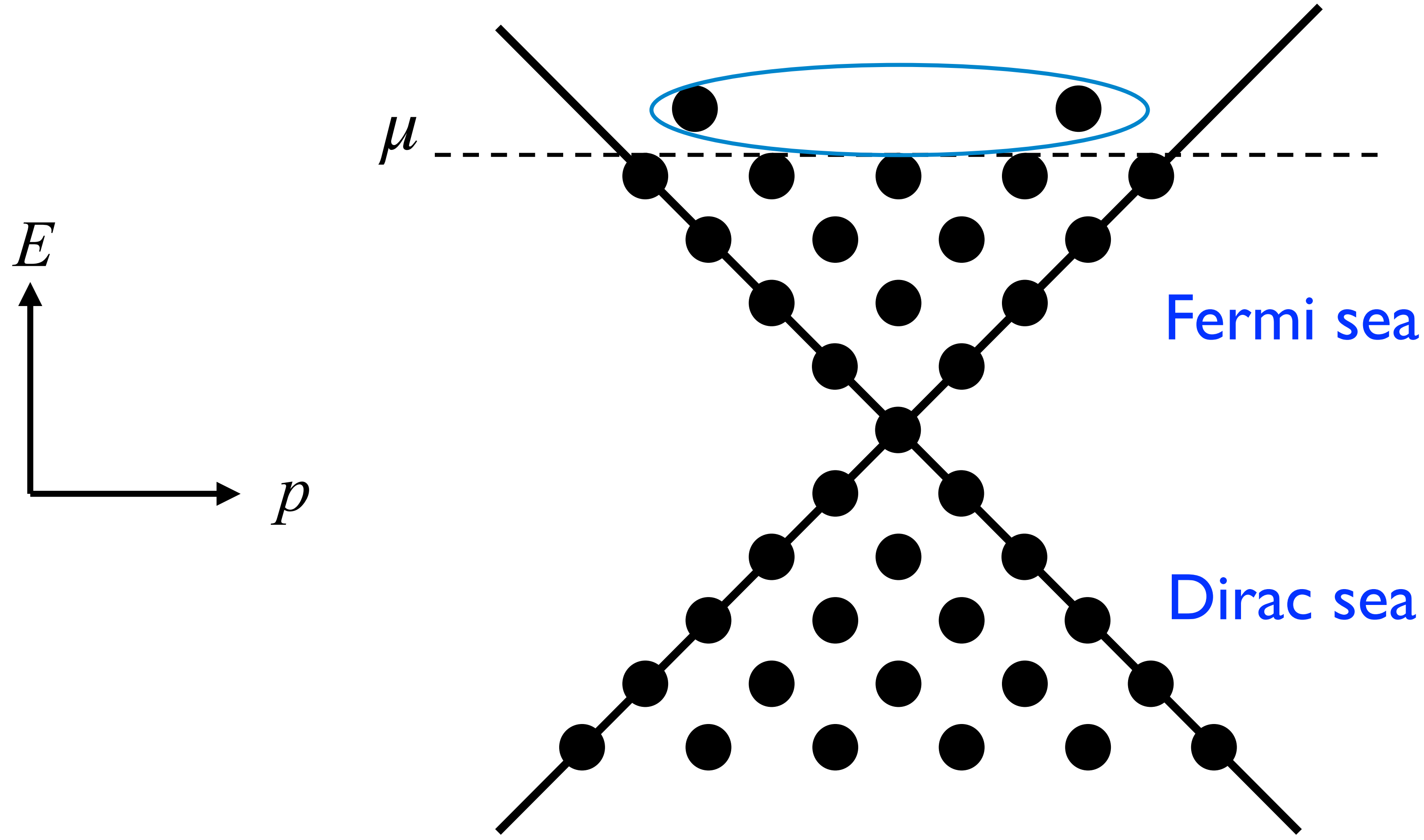
Non-Abelian nature of QCD is essential in all cases

Quark matter

higher density high-density limit

- Nuclear matter → Quark matter → Almost free quark gas

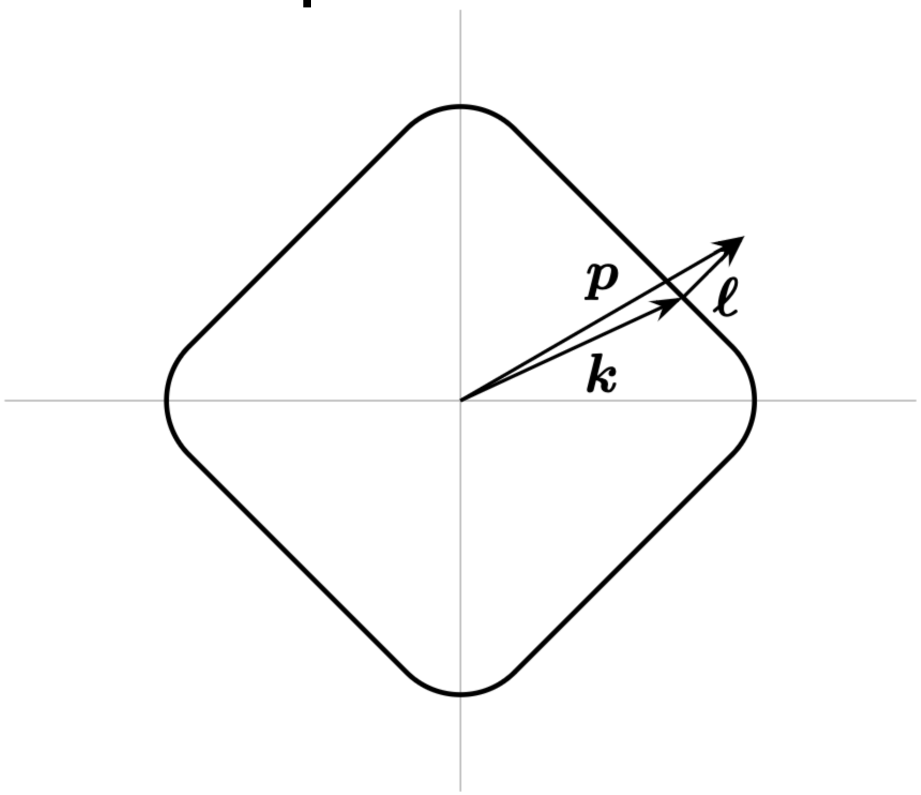
Collins-Perry, PRL (1975)



Fermi liquid theory

Polchinski, hep-th/9210046

- Decomposition of momentum near Fermi surface:



$$\boldsymbol{p} = \boldsymbol{k} + \boldsymbol{\ell}$$

$$\epsilon_{\boldsymbol{p}} - \epsilon_{\mathrm{F}} \simeq \boldsymbol{\ell} \cdot \boldsymbol{v}_{\mathrm{F}}$$

- Action of free theory: $S_{\text{free}} = \int \mathrm{d}t\, \mathrm{d}^3\boldsymbol{p}\, \psi^\dagger_{\boldsymbol{p}} [\mathrm{i}\partial_t - (\epsilon_{\boldsymbol{p}} - \epsilon_{\mathrm{F}})] \psi_{\boldsymbol{p}}$

- RG scaling close to the Fermi surface:

$$\boldsymbol{\ell} \to s\boldsymbol{\ell}, \quad \mathrm{d}^3\boldsymbol{p} \to s\, \mathrm{d}^3\boldsymbol{p}, \quad \mathrm{d}t \to s^{-1}\mathrm{d}t, \quad \psi \to s^{-1/2}\psi$$

BCS channel

- RG scaling of interactions:

$$S_{\rm int} = \int \mathrm{d}t \prod_{i=1}^{4} \mathrm{d}^3\boldsymbol{p}_i\, \delta^3(\boldsymbol{p}_1 + \boldsymbol{p}_2 - \boldsymbol{p}_3 - \boldsymbol{p}_4) V_{\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3,\boldsymbol{p}_4} \psi^\dagger_{\boldsymbol{p}_4} \psi^\dagger_{\boldsymbol{p}_3} \psi_{\boldsymbol{p}_2} \psi_{\boldsymbol{p}_1}$$

$$\boldsymbol{\ell} \to s\boldsymbol{\ell}, \quad \mathrm{d}^3\boldsymbol{p} \to s\,\mathrm{d}^3\boldsymbol{p}, \quad \mathrm{d}t \to s^{-1}\mathrm{d}t, \quad \psi \to s^{-1/2}\psi$$

$S_{\rm int} \to s^{-1+4-4/2} S_{\rm int} = s^1 S_{\rm int}$: irrelevant for general kinematics;

3-body interaction is more irrelevant

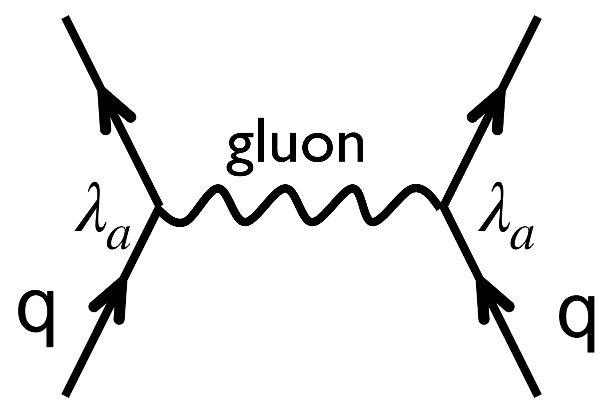
- Special kinematics (BCS channel): $\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{0}$

$$\delta(\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2) \to s^{-1}\delta(\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2), \quad S_{\rm int} \to s^{-1+4-4/2-1} S_{\rm int} = S_{\rm int}$$

marginally relevant for attractive interactions (BCS mechanism)

Color superconductivity

- $\mu_{\mathrm{B}} \gg \Lambda_{\mathrm{QCD}}$: one-gluon exchange int. (asymptotic freedom)
- Attractive interaction in the color asymmetric channel



$$(\lambda_a)_{ij}(\lambda_a)_{kl} = \frac{2}{3}(\lambda_{\mathrm{S}})_{ik}(\lambda_{\mathrm{S}})_{lj} - \frac{4}{3}(\lambda_{\mathrm{A}})_{ik}(\lambda_{\mathrm{A}})_{lj}$$

$$3 \otimes 3 = 6_{\mathrm{S}} \oplus \bar{3}_{\mathrm{A}}$$

- Attractive interaction leads to quark-quark pairing (BCS mechanism)

cf. metallic superconductivity: Coulomb repulsion between electrons

Color superconductivity originates from non-Abelian nature of QCD

Diquark condensate

- QCD has various quantum numbers:

$$\left\langle q^a_{\mathrm{R}i}(\boldsymbol{p}) q^b_{\mathrm{R}j}(-\boldsymbol{p}) \right\rangle = (d_{\mathrm{R}})^c_k \epsilon^{abc} \epsilon_{ijk}$$

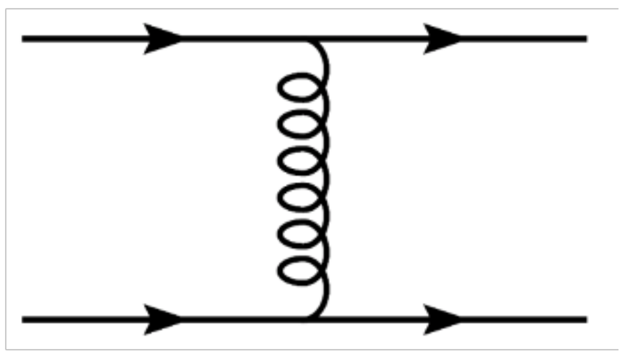
color a, b = R, G, B
flavor i, j = u, d, …

- Opposite momenta ***p*** and -***p*** (BCS instability)
- s-wave pairing (opposite spins) → same helicity/chirality
- Color antisymmetric (attractive interaction)

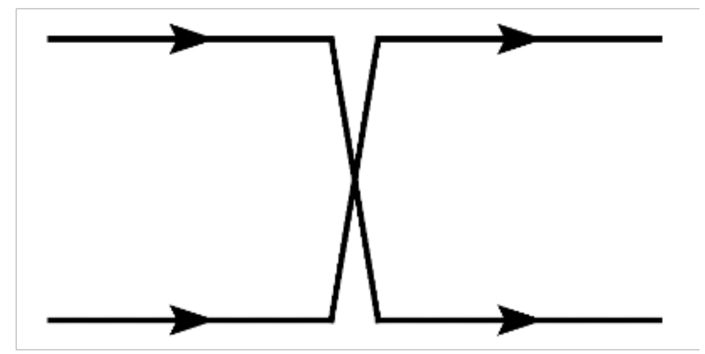
→ Flavor antisymmetric (Pauli principle)

Large-Nc limit

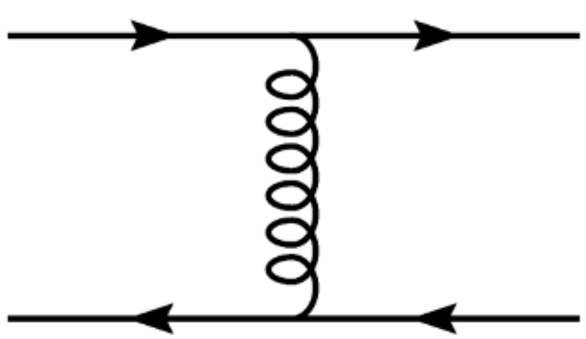
- Scattering in qq channel $3 \otimes 3 = 6_{\mathrm{S}} \oplus \bar{3}_{\mathrm{A}}$



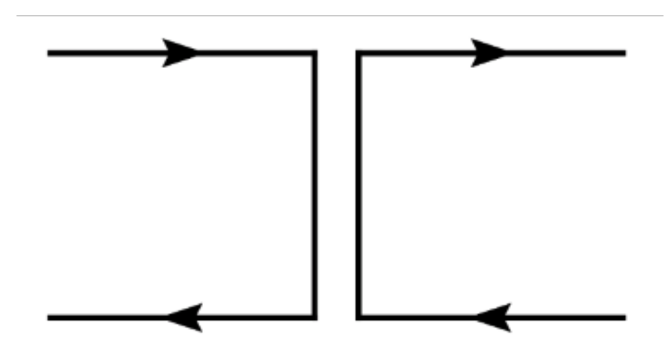
$$-\frac{N_{\mathrm{c}}+1}{N_{\mathrm{c}}}$$



- Scattering in $\bar{q}q$ channel $3 \otimes \bar{3} = 1 \oplus 8$



$$-\frac{N_{\mathrm{c}}^2-1}{N_{\mathrm{c}}}$$



$$\langle qq \rangle \propto \exp\left(-\frac{\#}{g}\sqrt{\frac{N_{\mathrm{c}}}{N_{\mathrm{c}}+1}}\right) \sim \mathrm{e}^{-\sqrt{N_{\mathrm{c}}}}$$

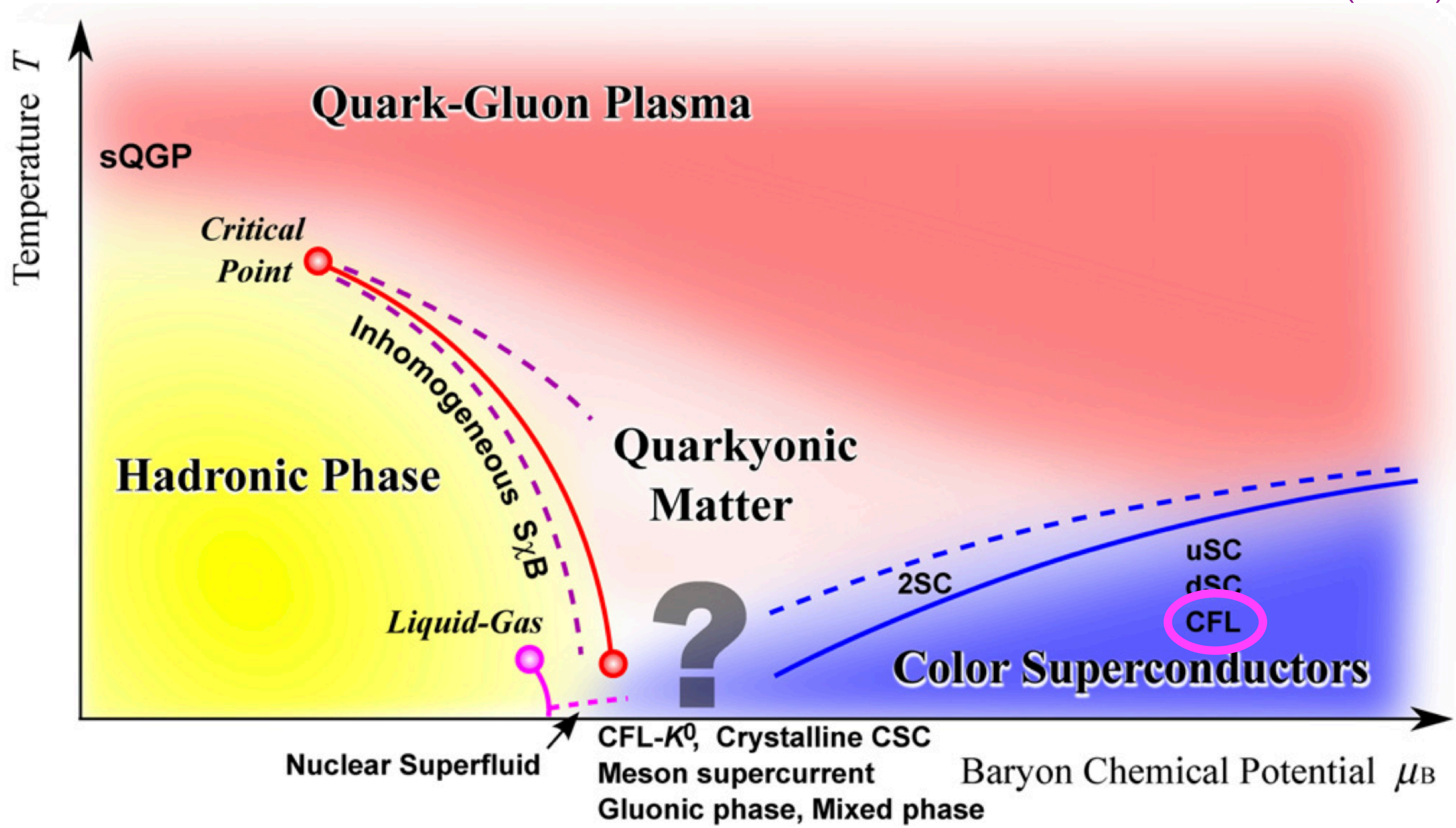
suppression as an artifact of large-N_c

Deryagin-Grigoriev-Rubakov (1992); Shuster-Son (2000)

Quark-hadron continuity and QCD anomaly

QCD phase diagram

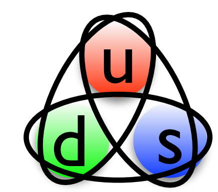
Fukushima-Hatsuda, PPNP (2010)



Color-flavor locking

color flavor

- 3-flavor limit: $\langle (q_{\mathrm{R}})_i^a (q_{\mathrm{R}})_j^b \rangle = (d_{\mathrm{R}})_k^c \epsilon^{abc} \epsilon_{ijk}$



$(d_{\mathrm{R}})_k^c \propto \delta_k^c$ color-flavor licking (CFL): ground state at $\mu_{\mathrm{B}} \gg \Lambda_{\mathrm{QCD}}$

Alford-Rajagopal-Wilczek, NPB (1999)

- Symmetry breaking: $SU(3)_C \times SU(3)_L \times SU(3)_R \times U(1)_B \to SU(3)_{C+L+R}$
- Color-singlet order parameter: $(\bar{d}_{\mathrm{L}}^a)(d_{\mathrm{R}}^a) \sim \bar{q}_{\mathrm{L}} \bar{q}_{\mathrm{L}} q_{\mathrm{R}} q_{\mathrm{R}}$ etc.
- Same symmetry as the hyper-nuclear matter at low density

$\to$ Quark-hadron continuity   Schäfer-Wilczek, PRL (1999)

Importance of QCD anomaly

- Without QCD anomaly, discrete symmetries are different:

$$\mathrm{U}(1)_\mathrm{A} \to \mathbb{Z}_2 \text{ by } \langle \bar{q}_\mathrm{L} q_\mathrm{R} \rangle \text{ in nuclear matter;}$$
$$\mathrm{U}(1)_\mathrm{A} \to \mathbb{Z}_4 \text{ by } \langle \bar{q}_\mathrm{L} \bar{q}_\mathrm{L} \rangle \langle q_\mathrm{R} q_\mathrm{R} \rangle \text{ in CFL}$$

Different symmetries → phase transition

- QCD anomaly explicitly breaks $\mathrm{U}(1)_\mathrm{A} \to \mathbb{Z}_{2N_\mathrm{f}}\ (=\mathbb{Z}_6)$:

$$\mathbb{Z}_6 \to \mathbb{Z}_2 \text{ by } \langle \bar{q}_\mathrm{L} q_\mathrm{R} \rangle \text{ in nuclear matter;}$$
$$\mathbb{Z}_6 \to \mathbb{Z}_{\mathrm{gcd}(6,4)} = \mathbb{Z}_2 \text{ by } \langle \bar{q}_\mathrm{L} \bar{q}_\mathrm{L} \rangle \langle q_\mathrm{R} q_\mathrm{R} \rangle \text{ in CFL}$$

QCD anomaly (instantons) in CFL is essential for the continuity

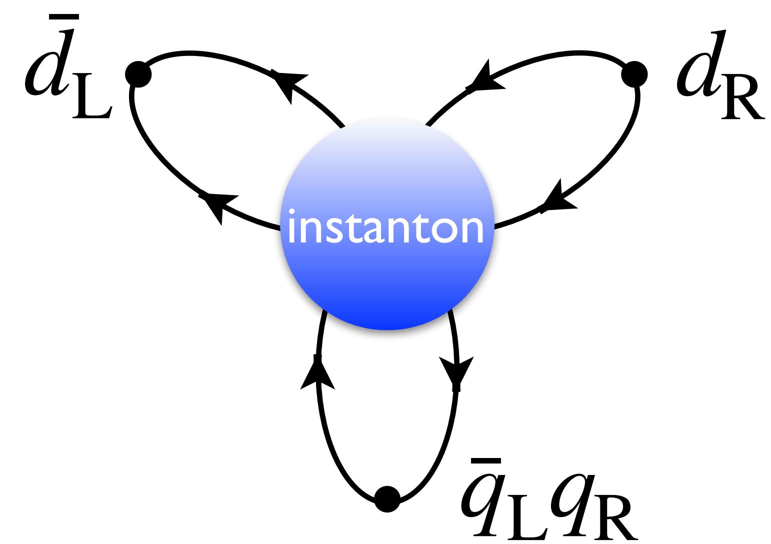
Hatsuda, Baym, Tachibana, Yamamoto, PRL (2006)

Anomaly-induced continuity

Hatsuda, Baym, Tachibana, Yamamoto, PRL (2006)

- Instanton-induced interaction in CFL

$$\det{}_{\mathrm{f}}(\bar{q}^i_{\mathrm{L}} q^j_{\mathrm{R}}) \sim \underbrace{(\bar{q}_{\mathrm{L}}\bar{q}_{\mathrm{L}})}_{\bar{d}_{\mathrm{L}}}\underbrace{(q_{\mathrm{R}}q_{\mathrm{R}})}_{d_{\mathrm{R}}}\ \underbrace{(\bar{q}_{\mathrm{L}}q_{\mathrm{R}})}_{\text{chiral cond.}}$$



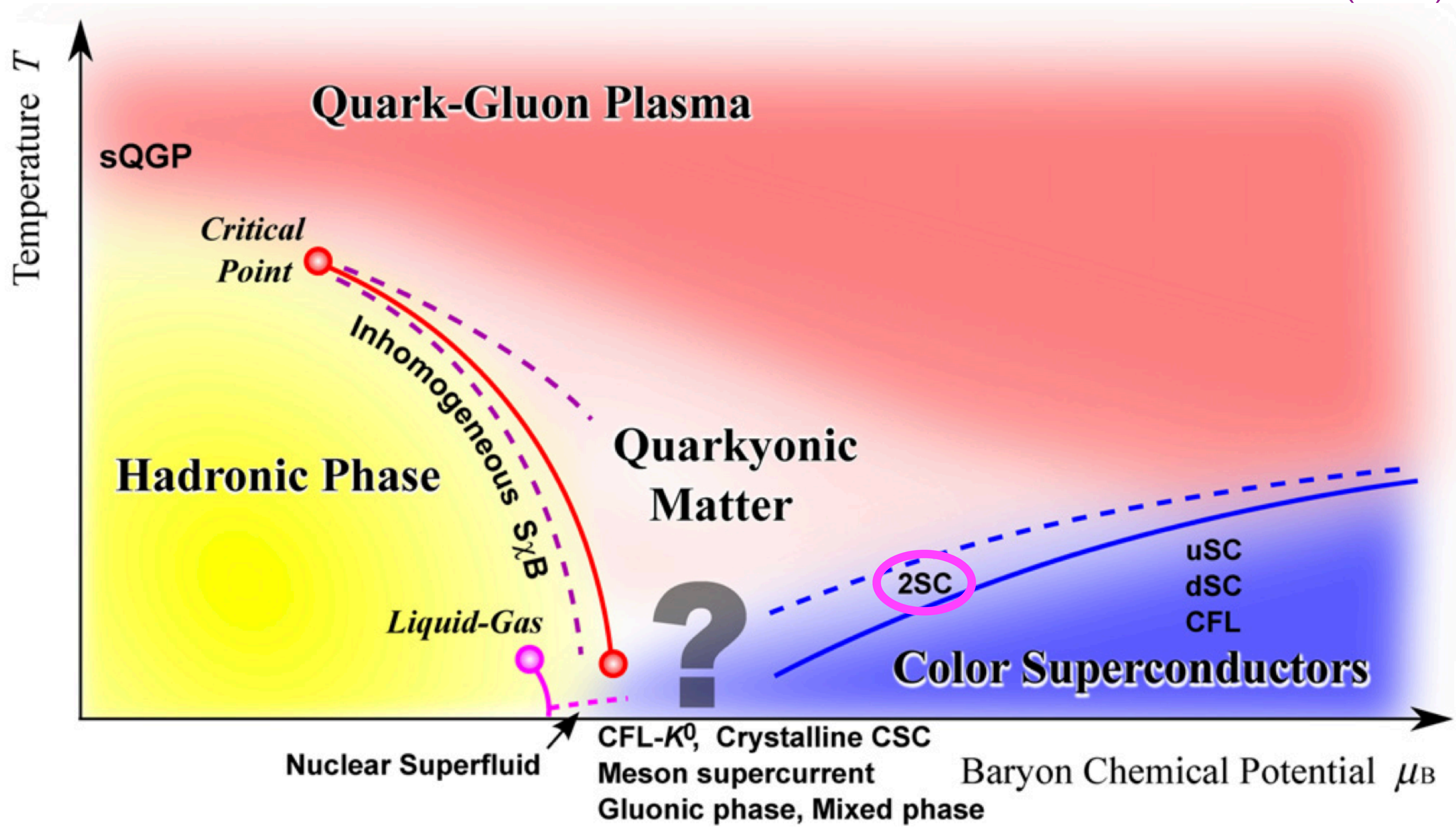
- It acts as a "source" for chiral condensate (like mass) even w/o mass

  → leads to crossover (like crossover in finite-T QCD)

Quantum Hall droplet in 2-flavor color super

QCD phase diagram

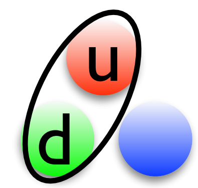
Fukushima-Hatsuda, PPNP (2010)



2-flavor color superconductivity

color flavor

- 2-flavor limit: $\langle (q_{\rm R})^a_i (q_{\rm R})^b_j \rangle = d_{\rm R}\epsilon^{ab3}\epsilon_{ij}$



see, e.g., Alford-Rajagopal-Wilczek, PLB (1998)

- "Symmetry breaking": $SU(3)_C \times U(1)_B \rightarrow SU(2)_C \times U(1)_{\tilde{B}}$

$$\tilde{B} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Baryon number is carried only by unpaired (blue) quarks

Low-energy excitations in 2SC

- η (q̄q̄qq state) becomes lighter at higher density

  $\Sigma \equiv d_R^\dagger d_L, \quad \Sigma = |\Sigma| e^{-i\eta}$ N.B. η is a q̄q state in the vacuum

- SU(2) gluon well below Δ: SU(2) pure Yang-Mills with emergent Λ'_{QCD}

  Son-Stephanov-Rischke, PRL (2001)

- Low-energy d.o.f.s: unpaired quarks, confined SU(2) gluons & η
  - Meson sector is essentially 1-flavor (even in 2-flavor QCD)
- Hierarchy at sufficiently high density: $\mu \gg \Delta \gg m_\eta \gg \Lambda'_{\mathrm{QCD}}$

N.B. η does not couple to unpaired quarks at leading order

Effective theory for η

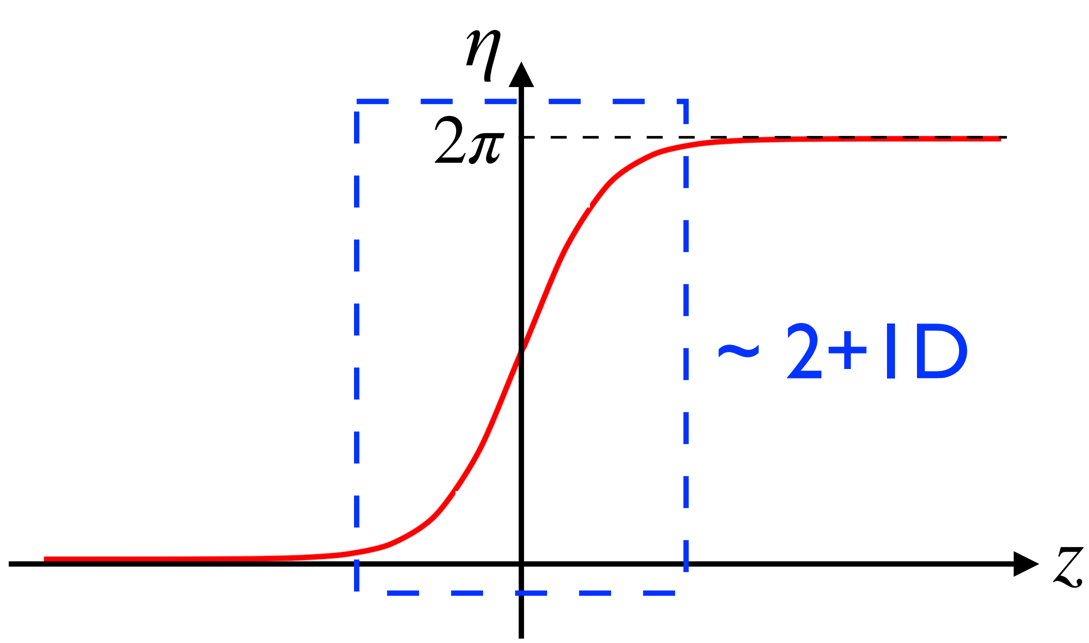
Son-Stephanov-Zhitnitsky, PRL (2001)

- $U(1)_A$ symmetry: $\eta \to \eta + \mathrm{const}\,.\ (\eta + 2\pi \sim \eta)$

$$\mathcal{L} = f^2[(\partial_t \eta)^2 - v^2(\boldsymbol{\nabla}\eta)^2] + A\cos\eta$$

(f, v, A computable by weak-coupling analysis at high density)

- Domain-wall solution from $\eta = 0\ (z = -\infty)$ to $\eta = 2\pi\ (z = \infty)$:



Topological charge:

$$Q_{\mathrm{DW}} := \frac{1}{2\pi}\int_{-\infty}^{\infty} \partial_z \eta = 1$$

EFT on 2+1D domain wall?
Low-energy d.o.f.s coupled to η?

Domain Walls of High-Density QCD

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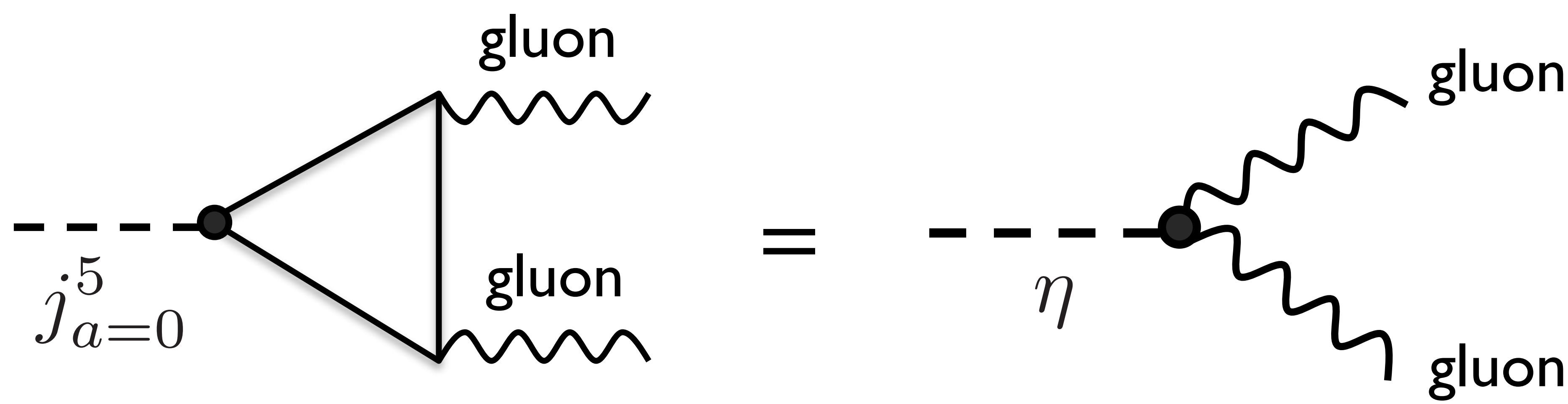
(Received 17 December 2000)

We show that in very dense quark matter there must exist metastable domain walls where the axial U(1) phase of the color-superconducting condensate changes by 2π. The decay rate of the domain walls is exponentially suppressed and we compute it semiclassically. We give an estimate of the critical chemical potential above which our analysis is under theoretical control.

Discussion.—It would be interesting to investigate possible astrophysical consequences of the high-density QCD walls. In particular, one would like to know if such walls can be created inside neutron stars. To describe the motion of the wall, one may need more than just the effective Lagrangian (8): the coupling of η to ungapped quarks and $\mathrm{SU}(2)_c$ gluons could be important. The moving wall may

QCD anomaly

Coupling of η to gluons emerges through the QCD anomaly



UV (quark)

IR (hadron)

$$\partial_\mu j^{\mu 5} = -\frac{1}{4\pi^2}\mathrm{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu})$$

$$S_{\mathrm{anom}} = -\frac{1}{16\pi^2}\int \mathrm{d}^4x\ \eta\ \mathrm{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu})$$

$$\because \eta \sim \bar{q}\bar{q}qq \rightarrow Q_5(\eta) = 4Q_5(q)$$

Topological field theory on the wall

Nishimura, Yamamoto, Yokokura PRD (2025)

- η couples to confined SU(2) gluons via QCD anomaly:

$$S_{\rm anom} = -\frac{1}{16\pi^2}\int \mathrm{d}^4x\ \eta\ \mathrm{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}) \qquad \boxed{F_{\mu\nu}\tilde{F}^{\mu\nu} = 2\epsilon^{\mu\nu\rho\sigma}\partial_\mu(A_\nu\partial_\rho A_\sigma + \cdots)}$$

$$\to -\frac{1}{8\pi^2}\underbrace{\int_{-\infty}^{\infty}\mathrm{d}z\partial_z\eta}_{2\pi \text{ for DW}}\int \mathrm{d}^3x\epsilon^{\mu\nu\rho}\mathrm{tr}(A_\mu\partial_\nu A_\rho + \cdots) =: S_{\rm CS}[A]$$

- $SU(2)_{-1}$ Chern-Simons (CS) theory on the wall:

$$S_{\rm CS}[A] = -\frac{1}{4\pi}\int \mathrm{d}^3x\ \epsilon^{\mu\nu\rho}\mathrm{tr}\left(A_\mu\partial_\nu A_\rho - \frac{2\mathrm{i}}{3}A_\mu A_\nu A_\rho\right)$$

SU(2) gauge field

$\rightleftarrows$ $U(1)_2$ CS theory (level-rank duality):

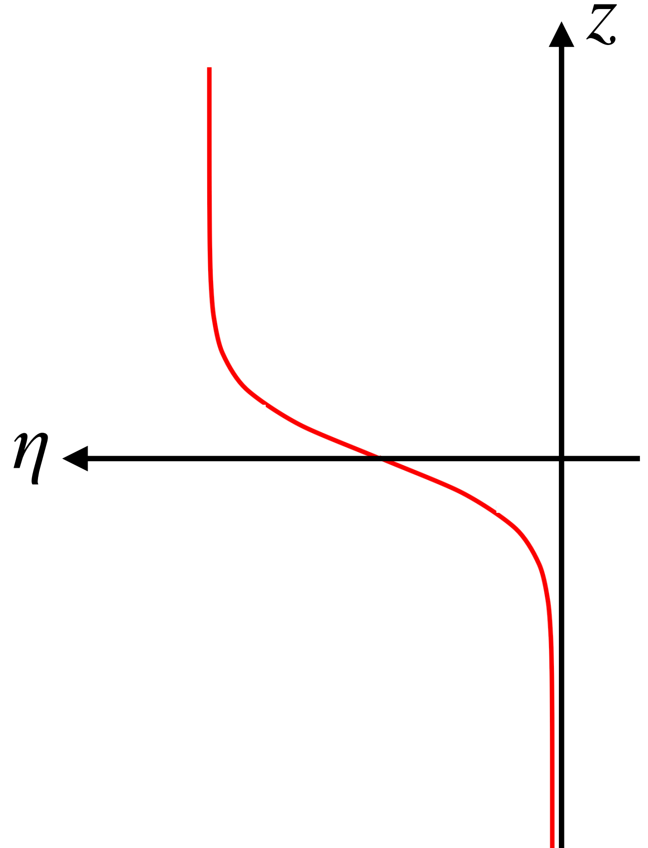
U(1) gauge field

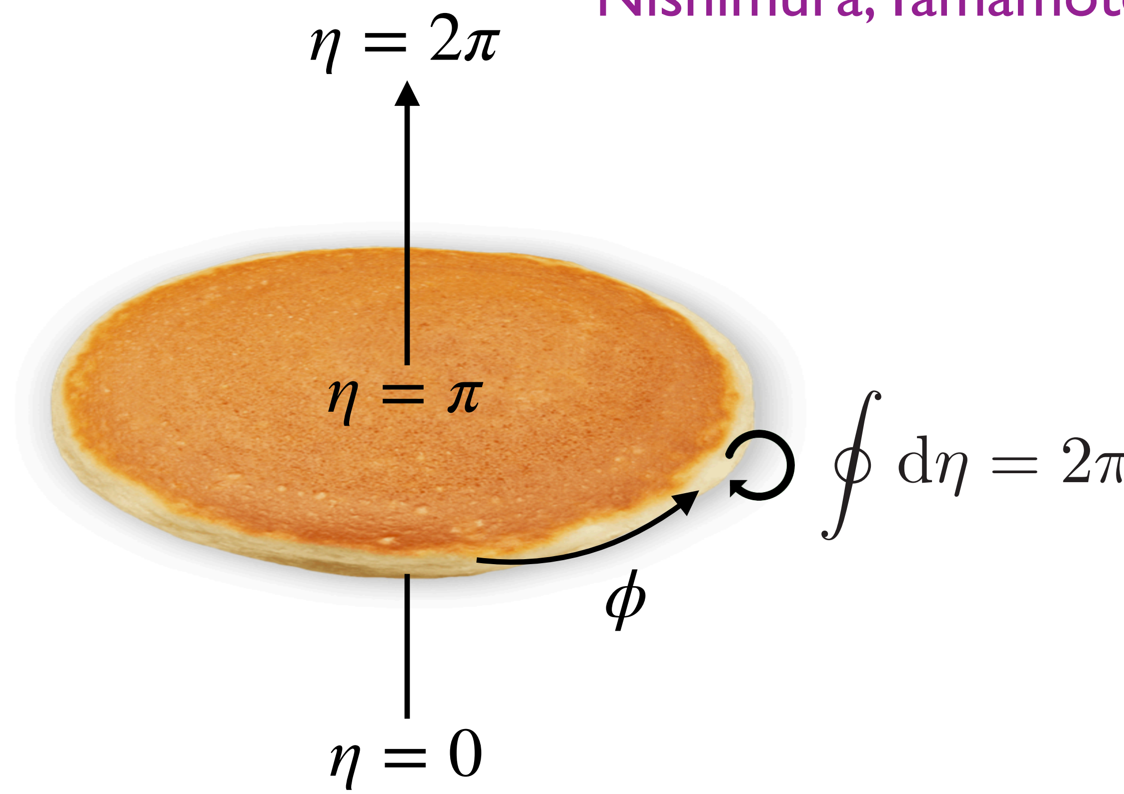
$$S_{\rm CS}[a] = \frac{2}{4\pi}\int \mathrm{d}^3x\ \epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho$$

Quantum Hall droplet

- Edge theory $S_{\mathrm{edge}}[\phi]$: 1+1D CFT for massless scalar ϕ
    - Energy ~ $1/R$, spin: 2/2 = 1
- Total energy of the system: $E(R) = \pi R^2 T_{\mathrm{DW}} + 2\pi R T_{\mathrm{string}} + \dfrac{C}{2\pi R}$
- Droplet stabilized by the edge mode ~ flavor-singlet vector meson

Nishimura, Yamamoto, Yokokura PRD (2025)

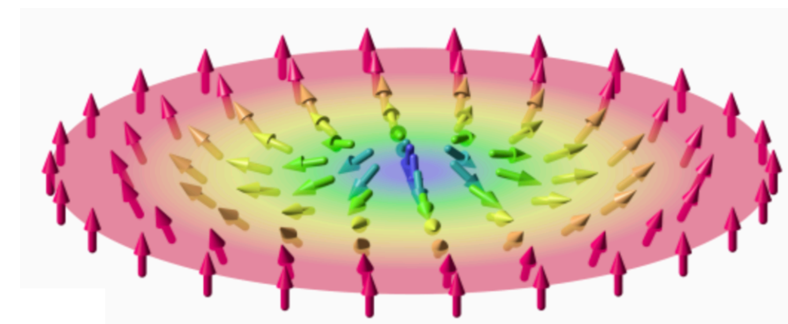




Historical remarks

- Skyrmion: Multi-flavor baryon in the QCD vacuum
  Skyrme (1961)

  nuclear physics → condensed matter physics



- Quantum Hall droplet: spin N/2 baryon in the large-N vacuum
  Komargodski, arXiv:1812.09253 (2018)

  condensed matter physics → particle physics

Our finding: quantum Hall droplet as a vector meson in 2SC

Conclusion and Outlook

- Quark-hadron continuity is possible thanks to QCD anomaly
- Large-Nc analysis does not generally work in high-density QCD
- Chern-Simons theory naturally emerges in 2SC phase
  - Quantum Hall droplet as a vector meson
- Quark-hadron continuity in 2-flavor QCD and real QCD?

Backup slides

Confined SU(2) sector

Rischke-Son-Stephanov, PRL (2001)

- Action: $S = \frac{1}{g^2}\int \mathrm{d}^4x \left(\frac{\epsilon}{2}\boldsymbol{E}^a \cdot \boldsymbol{E}^a - \frac{1}{2\lambda}\boldsymbol{B}^a \cdot \boldsymbol{B}^a\right)$
- Weak-coupling analysis: $\epsilon = 1 + \frac{g^2\mu^2}{18\pi^2\Delta^2} \gg 1\,, \quad \lambda = 1$
- Manifestly Lorentz-covariant form: $S = -\frac{1}{2g'^2}\int \mathrm{d}^4x' \mathrm{Tr}(F'_{\mu\nu}F'^{\mu\nu})$

  $(x^0)' := \epsilon^{-1/2}x^0$, $(A_0^a)' := \epsilon^{1/2}A_0^a$, and $g' := \epsilon^{-1/4}g$
- Effective coupling constant: $\alpha'_{\mathrm{s}} := \frac{g'^2}{4\pi} \simeq \frac{3}{2\sqrt{2}}\frac{g\Delta}{\mu}$
- Emergent confinement scale: $\Lambda'_{\mathrm{QCD}} \sim \Delta \exp\left(-\frac{3\pi}{11\alpha'_{\mathrm{s}}}\right) \simeq \Delta \exp\left(-\frac{2\sqrt{2}\pi}{11}\frac{\mu}{g\Delta}\right)$

Effective theory for η

Son-Stephanov-Zhitnitsky, PRL (2001)

$$\mathcal{L} = f^2[(\partial_t \eta)^2 - v^2(\boldsymbol{\nabla}\eta)^2] + A\cos\eta$$

- Weak-coupling analysis: $f^2 = \dfrac{\mu^2}{8\pi^2}, \quad v^2 = \dfrac{1}{3}$
- Debye screening of instantons $\rho \sim \mu^{-1}$

→ Instanton potential $V_{\rm inst}(\eta) = -a\mu^2\Delta^2\cos\eta,$

$$a \simeq 5\times 10^4 \left(\ln\frac{\mu}{\Lambda_{\rm QCD}}\right)^7 \left(\frac{\Lambda_{\rm QCD}}{\mu}\right)^{29/3}$$

- η mass: $m_\eta = 2\pi\sqrt{a}\Delta$

η domain-wall and string

Son-Stephanov-Zhitnitsky, PRL (2001)

- Bogomolny bound:

$$H = \int \mathrm{d}z \left[f^2 v^2 (\partial_z \eta)^2 + 2A \sin^2 \frac{\eta}{2} \right] \geq 2 \int \mathrm{d}z \sqrt{f^2 v^2 (\partial_z \eta)^2} \sqrt{2A \sin^2 \frac{\eta}{2}}$$

$$= 2\sqrt{2A} f v \int_0^{2\pi} \mathrm{d}\eta \left| \sin \frac{\eta}{2} \right| = 4\sqrt{2A} f v$$

- Domain-wall solution: $\eta(z) = 4 \tan^{-1} \mathrm{e}^{m_\eta z / v}$
- Domain-wall tension: $T_{\mathrm{DW}} = 8\sqrt{2a} f v \mu \Delta \simeq \frac{4}{\pi} \sqrt{\frac{a}{3}} \mu^2 \Delta$
- String tension at the DW boundary:

$$T_{\mathrm{string}} = 2\pi f^2 v^2 \ln \frac{R}{R_{\mathrm{core}}} \simeq \frac{\mu^2}{12\pi} \ln \left(\frac{1}{2\pi\sqrt{3a}} \right)$$

Physics at the edge

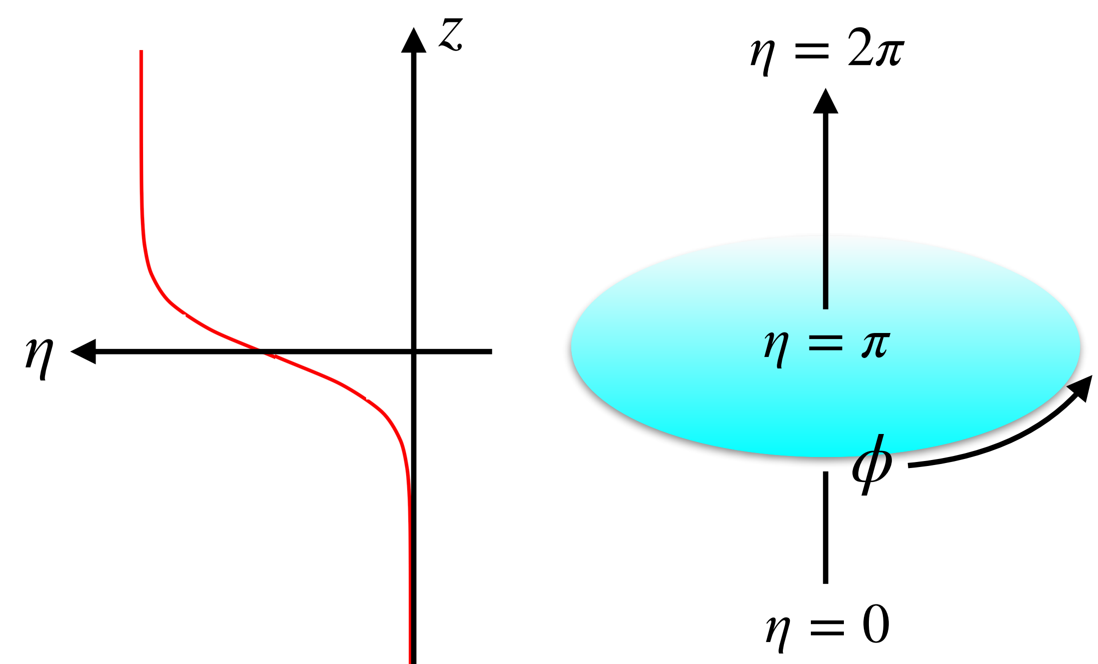
see, e.g., X. G. Wen, "Quantum Field Theory of Many-Body Systems"

- Gauge fixing $a_t + \omega a_\theta = 0$ in (r, θ) coordinates
- $\tilde{\theta} := \theta - \omega t,\ \tilde{t} := t,\ \tilde{r} := r \to \tilde{a}_{\tilde{t}} := a_t + \omega a_\theta = 0,\ \tilde{a}_{\tilde{\theta}} = a_\theta,\ \tilde{a}_{\tilde{r}} = a_r$

$$S_{\mathrm{CS}}[a] = \frac{2}{4\pi}\int \mathrm{d}^3x \epsilon^{\tilde{\mu}\tilde{\nu}\tilde{\rho}} \tilde{a}_{\tilde{\mu}} \partial_{\tilde{\nu}} \tilde{a}_{\tilde{\rho}}$$

- EOM for $\tilde{a}_{\tilde{t}}$: $\tilde{f}_{\tilde{\theta}\tilde{r}} = 0 \to \tilde{a}_{\tilde{i}} = \partial_{\tilde{i}}\phi\ (\tilde{i} = \tilde{\theta}, \tilde{r})$ as a constraint
- Edge theory = 1+1D CFT (chiral Tomonaga-Luttinger liquid):

$$\begin{aligned} S_{\mathrm{edge}} &= \frac{2}{4\pi}\int \mathrm{d}\tilde{t}\mathrm{d}\tilde{\theta} \partial_{\tilde{t}}\phi \partial_{\tilde{\theta}}\phi \\ &= \frac{2}{4\pi}\int \mathrm{d}t\mathrm{d}\theta (\partial_t + \omega\partial_\theta)\phi\partial_\theta\phi \end{aligned}$$



2D CFT analysis

- For generic O under $z \to w(z)$ in complex plane $z = x_1 + \mathrm{i}x_2$,

$$O'(w, \bar{w}) = \left(\frac{\mathrm{d}z}{\mathrm{d}w}\right)^{h} \left(\frac{\mathrm{d}\bar{z}}{\mathrm{d}\bar{w}}\right)^{\bar{h}} O(z, \bar{z})$$

$$\longrightarrow \; O'(\lambda z, \lambda \bar{z}) = \lambda^{-\underbrace{(h+\bar{h})}_{\text{scaling}}} O(z, \bar{z}), \quad O'(\mathrm{e}^{\mathrm{i}\theta} z, \mathrm{e}^{-\mathrm{i}\theta} \bar{z}) = \mathrm{e}^{-\mathrm{i}\underbrace{(h-\bar{h})}_{\text{spin}}\theta} O(z, \bar{z})$$

- Two-point function for the vertex operator $V_N = \mathrm{e}^{\mathrm{i}N\phi}$ ($N = 2$):

$$\langle V_N(z) V_M(0) \rangle \sim \mathrm{e}^{-NM\langle \phi(z)\phi(0)\rangle} \sim \frac{\delta_{N+M,0}}{z^N} \qquad \because \langle \phi(z)\phi(0)\rangle = -\tfrac{\#}{N} \ln|z|$$

- Scaling dimension & spin $N/2 = 1$, energy ~ C/R

Other theories

- 2-flavor QCD at finite isospin density:
  - $SU(3)_{-2}$ CS theory $\rightleftarrows$ $U(2)_3$ CS theory on the wall
  - Droplet = favored spin 3/2 baryon
- 2-color QCD at finite baryon density:
  - $SU(2)_{-2}$ CS theory $\rightleftarrows$ $U(2)_2$ CS theory on the wall
  - Droplet = favored spin 1 baryon