# Two-color QCD as a laboratory of cold and dense matter: Numerical experiments and effective models

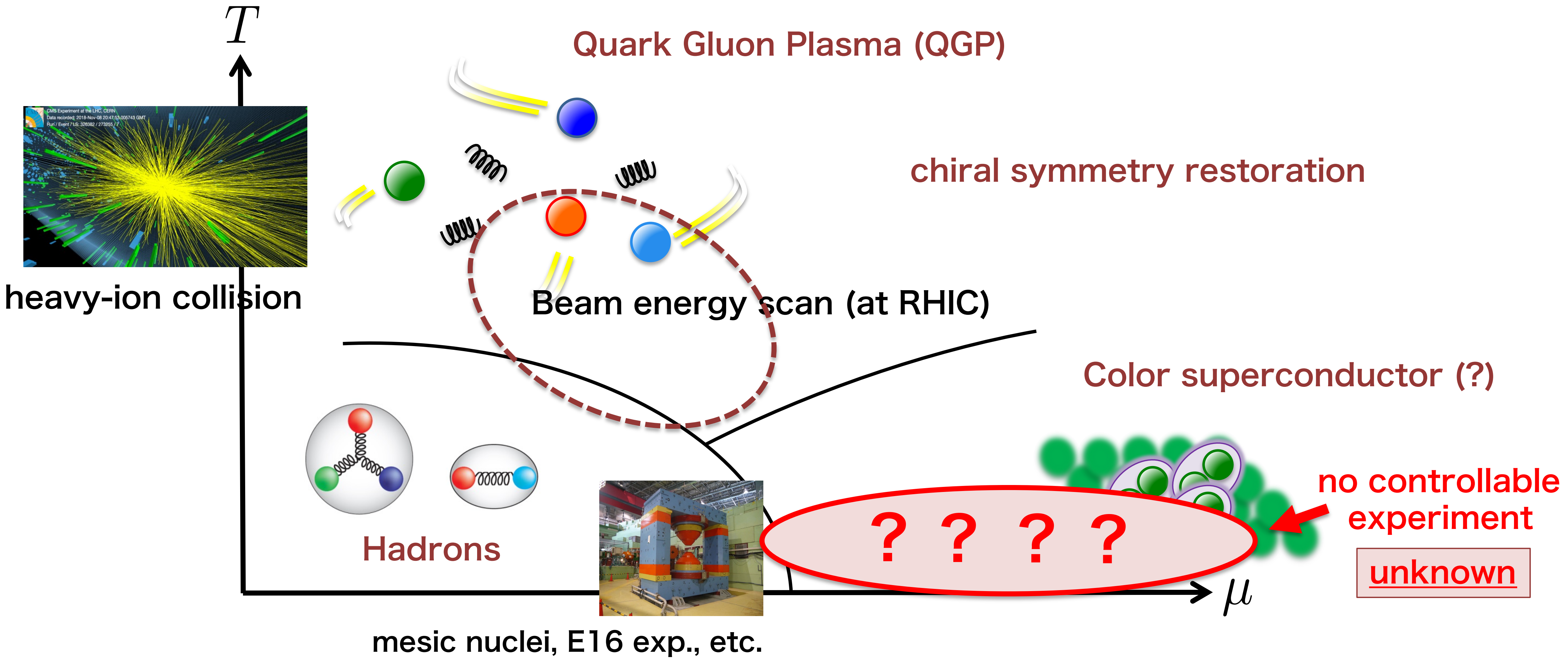
Daiki Suenaga (KMI/Nagoya U., Japan)

素粒子宇宙起源研究所
Kobayashi-Maskawa Institute
for the Origin of Particles and the Universe

# Introduction



- QCD phase diagram



# Introduction



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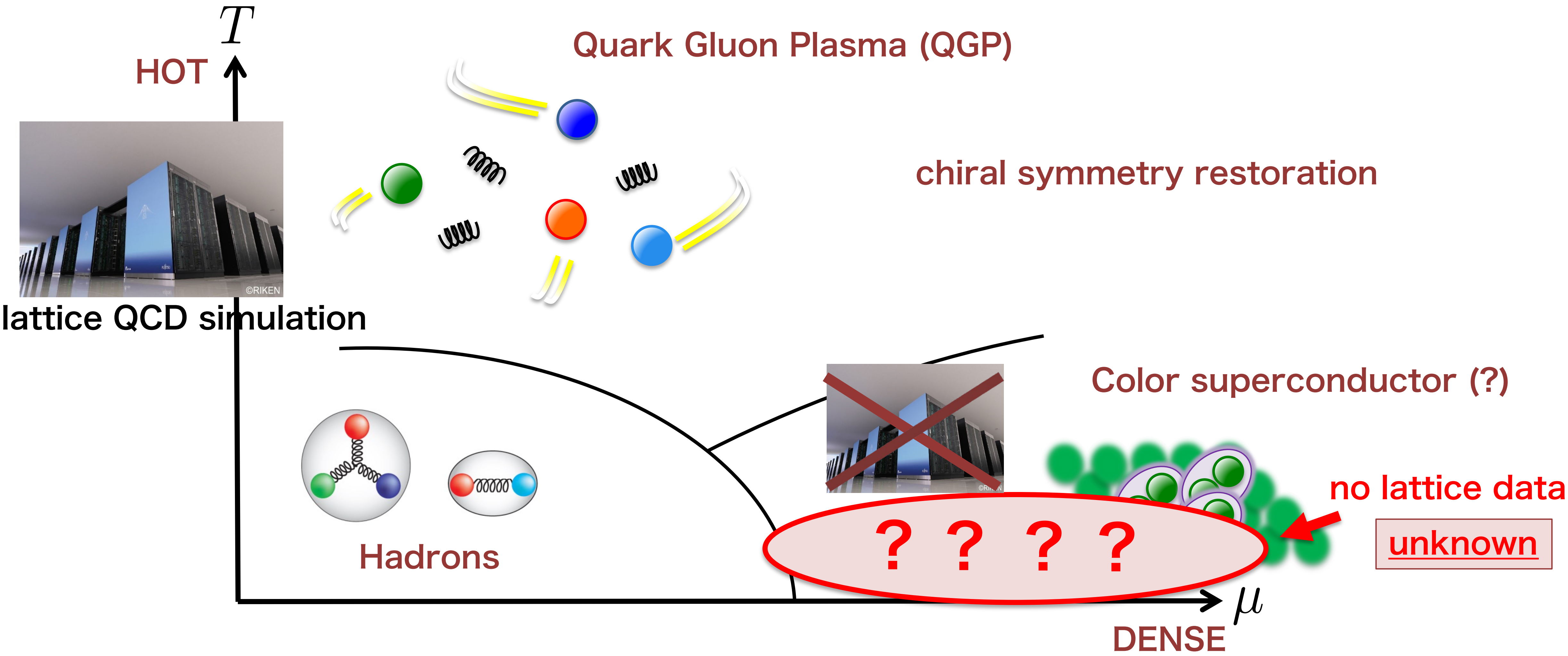
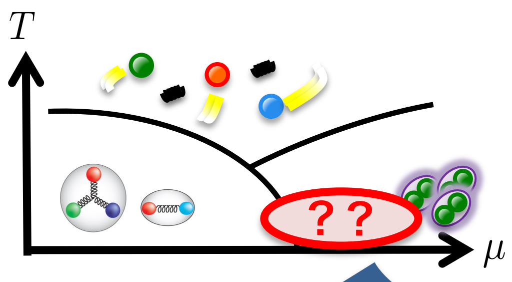


# Introduction



- **Lattice study in dense QCD?**




<u>three-color QCD (our world)</u>

- Lattice sim. at density is not easy

$$\mathrm{Det}(\gamma \cdot D - \mu\gamma_4 + m) \in \mathbb{C}$$ ✕

sign problem

with $N_f = \text{(even)}$

<u>two-color QCD (imaginary world)</u>

- Lattice sim. at density is possible!

$$\mathrm{Det}(\gamma \cdot D - \mu\gamma_4 + m)^{N_f} \geq 0$$ ◎

sign problem disappears
thanks to $SU(2)_c$ pseudoreality

# Introduction



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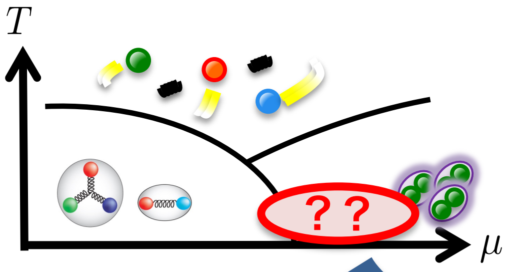


with $N_f = (\text{even})$

<u>three-color QCD (our world)</u>

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$$\mathrm{Det}(\gamma \cdot D - \mu\gamma_4 + m) \in \mathbb{C} \times$$

sign problem

<u>two-color QCD (imaginary world)</u>

- Lattice sim. at density is possible!

$$\mathrm{Det}(\gamma \cdot D - \mu\gamma_4 + m)^{N_f} \geq 0 \circledcirc$$

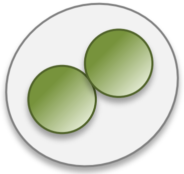
sign problem disappears thanks to $SU(2)_c$ pseudoreality

- Baryon is made of three quarks

nucleon

some differences

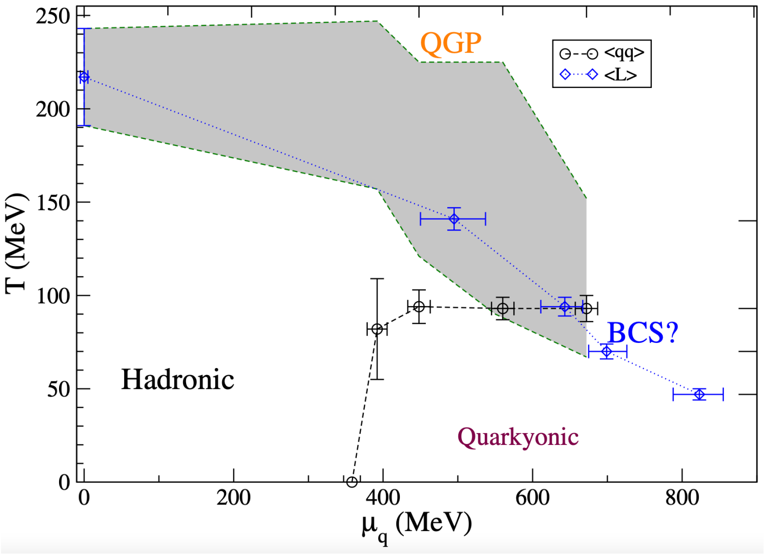
- Baryon is made of two quarks



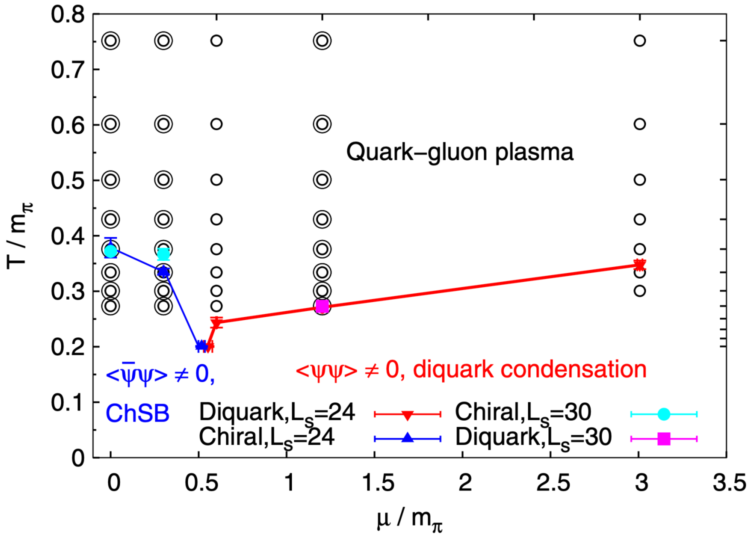
diquark baryon

- Two-color QCD is a useful laboratory to explore cold-dense medium with lattice simulations

- **Phase diagram in two-color QCD (QC$_2$D)**
  - Examples of simulation results of phase diagram in QC$_2$D

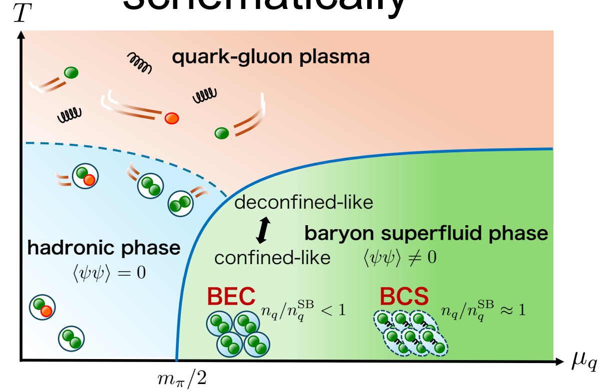



Boz-Cotter-Fister-Mehta-Skullerud (2013)




Buividovich-Smith-Smekal (2020)

schematically



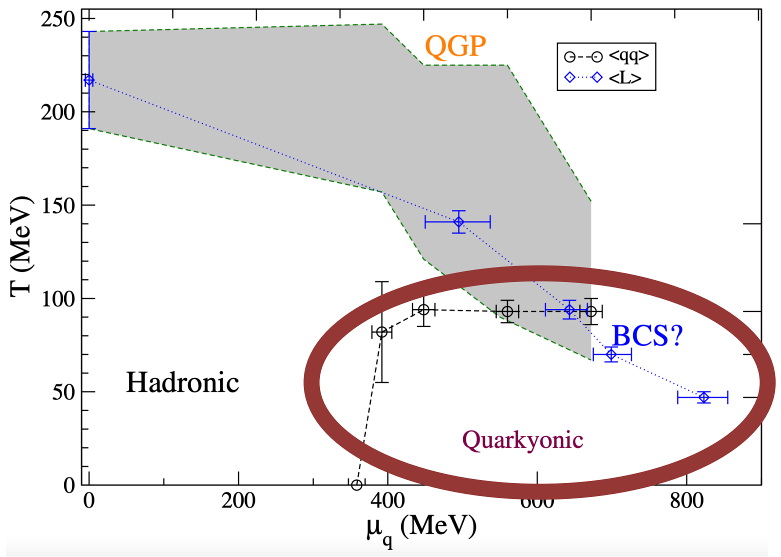

…

- Currently at least four lattice simulation groups are active
  - Ireland/UK group (Hands, Skullerud, …)
  - UK group (Buividovich, ...)
  - Russian group (Bornyakov, ...)
  - Japanese group (Iida-san, Itou-san, …)

# Introduction



- **Phase diagram in two-color QCD ($QC_2D$)**
  - Examples of simulation results of phase diagram in $QC_2D$

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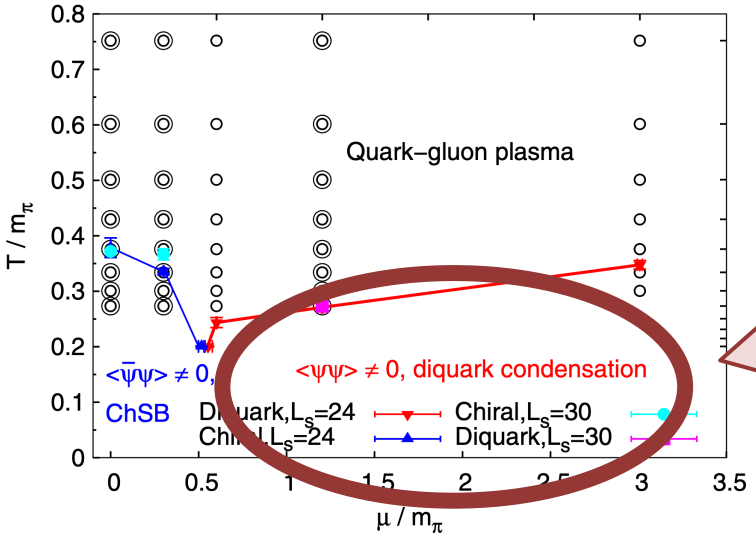


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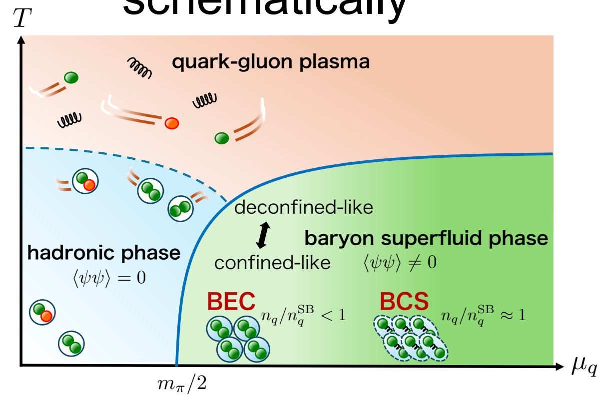


…

diquark condensed phase (baryon superfluid phase)

$\langle \psi\psi \rangle \neq 0$

- Currently at least four lattice simulation groups are active
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# Introduction

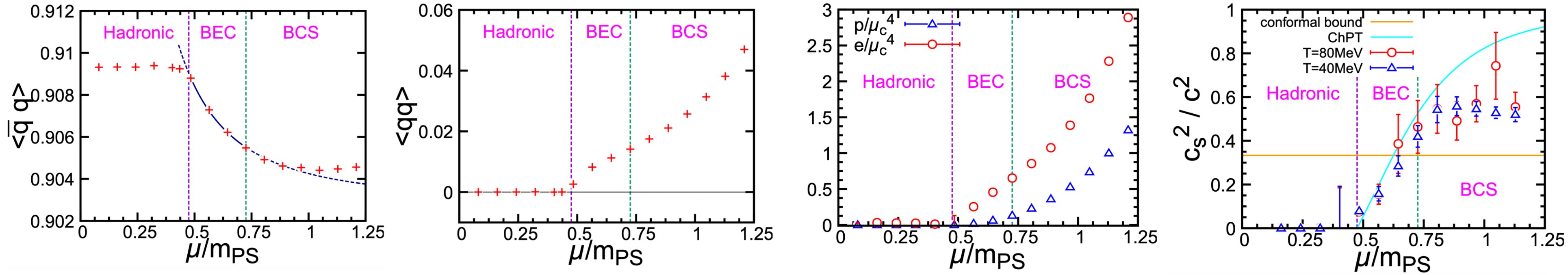


## • Lattice results

- In addition to phase diagram, hadron mass spectrum, gluon propagator, transport coefficient, EoS, sound velocity, $\langle\bar{\psi}\psi\rangle, \langle\psi\psi\rangle, \langle L\rangle$, etc. have been simulated

Japanese group

Talk by E. Itou on May 28




• • •

### My approach

– (i) Regard $QC_2D$ lattice simulations as useful "numerical experiments" of cold and dense QCD, and (ii) give interpretation from symmetry viewpoints based on effective models

**Suenaga** (2025) (Review paper)

<u>My publications on $QC_2D$</u>

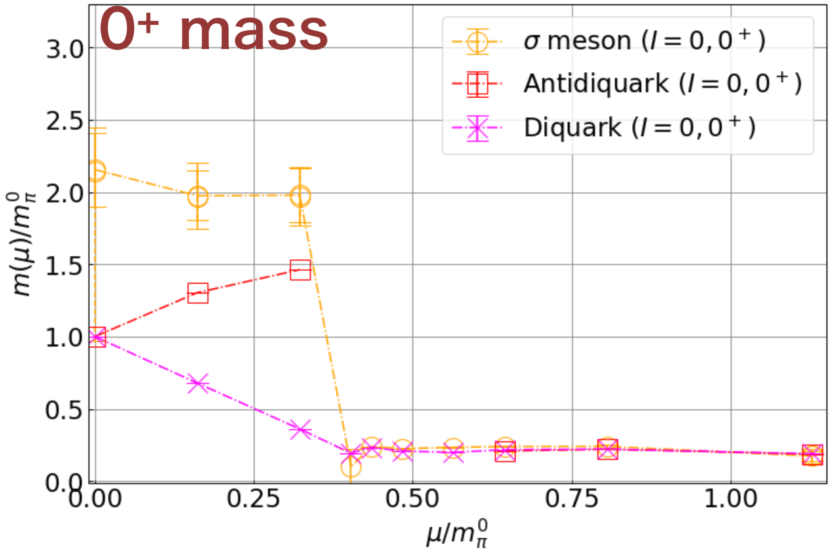
Gluon propagator: **Suenaga**-Kojo(2019), Kojo-**Suenaga**(2021), CSE effect: **Suenaga**-Kojo(2021), Sound velocity: Kojo-**Suenaga**(2022), Kawaguchi-**Suenaga**(2024), Topological susceptibility: Kawaguchi-**Suenaga**(2023), Hadron mass: **Suenaga**-Murakami-Itou-Iida (2023, 2024), FRG analysis: Fejos-**Suenaga**(2025, 2025), LSM with Nf=2+2: Sakai-**Suenaga** (2025), in preparations

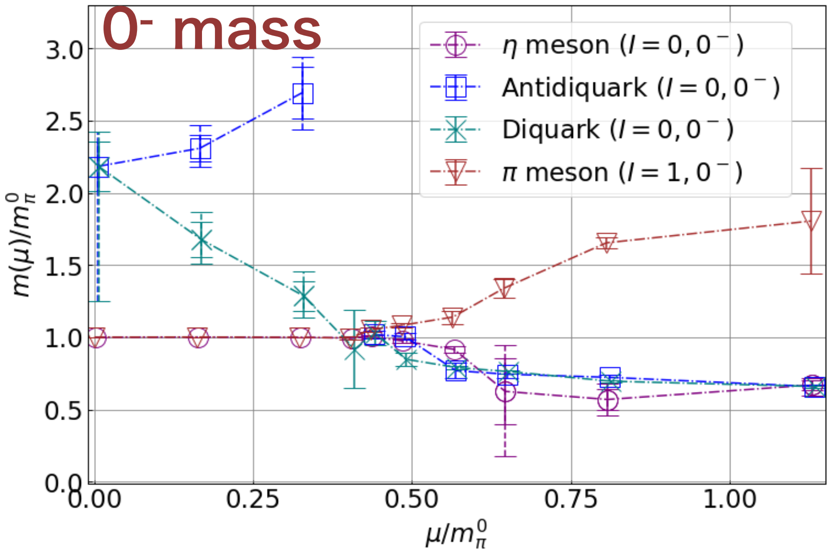
# Introduction



## Lattice results on hadron masses

Murakami et al (2023)




← pion $(I = 1, 0^-)$

$I = 0, 0^-$ hadron

lighter than pion!

# Introduction

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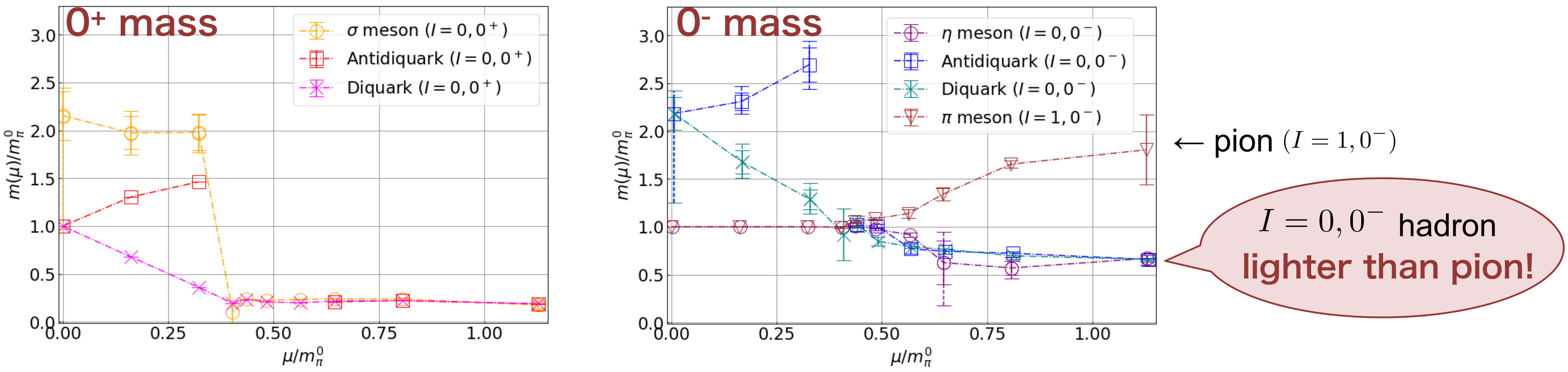



- Pion is no longer light in superfluid phase (for $m_\pi^0/2 \lesssim \mu$) !

- ChPT is powerful but cannot treat $I=0,0^-$ hadron ⇒ ChPT is not the useful "low-energy model" in superfluid phase of $QC_2D$!

Kogut et al (2000)

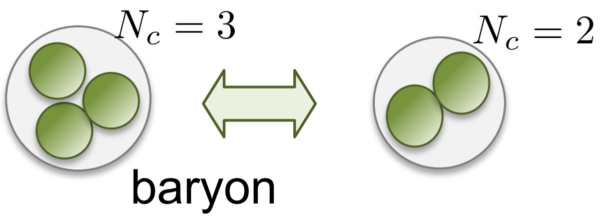
⇒ I constructed another model (linear sigma model) as a reasonable EFT in (dense) $QC_2D$

# Q: What can we learn from $QC_2D$?

- It is too naïve to bring all obtained information to Nc=3 world

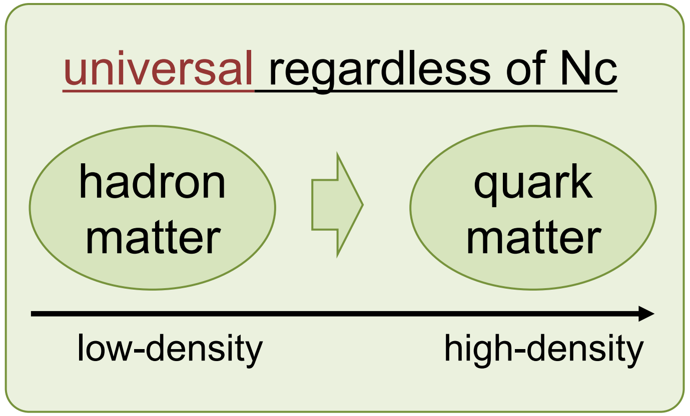
$\longrightarrow$ detailed dynamics is different

eg




- But we can get information on
  - to what extent hadron model description can apply
  - Which representation of hadrons is useful in medium
  - how to incorporate quark dof into hadron model → "unified model"
  - deep understanding of quark matter in high-dense regime

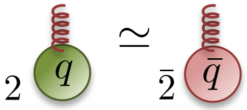
⋮

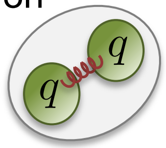


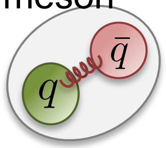
# Model



- **Linear sigma model (LSM) with SU(4)**

  - Pseudo reality of $SU(2)_c$ means "blindness" of gluons: 


⇒ baryon 

and meson 

can be treated in a unified way

Pauli (1957), Gursey (1958)

- $SU(2)_L \times SU(2)_R$ chiral symmetry is extended to the Pauli-Gursey $SU(4)$ symmetry

# Model



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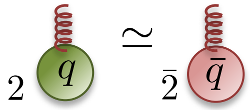
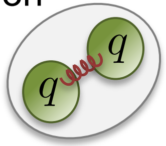
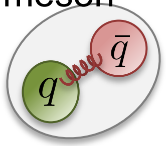



⇒ baryon




and meson




can be treated in a unified way

Pauli (1957), Gursey (1958)

- $SU(2)_L \times SU(2)_R$ chiral symmetry is extended to the Pauli-Gursey $SU(4)$ symmetry

⇓ LSM is constructed with SU(4)

chemical potential ↘ quark mass ↙ $U(1)_A$ anomaly

$$\mathcal{L}_{\rm LSM} = {\rm tr}[D_\mu \Sigma^\dagger D^\mu \Sigma] - m_0^2 {\rm tr}[\Sigma^\dagger\Sigma] - \lambda_1 \left({\rm tr}[\Sigma^\dagger\Sigma]\right)^2 - \lambda_2 {\rm tr}[(\Sigma^\dagger\Sigma)^2] + {\rm tr}[H^\dagger\Sigma + \Sigma^\dagger H] + c({\rm det}\Sigma + {\rm det}\Sigma^\dagger)$$

with
$$\Sigma = \frac{1}{2}\begin{pmatrix} 0 & -\frac{B'-iB}{2\sqrt{2}} & \frac{\sigma-i\eta+a_0^0-i\pi^0}{4} & \frac{a_0^+-i\pi^+}{2\sqrt{2}} \\ \frac{B'-iB}{2\sqrt{2}} & 0 & \frac{a_0^--i\pi^-}{2\sqrt{2}} & \frac{\sigma-i\eta-a_0^0+i\pi^0}{4} \\ -\frac{\sigma-i\eta+a_0^0-i\pi^0}{4} & -\frac{a_0^--i\pi^-}{2\sqrt{2}} & 0 & -\frac{\bar{B}'-i\bar{B}}{2\sqrt{2}} \\ -\frac{a_0^+-i\pi^+}{2\sqrt{2}} & -\frac{\sigma-i\eta-a_0^0+i\pi^0}{4} & \frac{\bar{B}'-i\bar{B}}{2\sqrt{2}} & 0 \end{pmatrix}$$

⇔

| Hadron | $J^P$ | Quark number | Isospin |
|---|---|---|---|
| $\sigma$ | $0^+$ | 0 | 0 |
| $a_0$ | $0^+$ | 0 | 1 |
| $\eta$ | $0^-$ | 0 | 0 |
| $\pi$ | $0^-$ | 0 | 1 |
| $B$ ($\bar{B}$) | $0^+$ | $+2(-2)$ | 0 |
| $B'$ ($\bar{B}'$) | $0^-$ | $+2(-2)$ | 0 |

← $I = 0, 0^-$ hadrons (mandatory from lattice)

# Mean fields



- **Mean field**

- The mean fields are $\sigma_0 \equiv \langle \sigma \rangle$ and $\Delta \equiv \left\langle \frac{B+\bar{B}}{\sqrt{2}} \right\rangle$

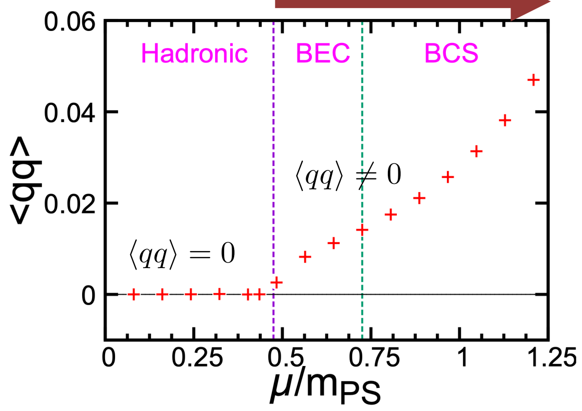
$\sigma_0 \sim \langle \bar{\psi}\psi \rangle$ : chiral condensate

$\Delta \sim -\frac{i}{2}\langle \psi^T C\gamma_5 \tau_c^2 \tau_f^2 \psi \rangle + \mathrm{h.c.}$ : diquark condensate

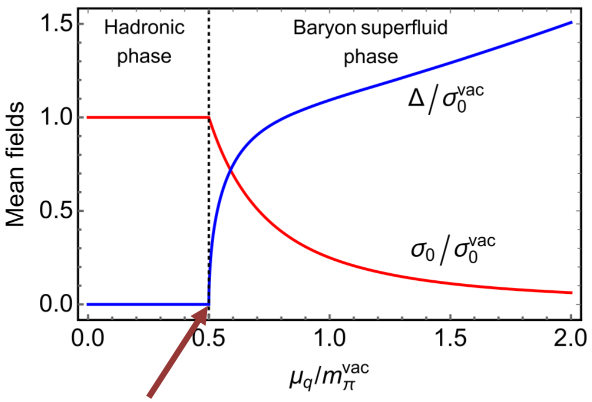
diquark condensate by lattice

Iida et al (2024)

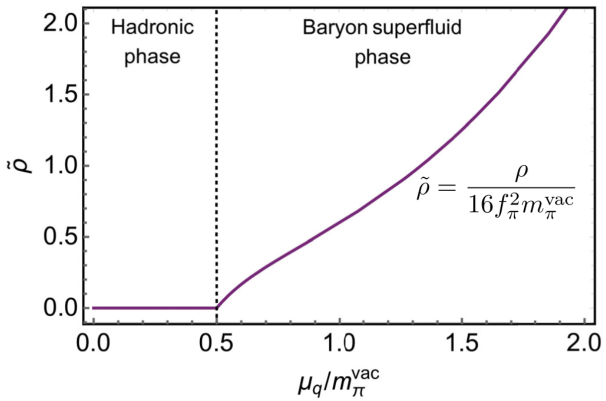
baryon superfluidity




$\sigma_0$ and $\Delta$ vs $\mu_q$




density $\rho$ vs $\mu_q$




2nd order phase transition at $\mu_q = m_\pi^{\rm vac}/2$

**Input here**

$\sigma_0^{\rm vac} = 250\,\mathrm{MeV}$ (put by hand)

$\lambda_1 = c = 0$ (large Nc)

$m_\pi^{\rm vac} = 738\,\mathrm{MeV}$

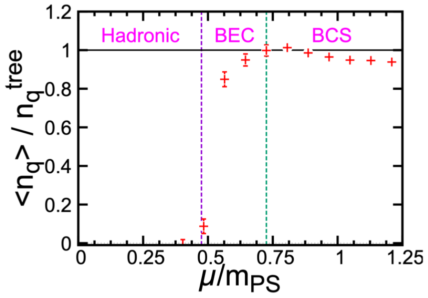
$m_{a_0}^{\rm vac}/m_\pi^{\rm vac} = 2.18$

} lattice

Murakami et al

BEC : $\langle n_q \rangle / n_q^{\rm tree} \lesssim 1$

BCS : $\langle n_q \rangle / n_q^{\rm tree} \approx 1$

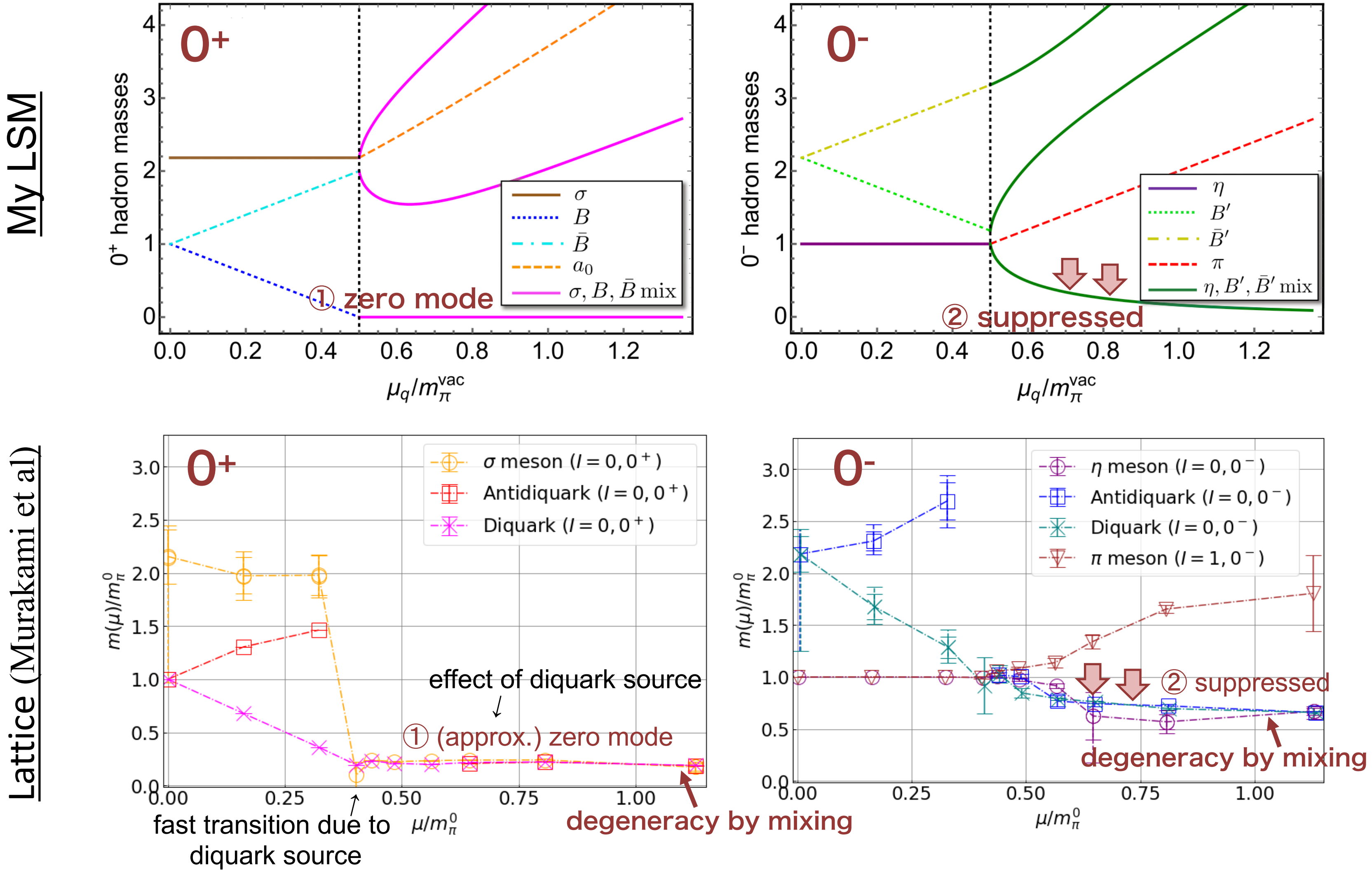


# Hadron masses



- Mass spectrum

U(1)$_B$ violation in the superfluid phase



# Hadron masses



- **Mass spectrum**

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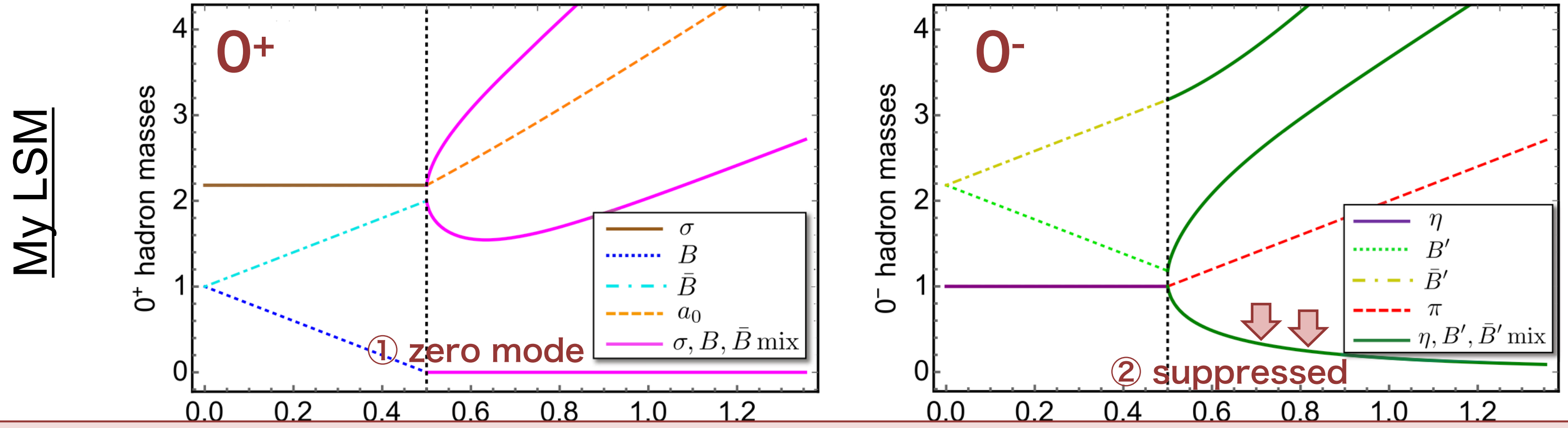
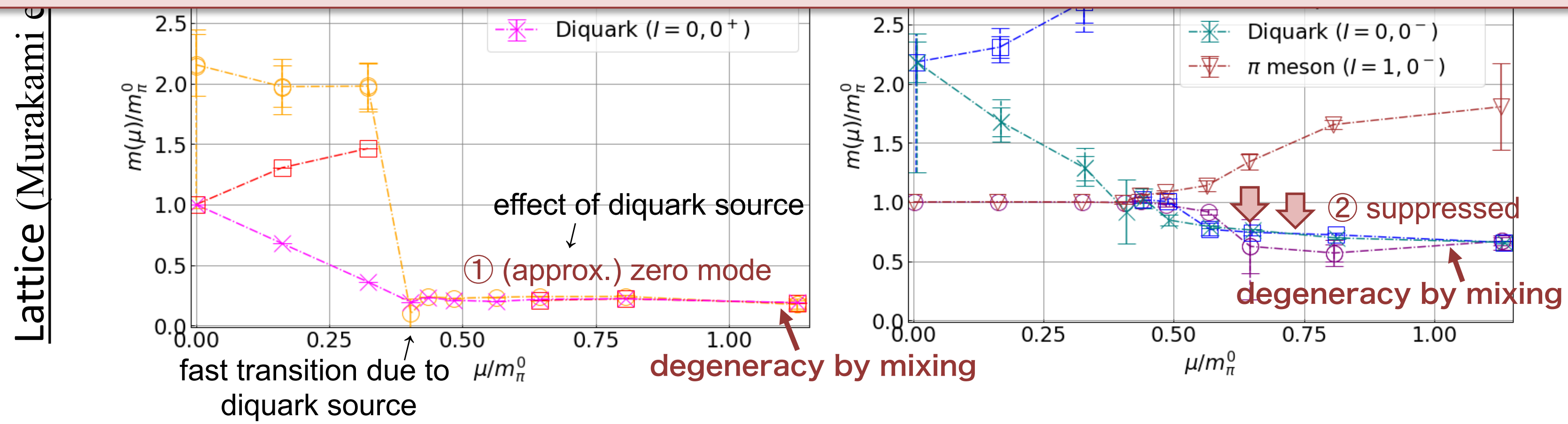



NOTE
ChPT does not treat $\eta, B', \bar{B}'$
Kogut et al (2000)
linear rep. is essential!

**The lowest excitations are qualitatively described by my LSM well!**



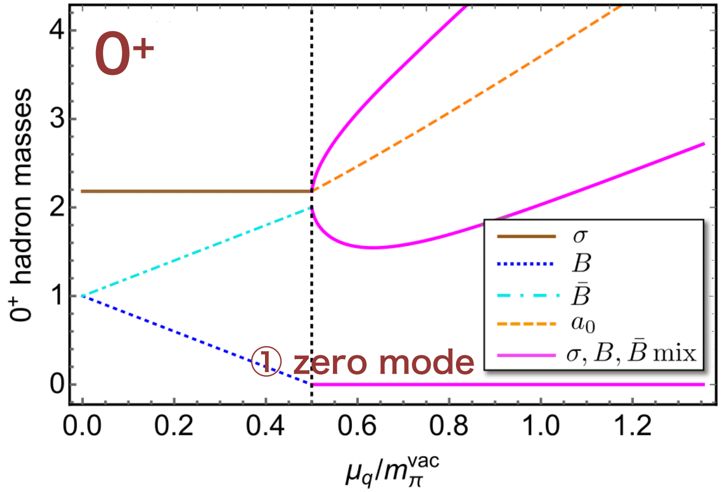
# Hadron masses

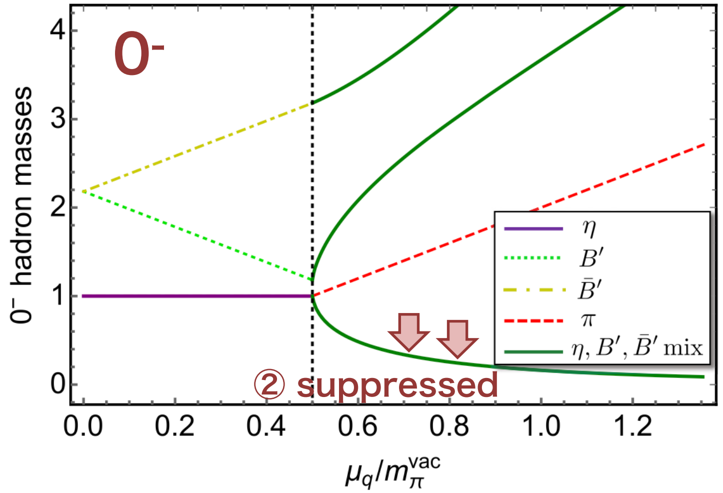


- Mass spectrum

U(1)$_B$ violation in the superfluid phase

My LSM

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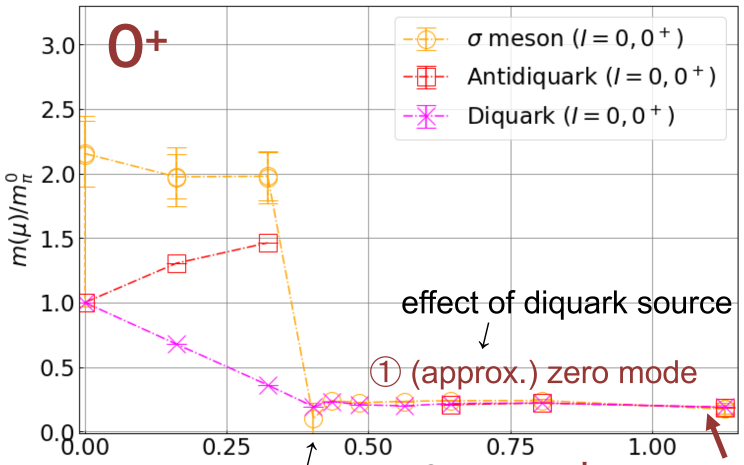
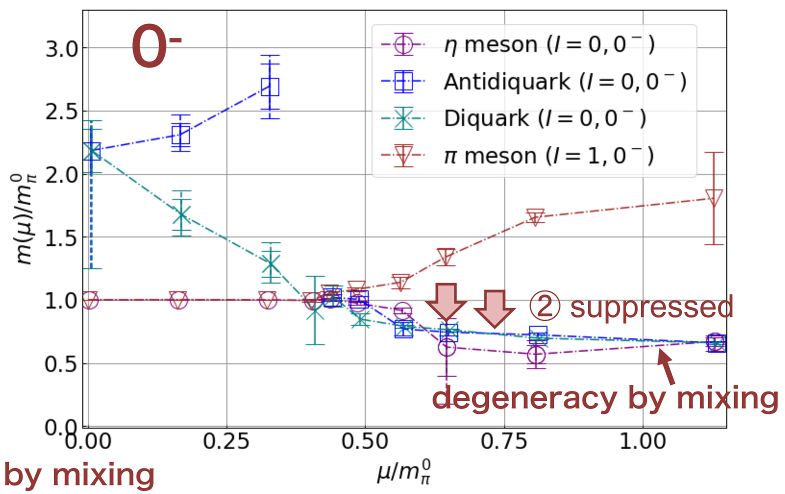
Lattice (Murakami et al)




fast transition due to diquark source

degeneracy by mixing




Updated lattice data is coming soon!

Talk by E. Itou on May 28

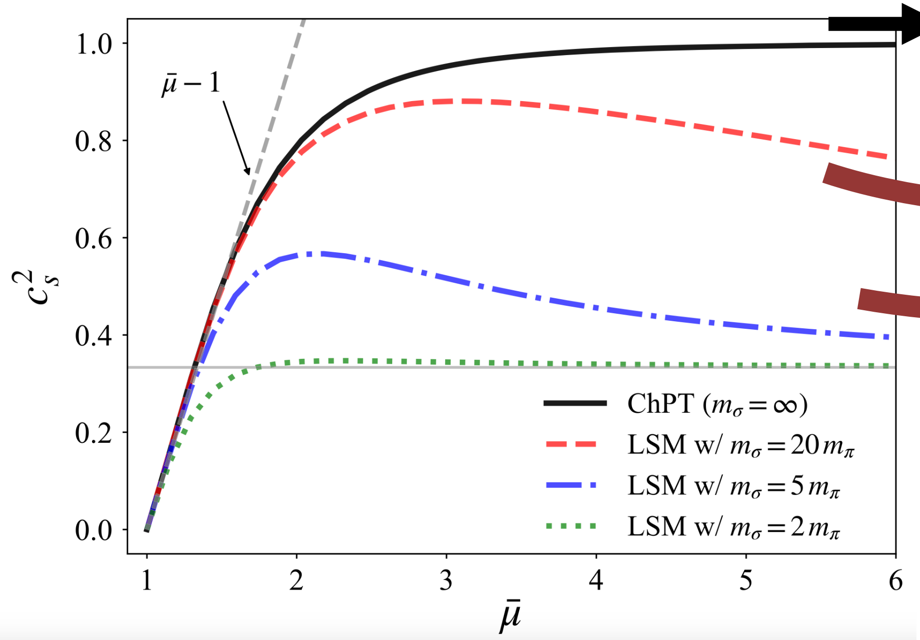
# Sound velocity



- **Sound velocity peak**

Kawaguchi-**Suenaga** (2024)

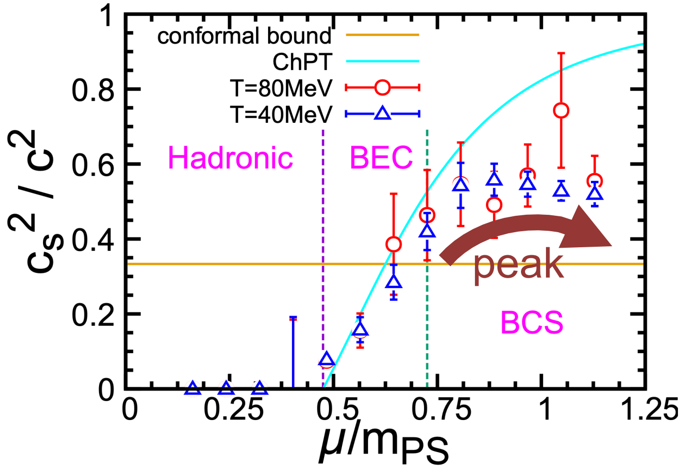
$\bar{\mu} = \mu/\mu_{\rm cr} = 2\mu/m_\pi$

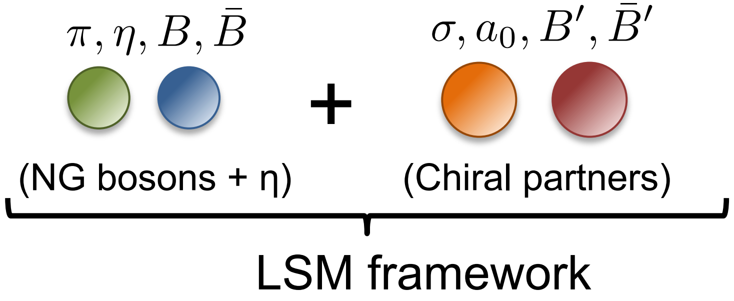



**ChPT result** $(c_s^2 \to 1)$ : no peak

peak when $m_\sigma < \infty$
$(c_s^2 \to 1/3)$

Lattice: Iida et al (2024)




- The peak structure is driven by contributions from parity partners

- Any connection with crossover to quark matter?

# How about finite T?



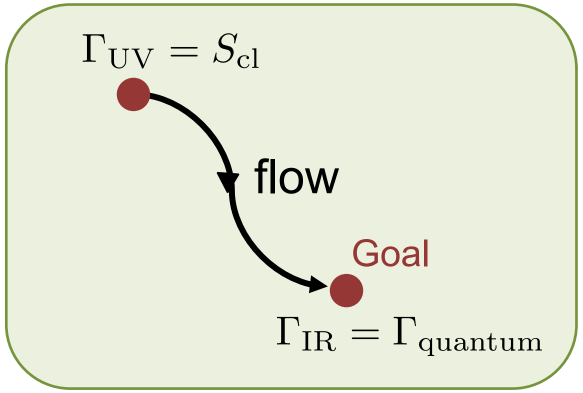
- **FRG analysis with the LSM in medium**

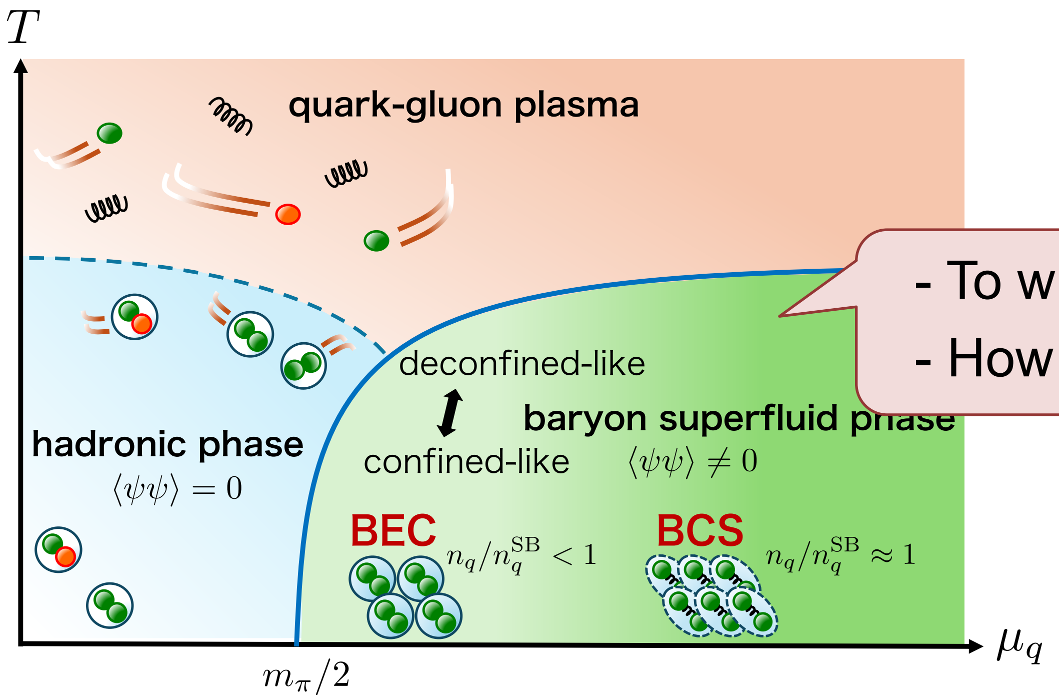
Fejos-**Suenaga** (2025, 2025, in preparation)

- The flow equation is $\partial_k \Gamma_k = \frac{1}{2}\tilde{\partial}_k \mathrm{Tr}\,\mathrm{Log}\,[\Gamma_k'' + R_k]$, with $\tilde{\partial}_k \equiv \partial_k R_k \frac{\partial}{\partial R_k}$

→ Regulator $R_k$ allows to include fluctuations of $p \gtrsim k$

3D Litim regulator $R_k(\boldsymbol{p}) = (k^2 - \boldsymbol{p}^2)\theta(k^2 - \boldsymbol{p}^2)$




Fluctuations are hadronic (no quarks)

- To what extent can we adopt the hadronic model in medium ?
- How the anomaly couplings are modified in medium ?

anomaly is enhancement in hadronic medium?

[eg, Indication in $QC_2D$: Suenaga et al (2023)
FRG in three-color QCD: Fejos-Hosaka(2015)]

# FRG analysis



- **Ansatz of effective action in medium**

$$\Gamma = \int d^4x \Big( \mathrm{Tr}[D_\mu \Sigma^\dagger D^\mu \Sigma] + \bar{c} m_q \mathrm{tr}[E^T \Sigma + \Sigma^\dagger E] - (V_{\rm AF} + V_{\rm A}) \Big)$$

kinetic term with $\mu_q$ — quark mass effect

Symmetry with $\mu_q$ [Not SU(4)]

$SU(2)_L \times SU(2)_R \times U(1)_B$

parity and time-reversal

with

non-anomalous

$$\begin{aligned} V_{\rm AF} &= m_M^2 \mathrm{tr}\left[\Sigma_M^\dagger \Sigma_M\right] + m_B^2 \mathrm{tr}\left[\Sigma_{B_R}^\dagger \Sigma_{B_R} + \Sigma_{B_L}^\dagger \Sigma_{B_L}\right] + \lambda_{M1}\left(\mathrm{tr}\left[\Sigma_M^\dagger \Sigma_M\right]\right)^2 \\ &+ \lambda_{M2} \mathrm{tr}\left[\left(\Sigma_M^\dagger \Sigma_M - \frac{1}{2}\mathrm{tr}[\Sigma_M^\dagger \Sigma_M]\right)^2\right] + \lambda_{B1}\left(\mathrm{tr}\left[\Sigma_{B_R}^\dagger \Sigma_{B_R} + \Sigma_{B_L}^\dagger \Sigma_{B_L}\right]\right)^2 \\ &+ \lambda_{B2}\left(\mathrm{tr}\left[\Sigma_{B_R}^\dagger \Sigma_{B_R} - \Sigma_{B_L}^\dagger \Sigma_{B_L}\right]\right)^2 + \gamma_1 \mathrm{tr}\left[\Sigma_M^\dagger \Sigma_M\right] \mathrm{tr}\left[\Sigma_{B_R}^\dagger \Sigma_{B_R} + \Sigma_{B_L}^\dagger \Sigma_{B_L}\right] \\ &+ \gamma_2 \left(\mathrm{tr}\left[\Sigma_{B_R}^\dagger \Sigma_{B_L}\right] \det\Sigma_M^\dagger + \mathrm{tr}\left[\Sigma_{B_L}^\dagger \Sigma_{B_R}\right] \det\Sigma_M\right) , \end{aligned}$$

anomalous

$$\begin{aligned} V_{\rm A} &= a_M\left(\det\Sigma_M + \det\Sigma_M^\dagger\right) + a_B \mathrm{tr}\left[\Sigma_{B_R}^\dagger \Sigma_{B_L} + \Sigma_{B_L}^\dagger \Sigma_{B_R}\right] + c_{M1}\left((\det\Sigma_M)^2 + (\det\Sigma_M^\dagger)^2\right) \\ &+ c_{M2} \mathrm{tr}\left[\Sigma_M^\dagger \Sigma_M\right]\left(\det\Sigma_M + \det\Sigma_M^\dagger\right) + c_{B1}\left((\mathrm{tr}[\Sigma_{B_R}^\dagger \Sigma_{B_L}])^2 + (\mathrm{tr}[\Sigma_{B_L}^\dagger \Sigma_{B_R}])^2\right) , \\ &+ c_{B2} \mathrm{tr}\left[\Sigma_{B_R}^\dagger \Sigma_{B_R} + \Sigma_{B_L}^\dagger \Sigma_{B_L}\right] \mathrm{tr}\left[\Sigma_{B_R}^\dagger \Sigma_{B_L} + \Sigma_{B_L}^\dagger \Sigma_{B_R}\right] + d_1 \mathrm{tr}\left[\Sigma_M^\dagger \Sigma_M\right] \mathrm{tr}\left[\Sigma_{B_R}^\dagger \Sigma_{B_L} + \Sigma_{B_L}^\dagger \Sigma_{B_R}\right] \\ &+ d_2 \mathrm{tr}\left[\Sigma_{B_R}^\dagger \Sigma_{B_R} + \Sigma_{B_L}^\dagger \Sigma_{B_L}\right]\left(\det\Sigma_M + \det\Sigma_M^\dagger\right) + d_3\left(\mathrm{tr}\left[\Sigma_{B_R}^\dagger \Sigma_{B_L}\right]\det\Sigma_M + \mathrm{tr}\left[\Sigma_{B_L}^\dagger \Sigma_{B_R}\right]\det\Sigma_M^\dagger\right) \end{aligned}$$

$$\Sigma = \frac{1}{2}\begin{pmatrix} 0 & -\frac{B'-iB}{2\sqrt{2}} & \frac{\sigma-i\eta+a_0^0-i\pi^0}{4} & \frac{a_0^+-i\pi^+}{2\sqrt{2}} \\ \frac{B'-iB}{2\sqrt{2}} & 0 & \frac{a_0^- - i\pi^-}{2\sqrt{2}} & \frac{\sigma-i\eta-a_0^0+i\pi^0}{4} \\ -\frac{\sigma-i\eta+a_0^0-i\pi^0}{4} & -\frac{a_0^- - i\pi^-}{2\sqrt{2}} & 0 & -\frac{\bar{B}'-i\bar{B}}{2\sqrt{2}} \\ -\frac{a_0^+-i\pi^+}{2\sqrt{2}} & -\frac{\sigma-i\eta-a_0^0+i\pi^0}{4} & \frac{\bar{B}'-i\bar{B}}{2\sqrt{2}} & 0 \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} i\Sigma_{B_R} & \Sigma_M^\dagger \\ -\Sigma_M^* & -i\Sigma_{B_L}^\dagger \end{pmatrix} ,$$

-We assume this “classical” form at any flow momentum k

# FRG analysis



- **Phase diagram**

<u>input/initial condition</u>

Fejos-**Suenaga** (2025)

$M_\pi \approx 0.14\,\mathrm{GeV}$

$f_\pi = 0.093\,\mathrm{GeV}$

large-Nc sup. at UV

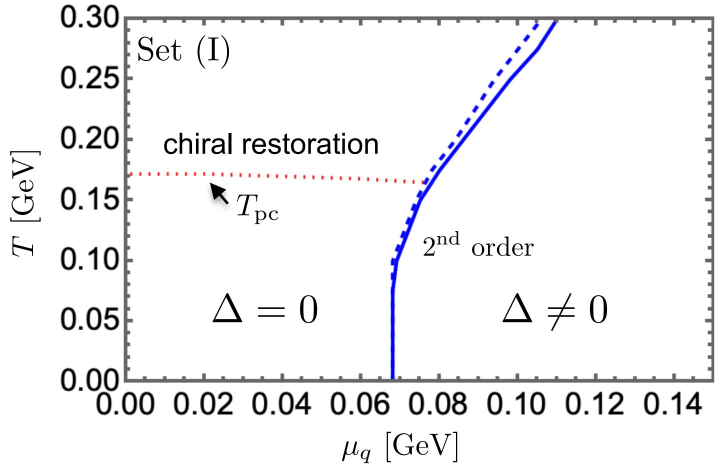
SU(4) relation at UV

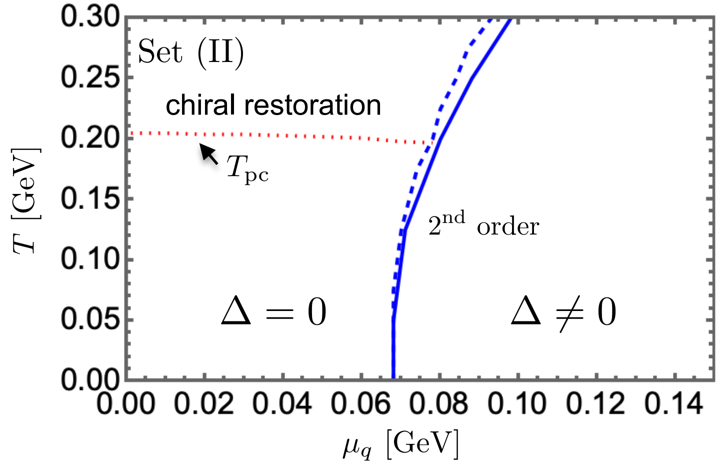
$+$

$M_\eta \approx 0.3\,\mathrm{GeV}$ → Set (I) (small anomaly)

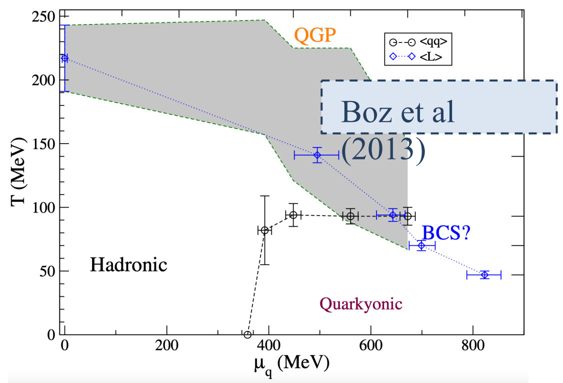
$M_\eta \approx 0.95\,\mathrm{GeV}$ → Set (II) (large anomaly)

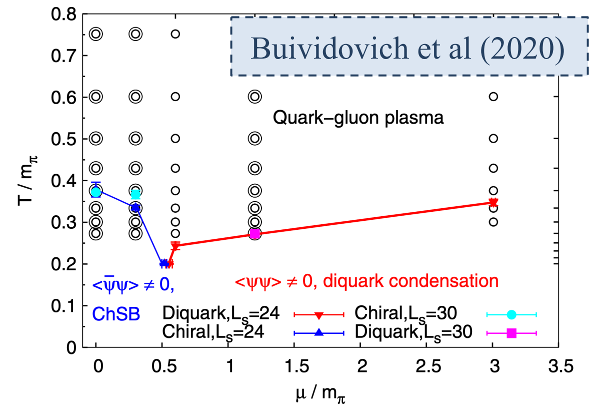
$(k_{\mathrm{UV}} = 1\,\mathrm{GeV}\,,\ k_{\mathrm{IR}} = 0.3\,\mathrm{GeV})$




<u>lattice data</u>



# FRG analysis

- **Phase diagram**

input/initial condition — Fejos-Suenaga (2025)

$M_\pi \approx 0.14\,\mathrm{GeV}$
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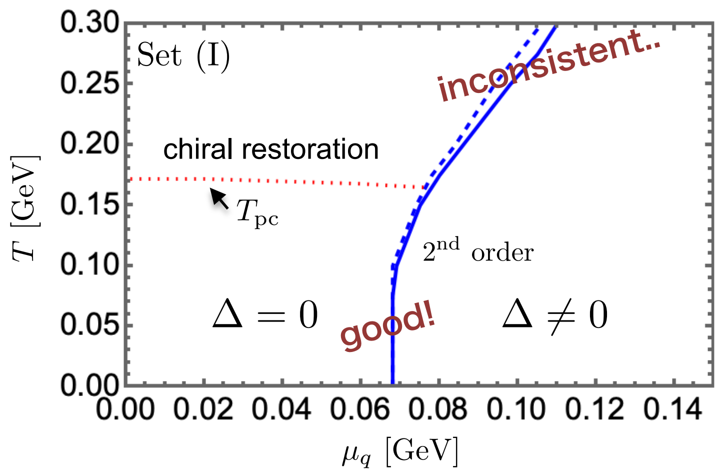
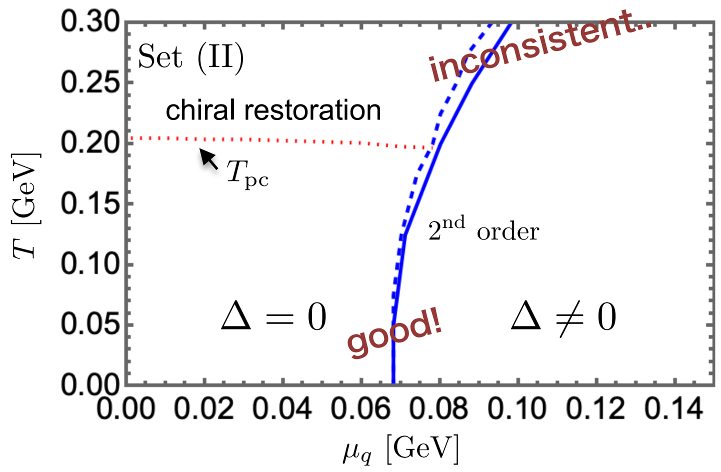
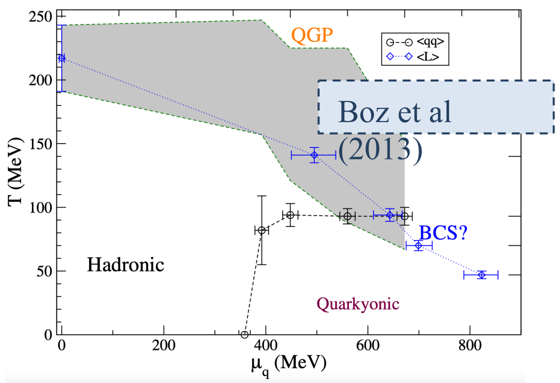
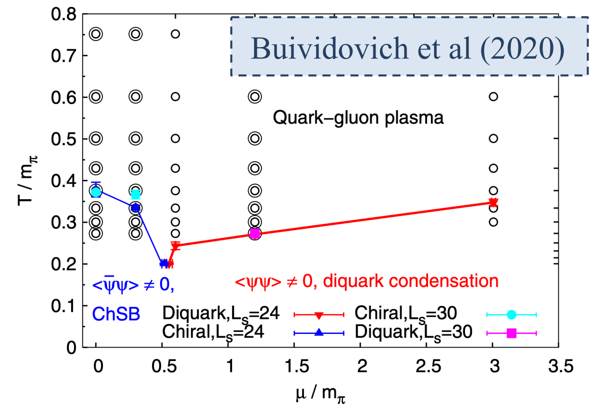



lattice data



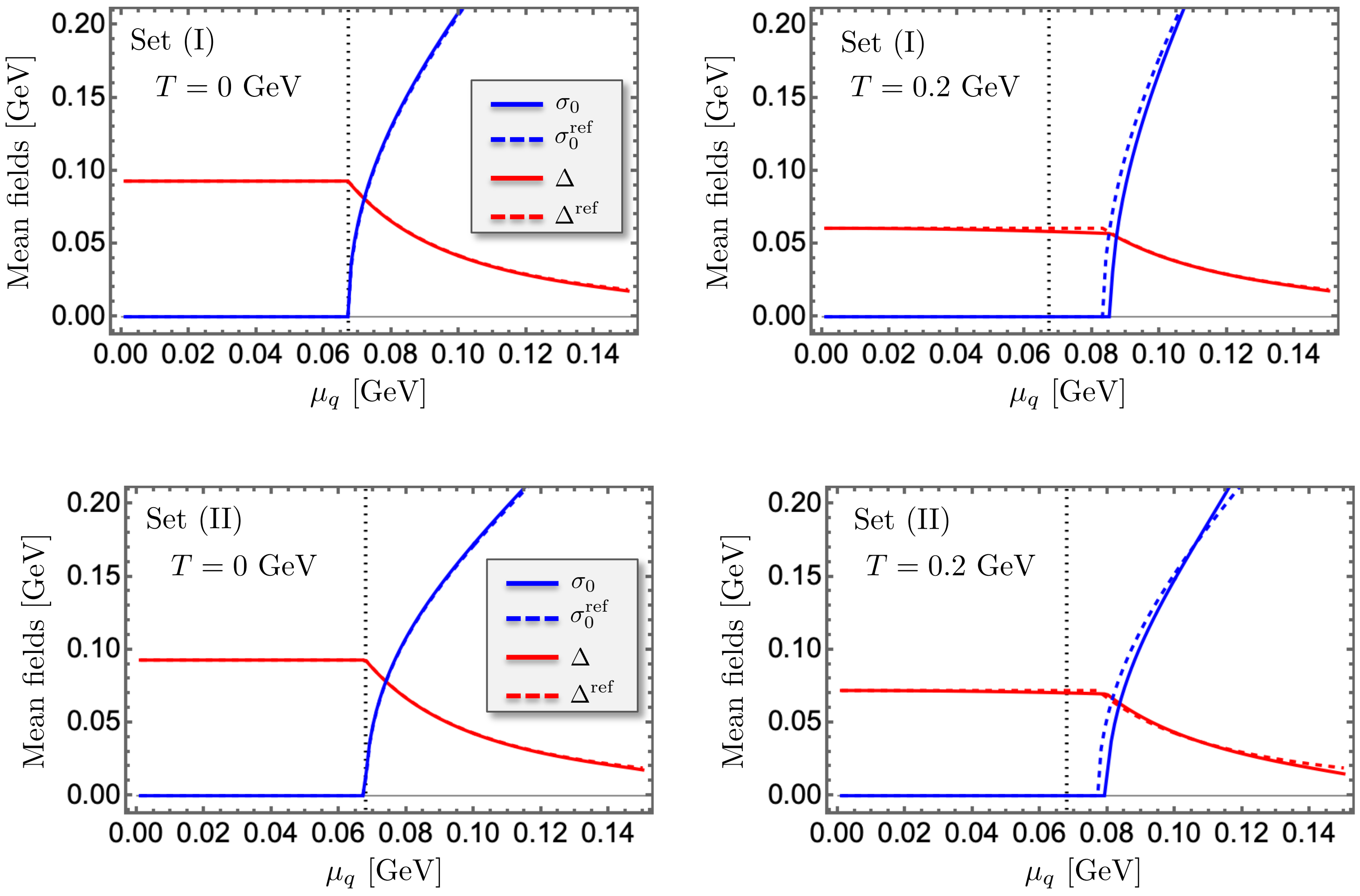




- For $T \lesssim T_{\rm pc}$, our treatment is reasonable
- Phase boundary at $T \gtrsim T_{\rm pc}$ is inconsistent with lattice data → explicit quark d.o.f. is necessary

# FRG analysis



- **Mean fields**




Set (I): small anomaly

Set (II): large anomaly

- Phase transition is always second order

# FRG analysis



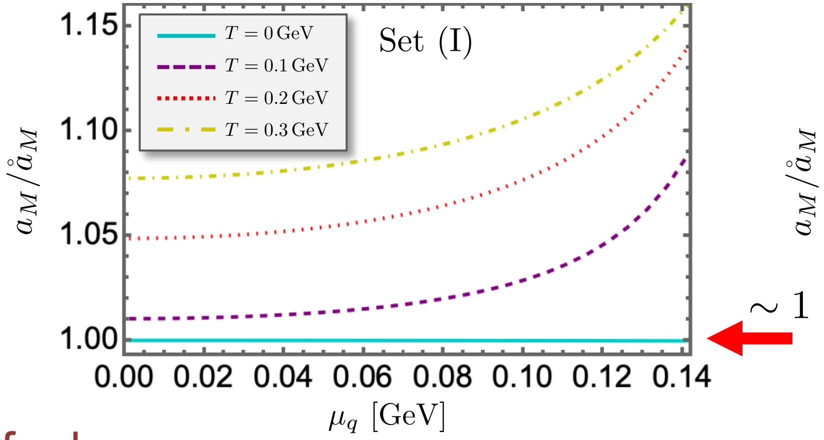
- **Anomaly enhancement**
  - (Normalized) anomaly coefficients for meson and baryon: $a_M$ and $a_B$

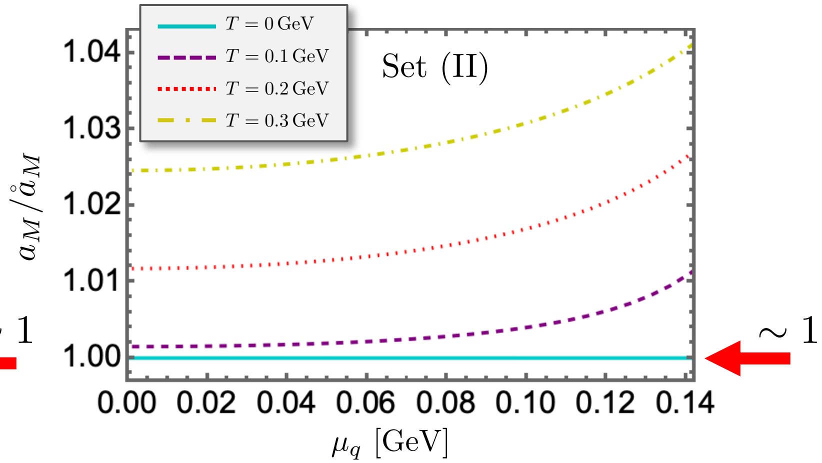
$$
V_{\mathrm{A}} = a_M\left(\det\Sigma_M + \det\Sigma_M^\dagger\right) + a_B \mathrm{tr}\left[\Sigma_{B_R}^\dagger \Sigma_{B_L} + \Sigma_{B_L}^\dagger \Sigma_{B_R}\right] + \cdots
$$

$\mathring{a}_M, \mathring{a}_B$ : vacuum values

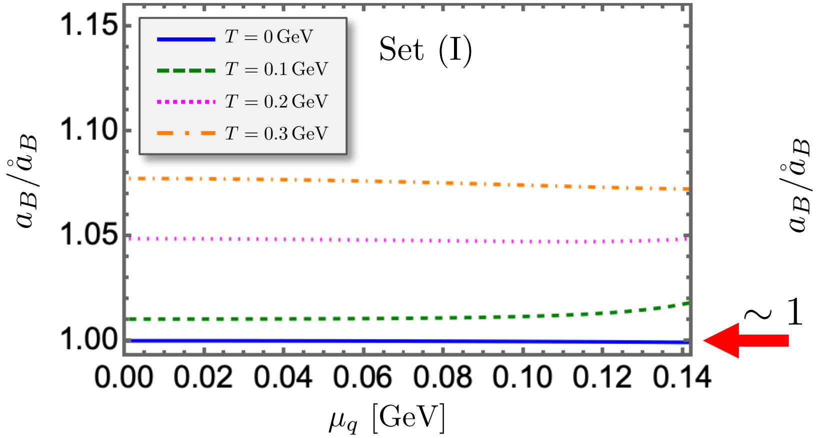
Set (I): small anomaly
Set (II): large anomaly

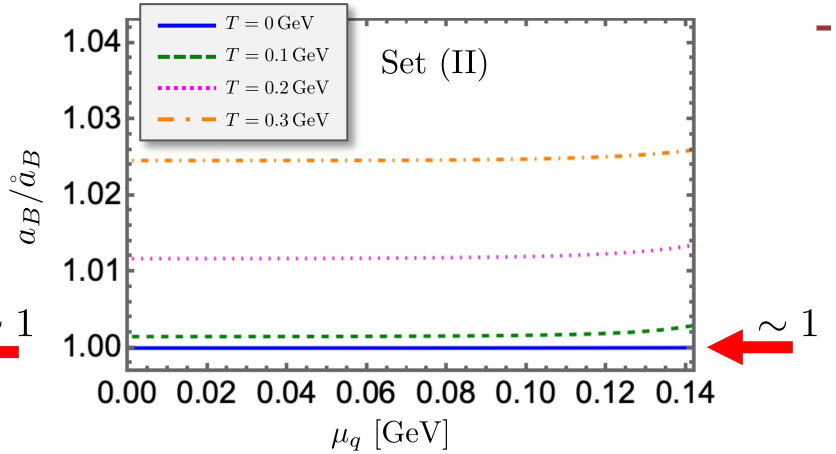
for meson




for baryon




- At T=0, no anomaly enhancement in finite $\mu_q$

# FRG analysis



- **Anomaly enhancement**
  - (Normalized) anomaly coefficients for meson and baryon: $a_M$ and $a_B$

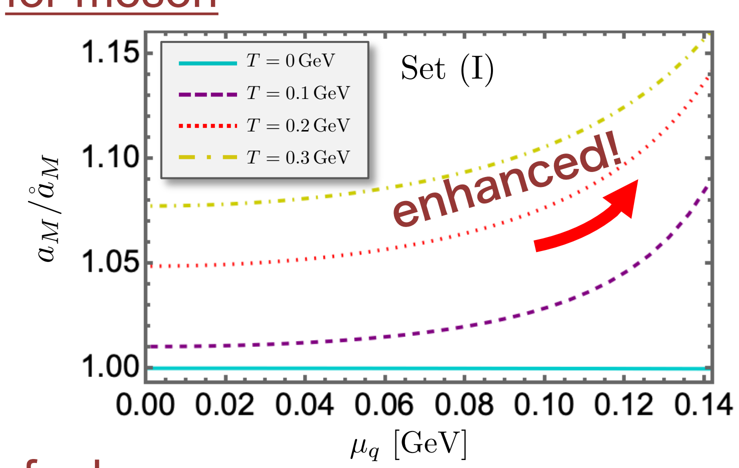
$$V_{\rm A} = a_M\left(\det\Sigma_M + \det\Sigma_M^\dagger\right) + a_B {\rm tr}\left[\Sigma^\dagger_{B_R}\Sigma_{B_L} + \Sigma^\dagger_{B_L}\Sigma_{B_R}\right] + \cdots$$

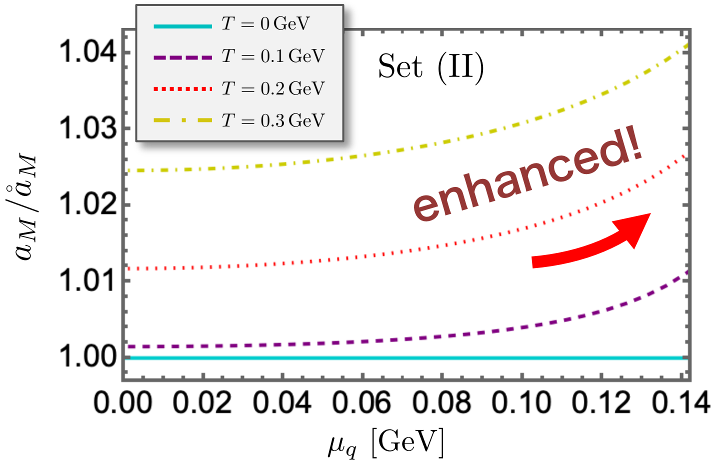
$\mathring{a}_M, \mathring{a}_B$ : vacuum values

Set (I): small anomaly

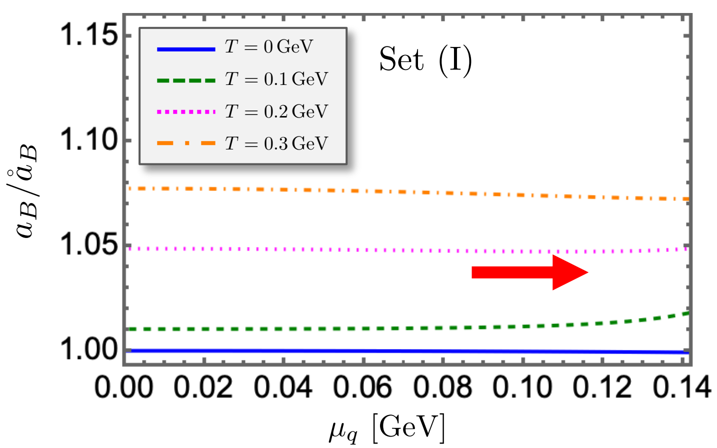
Set (II): large anomaly

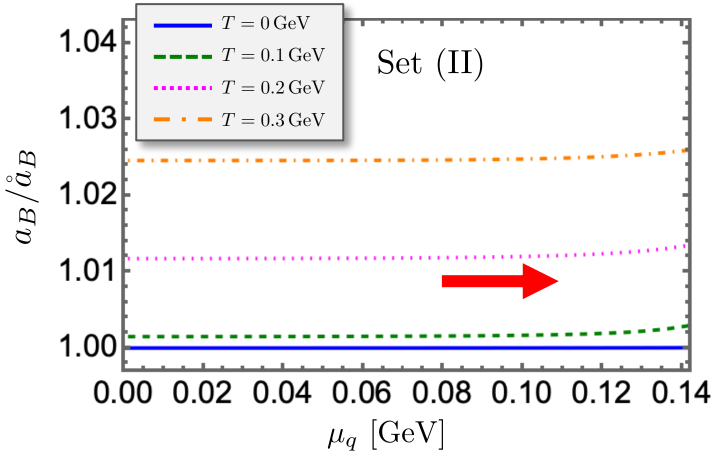
for meson




for baryon




- At T=0, no anomaly enhancement in finite $\mu_q$

- At finite T
  - $a_M$ is enhanced
  - $a_B$ is not enhanced

# FRG analysis



- **Topological susceptibility**

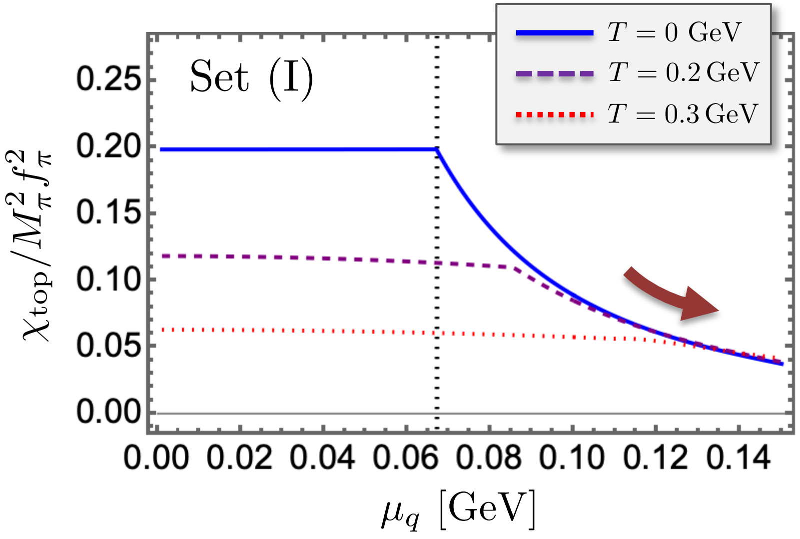
propagator with $p = 0$

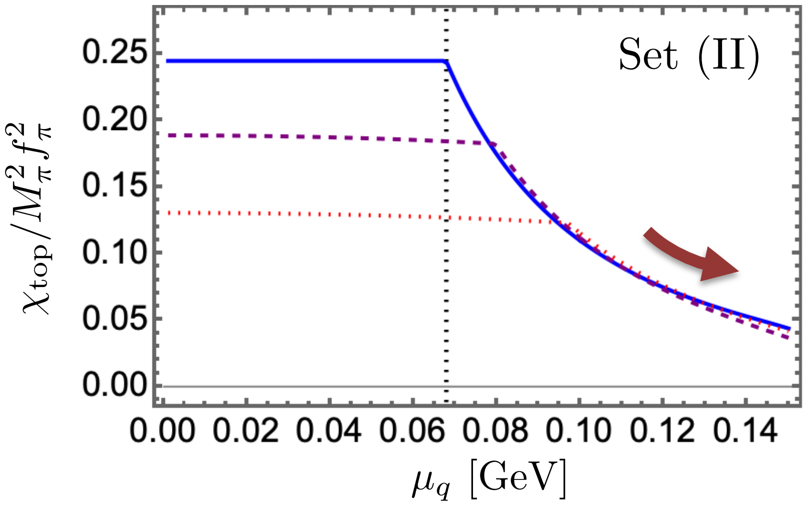
- Within our model, $\chi_{\text{top}}/M_\pi^2 f_\pi^2 = (M_\pi^2/4)\,[-D_{\pi\pi}(0) + D_{\eta\eta}(0)]$

Set (I): small anomaly

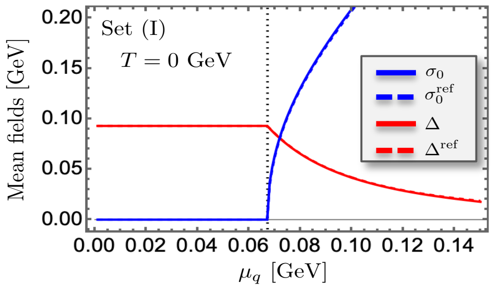
≈

Set (II): large anomaly




$\sigma_0$ at finite $\mu_q$




- The topological susceptibility is suppressed simply following chiral restoration of $\sigma_0 \propto \mu_q^{-2}$
- Anomaly enhancement is not seen; it is hidden by chiral restoration effect

# FRG analysis

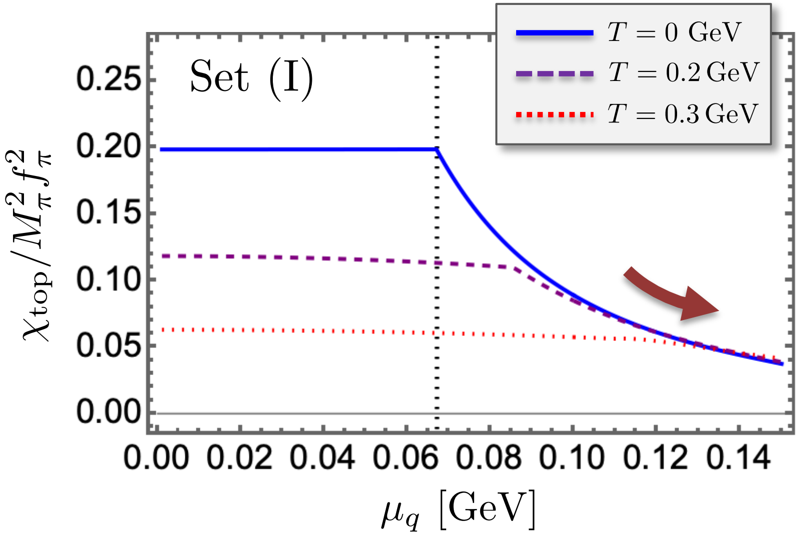


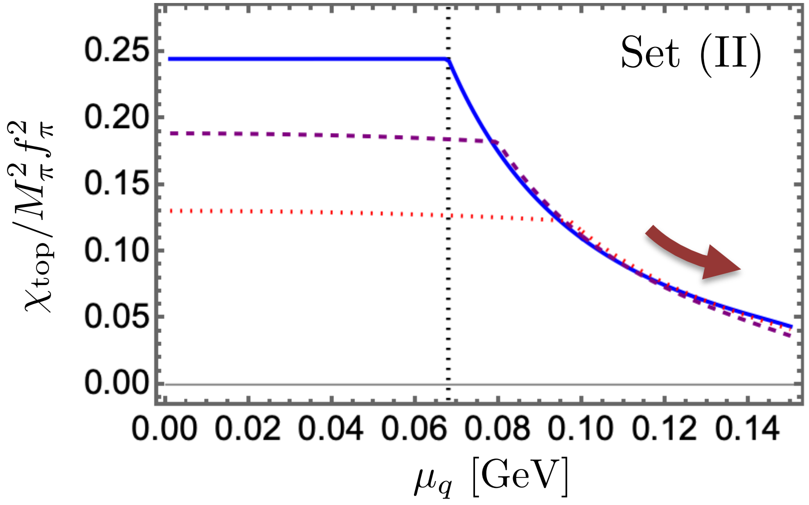
- **Topological susceptibility**

propagator with $p = 0$

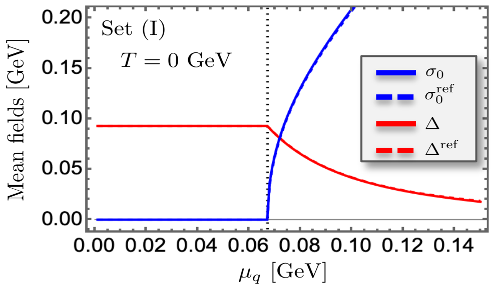
- Within our model, $\chi_{\rm top}/M_\pi^2 f_\pi^2 = (M_\pi^2/4)\,[-D_{\pi\pi}(0) + D_{\eta\eta}(0)]$

Set (I): small anomaly
≈
Set (II): large anomaly

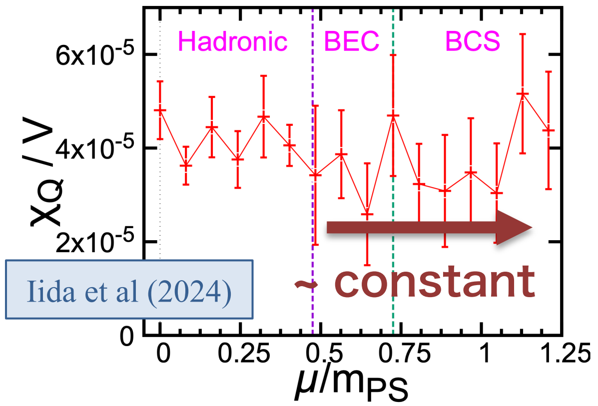


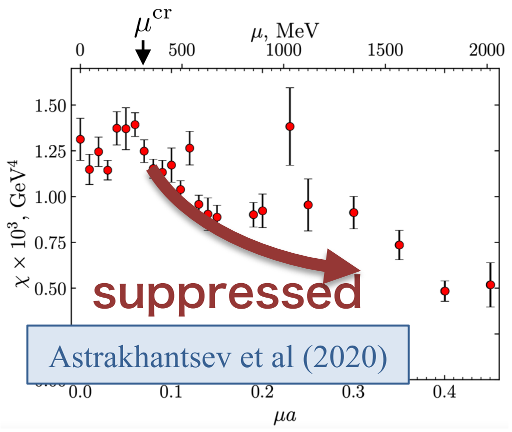




$\sigma_0$ at finite $\mu_q$




lattice




No conclusive statement yet from lattice
→ Needs more refined simulations

# Conclusions



- Constructed Linear sigma model (LSM) in $QC_2D$ to study spin-0 hadrons including *parity partners* at $\mu_q$

⇩ Succeeded in explaining lattice mass spectrum qualitatively!

- FRG analysis with LSM in finite T and density
  - Enhancement of anomaly effect to mesons in (hadronic) medium
  - Topological susceptibility is suppressed in medium with chiral restoration

$QC_2D$ lattice sim.

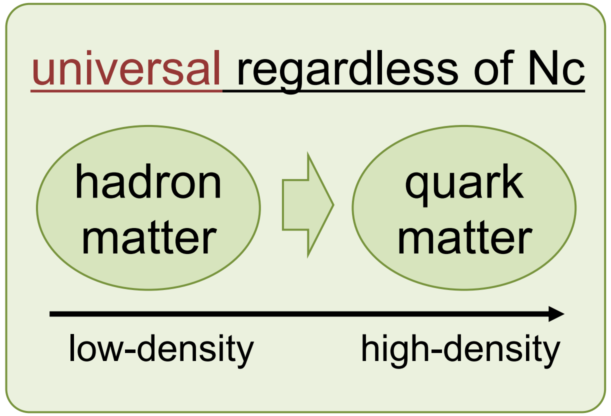

"numerical experiments"

- Next step: Inclusion of quark d.o.f.

**From $QC_2D$ study we can learn**

- to what extent hadron model description can apply
- Which representation of hadrons is useful in medium
- how to incorporate quark dof into hadron model → "unified model"
- deep understanding of quark matter in high-dense regime

⋮



# Thank you!

# Sound velocity



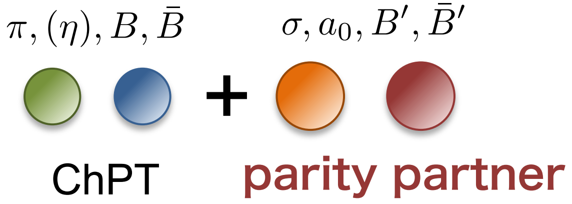
- **Sound velocity at mean-field level within the LSM**

pressure: $p = \underbrace{f_\pi^2 m_\pi^2 \left(\bar{\mu}^2 + \frac{1}{\bar{\mu}^2}\right)}_{\text{ChPT result}} + f_\pi^2 m_\pi^2 \left[\frac{4}{\delta\bar{m}^2_{\sigma-\pi}}(\bar{\mu}^2 - 1)^2\right]$

energy: $\epsilon = \underbrace{f_\pi^2 m_\pi^2 \left[\frac{(\bar{\mu}^2+3)(\bar{\mu}^2-1)}{\bar{\mu}^2}\right]}_{\text{ChPT result}} + f_\pi^2 m_\pi^2 \left[\frac{4}{\delta\bar{m}^2_{\sigma-\pi}}(3\bar{\mu}^2+1)(\bar{\mu}^2-1)\right]$

sound velocity: $c_s^2 = \frac{(1 - 1/\bar{\mu}^4) + 8(\bar{\mu}^2-1)/\delta\bar{m}^2_{\sigma-\pi}}{\underbrace{(1+3/\bar{\mu}^4)}_{\text{ChPT result}} + 8(3\bar{\mu}^2-1)/\delta\bar{m}^2_{\sigma-\pi}}$

$\bar{\mu} = \mu/\mu_{\rm cr} = 2\mu/m_\pi$
$\delta\bar{m}^2_{\sigma-\pi} = (m_\sigma^2 - m_\pi^2)/\mu_{\rm cr}^2$




Universal structure: $(\text{LSM result}) = (\text{ChPT result}) + (1/\delta\bar{m}^2_{\sigma-\pi} \text{ contribution})$

- Integrating out the parity partners $(m_\sigma \to \infty)$ yields the ChPT results $(1/\delta\bar{m}^2_{\sigma-\pi} \to 0)$

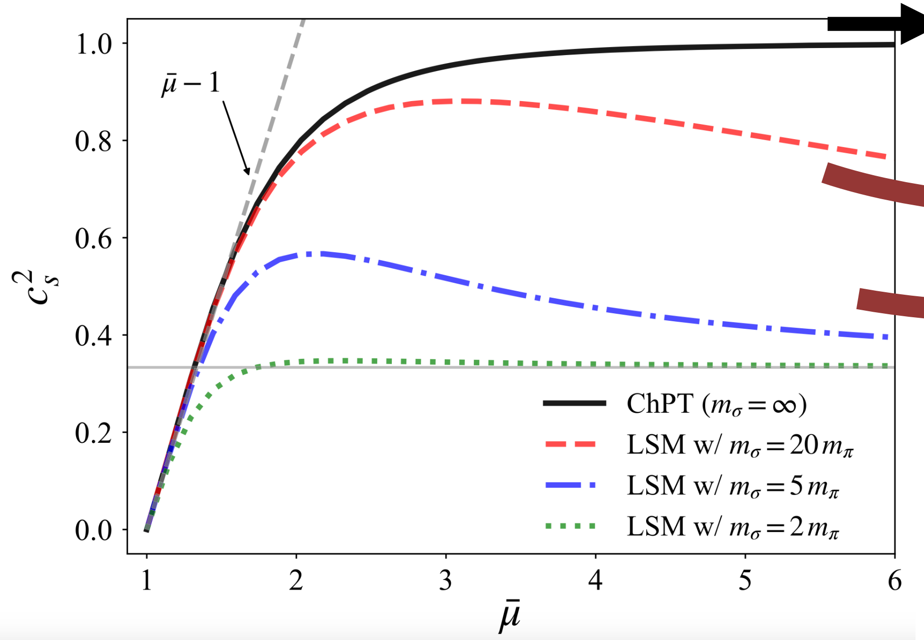
# Sound velocity



- **Sound velocity peak**

Kawaguchi-**Suenaga** (2024)

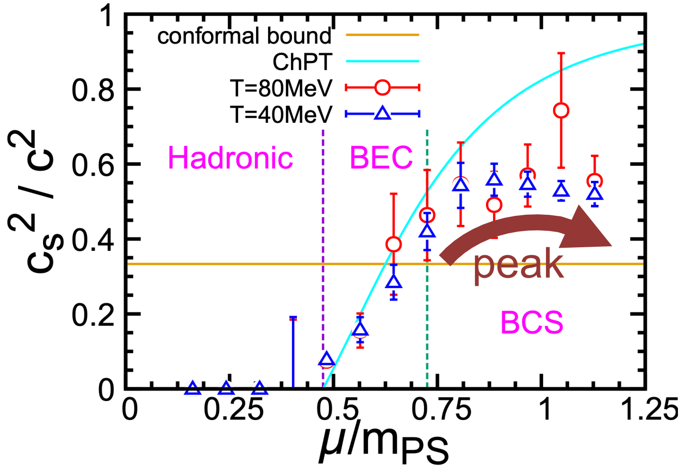
$\bar{\mu} = \mu/\mu_{\rm cr} = 2\mu/m_\pi$

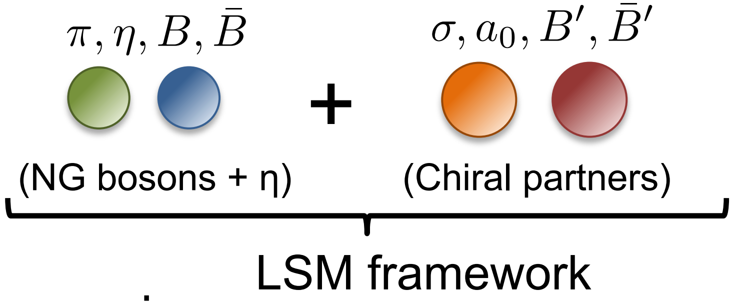



**ChPT result** $(c_s^2 \to 1)$ : no peak

peak when $m_\sigma < \infty$ $(c_s^2 \to 1/3)$

Lattice: Iida et al (2024)




- The peak structure is driven by contributions from parity partners

- Fluctuation and spin-1 hadron effect are needed for more quantitative comparison

- Any connection with crossover to quark matter?

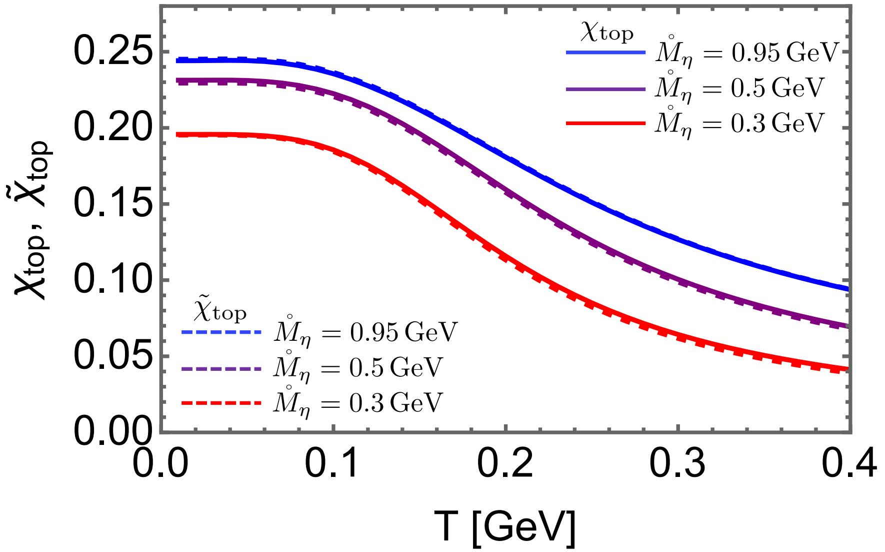
# FRG for anomaly effect



- **Topological susceptibility at finite T**

Definition: $\bar{\chi}_{\rm top} \equiv -i\int d^4x \langle 0|\mathrm{T}Q(x)Q(0)|0\rangle = \frac{\mathring{M}_\pi^4 \mathring{\sigma}_0^2}{4}\left(\frac{1}{M_\pi^2} - \frac{1}{M_\eta^2}\right)$ with $Q = (g_s^2/64\pi^2)\epsilon^{\mu\nu\rho\sigma}G^a_{\mu\nu}G^a_{\rho\sigma}$

$\bar{\chi}_{\rm top} = -\frac{m_l\langle\bar{\psi}\psi\rangle}{4}\delta_m$ with $\delta_m = 1 - \frac{\chi_\eta}{\chi_\pi}$




$$\chi_{\rm top} \equiv \frac{\bar{\chi}_{\rm top}}{\mathring{M}_\pi^2\mathring{\sigma}_0^2} = \frac{\mathring{M}_\pi^2}{4}\left(\frac{1}{M_\pi^2} - \frac{1}{M_\pi^2 + (M_\eta^2 - M_\pi^2)}\right) \quad (\leftarrow \text{full result})$$

$$\tilde{\chi}_{\rm top} \equiv \frac{\mathring{M}_\pi^2}{4}\left(\frac{1}{M_\pi^2} - \frac{1}{M_\pi^2 + (\mathring{M}_\eta^2 - \mathring{M}_\pi^2)}\right) \quad (\leftarrow \text{reference})$$

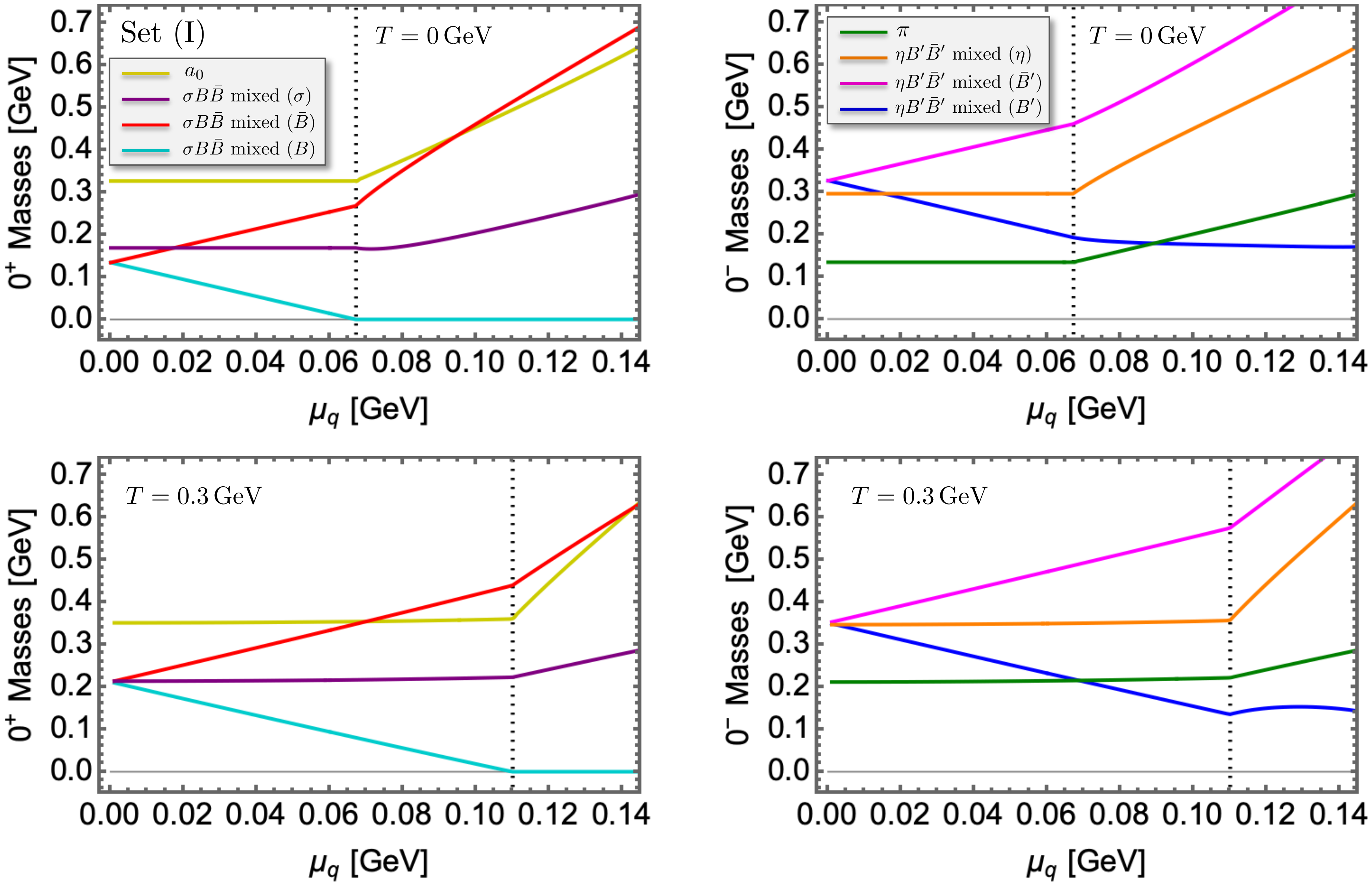
NOTE: $M_\eta^2 - M_\pi^2 = -2a - (2c_1 + c_2)\sigma_0^2 \propto (\text{anomaly effect})$

- $\chi_{\rm top}$ is suppressed at higher temperature followed by chiral restoration: $\chi_{\rm top} \sim \frac{\mathring{M}_\pi^2}{4}\frac{1}{M_\pi^2} \propto \sigma_0$
- No difference between $\chi_{\rm top}$ and $\tilde{\chi}_{\rm top}$

  → topological susceptibility is not suitable probe to see corrections of anomaly effects

# Hadron mass

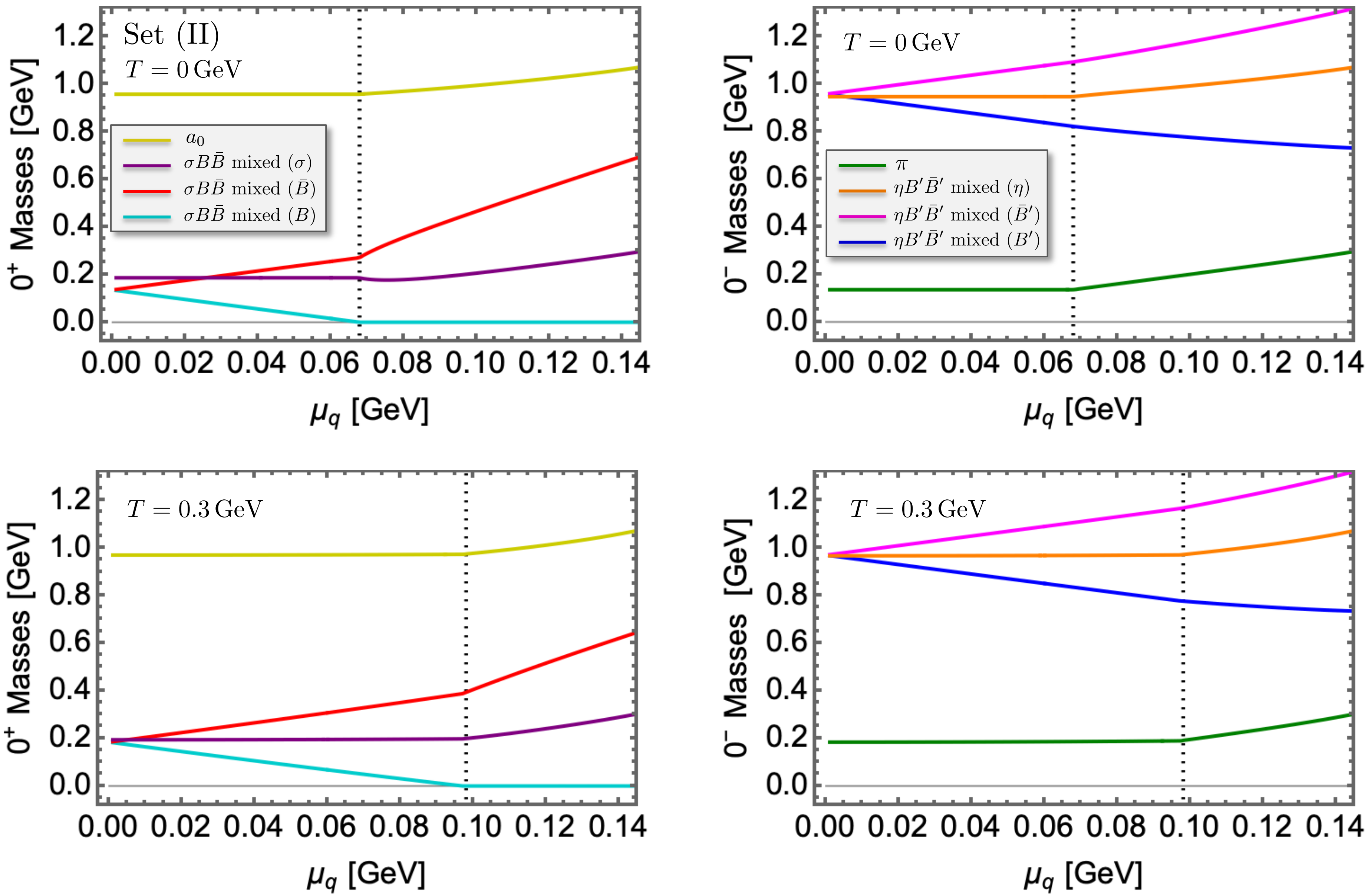




Set (I)
$T = 0\,\mathrm{GeV}$
$a_0$
$\sigma B\bar{B}$ mixed $(\sigma)$
$\sigma B\bar{B}$ mixed $(\bar{B})$
$\sigma B\bar{B}$ mixed $(B)$
$0^+$ Masses [GeV]
$\mu_q$ [GeV]
$\pi$
$\eta B'\bar{B}'$ mixed $(\eta)$
$\eta B'\bar{B}'$ mixed $(\bar{B}')$
$\eta B'\bar{B}'$ mixed $(B')$
$0^-$ Masses [GeV]
$T = 0.3\,\mathrm{GeV}$
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7
0.00 0.02 0.04 0.06 0.08 0.10 0.12 0.14

# Hadron mass



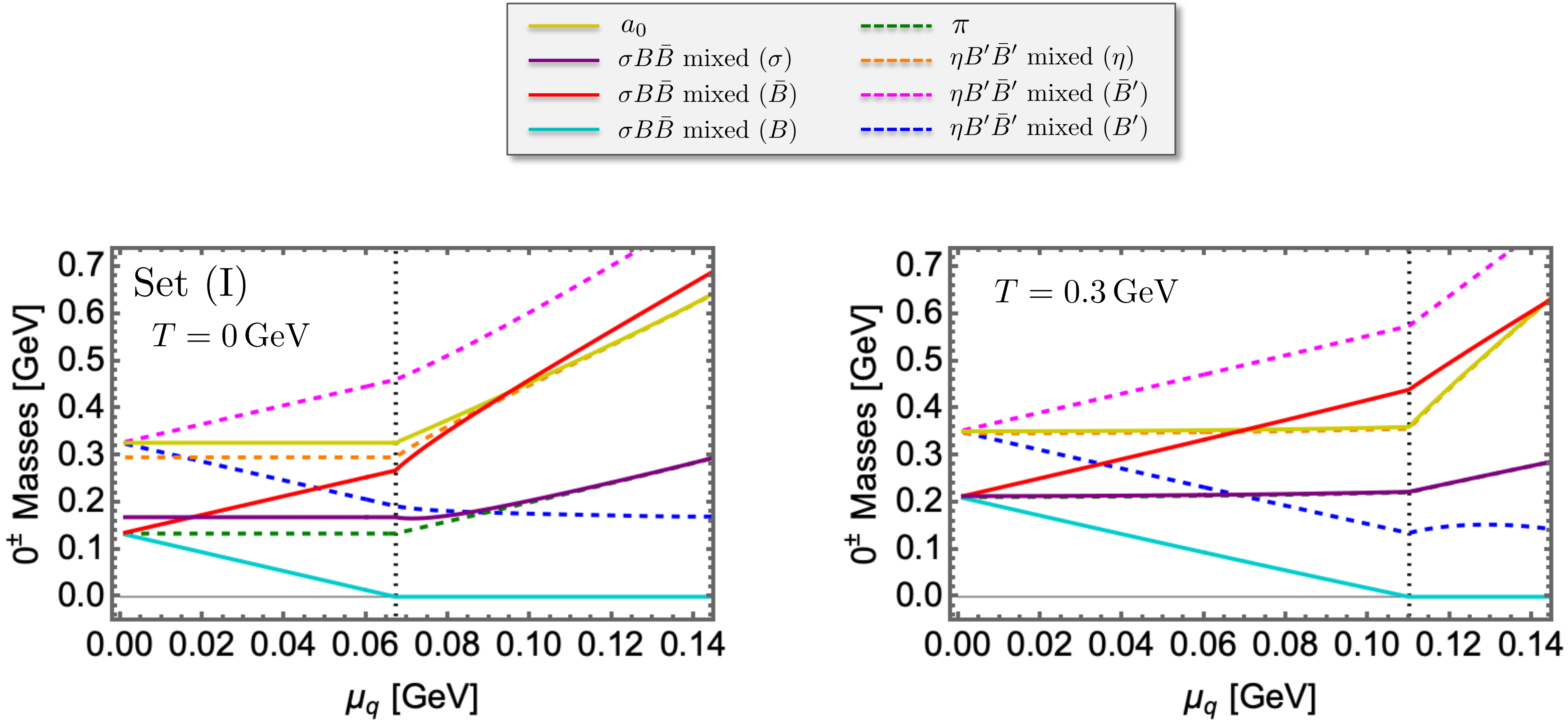


Set (II)
$T = 0\,\mathrm{GeV}$
$a_0$
$\sigma B\bar{B}$ mixed ($\sigma$)
$\sigma B\bar{B}$ mixed ($\bar{B}$)
$\sigma B\bar{B}$ mixed ($B$)
$0^+$ Masses [GeV]
$\mu_q$ [GeV]
$T = 0\,\mathrm{GeV}$
$\pi$
$\eta B'\bar{B}'$ mixed ($\eta$)
$\eta B'\bar{B}'$ mixed ($\bar{B}'$)
$\eta B'\bar{B}'$ mixed ($B'$)
$0^-$ Masses [GeV]
$\mu_q$ [GeV]
$T = 0.3\,\mathrm{GeV}$
$0^+$ Masses [GeV]
$\mu_q$ [GeV]
$T = 0.3\,\mathrm{GeV}$
$0^-$ Masses [GeV]
$\mu_q$ [GeV]

# Hadron mass

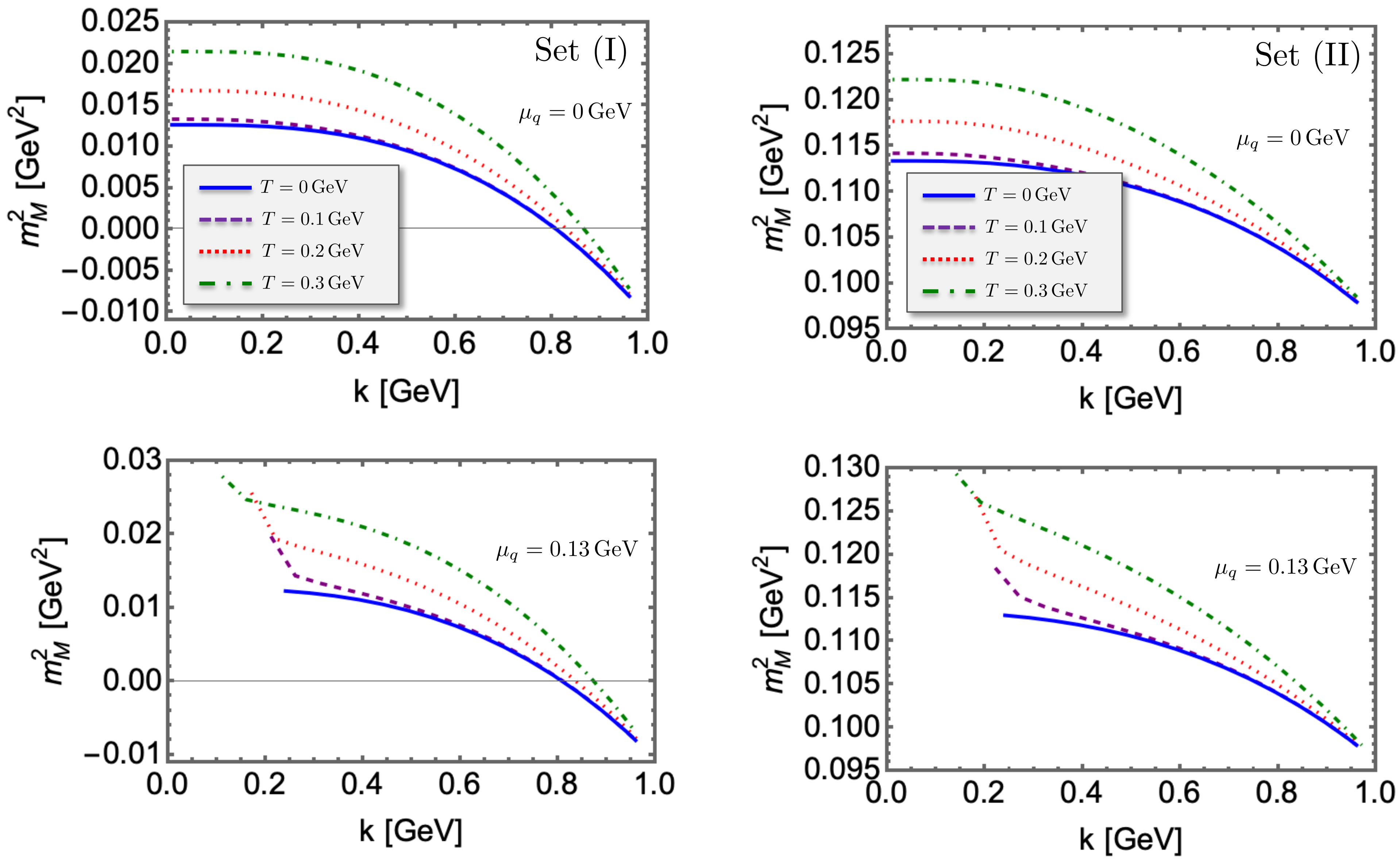


- **Mass degeneracies of chiral partners**



# Determination of IR cutoff






$$k_{\mathrm{IR}} = 0.3\,\mathrm{GeV}$$