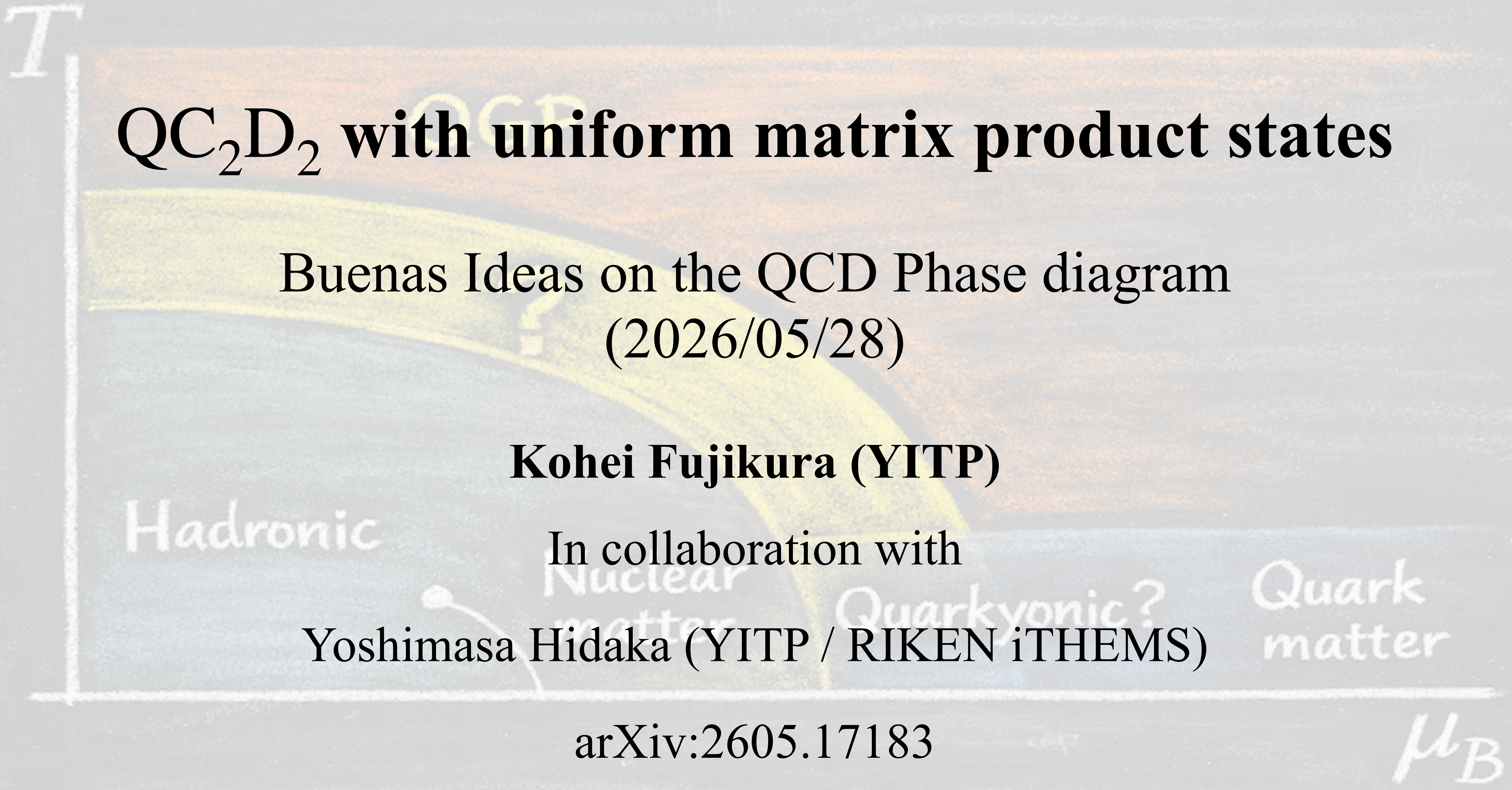


QC_2D_2 with uniform matrix product states

Buenas Ideas on the QCD Phase diagram
(2026/05/28)

Kohei Fujikura (YITP)

In collaboration with

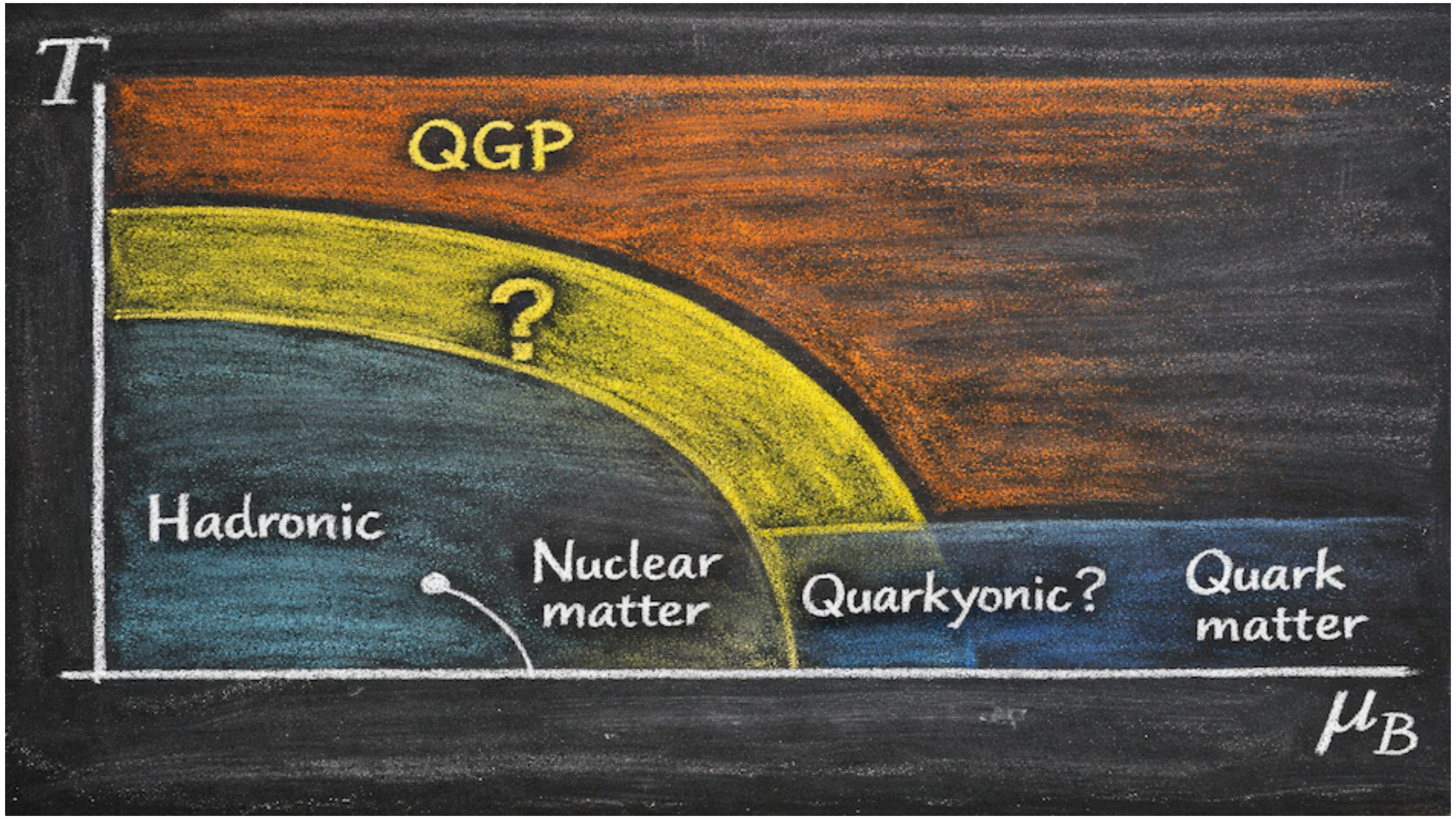
Yoshimasa Hidaka (YITP / RIKEN iTHEMS)

arXiv:2605.17183

Buenos dîas!

Buenas Ideas from Hamiltonian formulation

I claim that QCD in (1 + 1) dimensions may be solvable using the Hamiltonian formulation of lattice gauge theories.



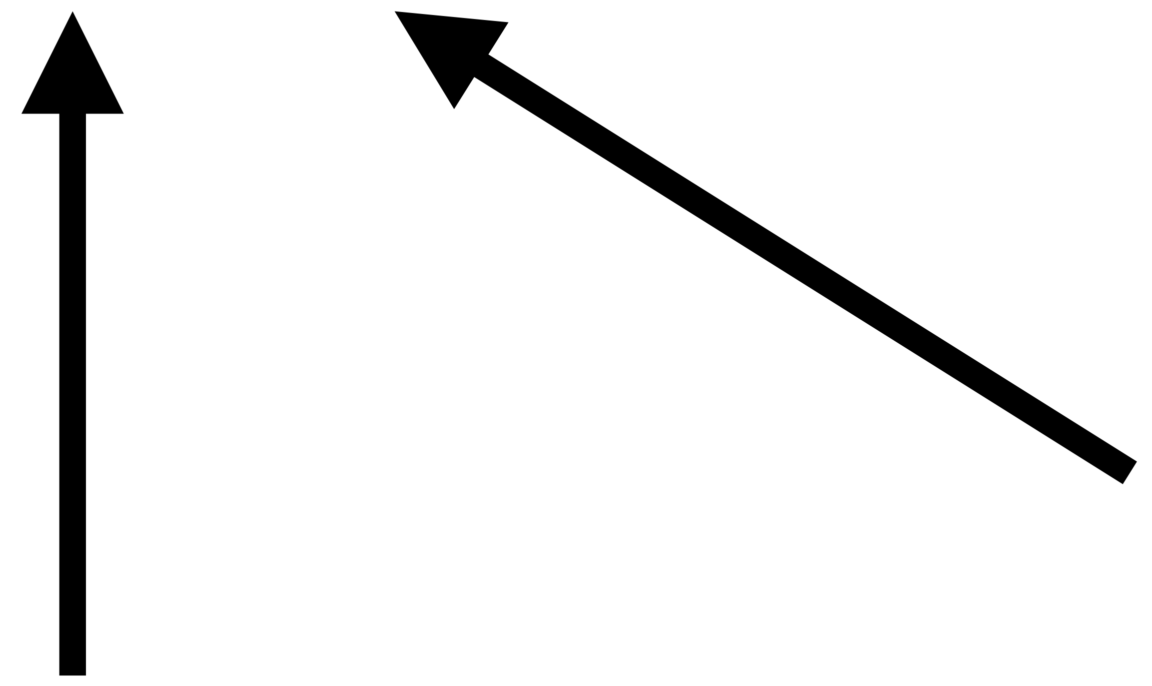
¡Disfruta!

Hamiltonian (Operator) formalism

In the Hamiltonian formalism, physical states can be constructed directly.

$$\langle \Psi | \hat{O} | \Psi \rangle$$

$|\Psi\rangle$: Physical state



Huge vector
(Hilbert space)

Huge and usually sparse matrix
(Operator)

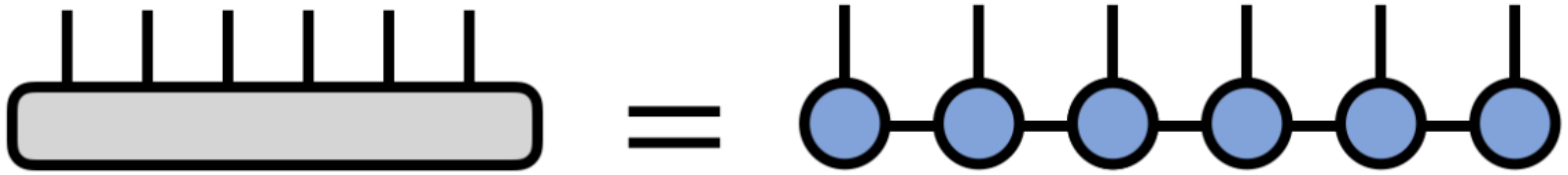
No sign problem, but dimension of $|\Psi\rangle$ is quite large…

Ex) $\dim|\Psi\rangle = 2^N$ even for N-site Ising model.

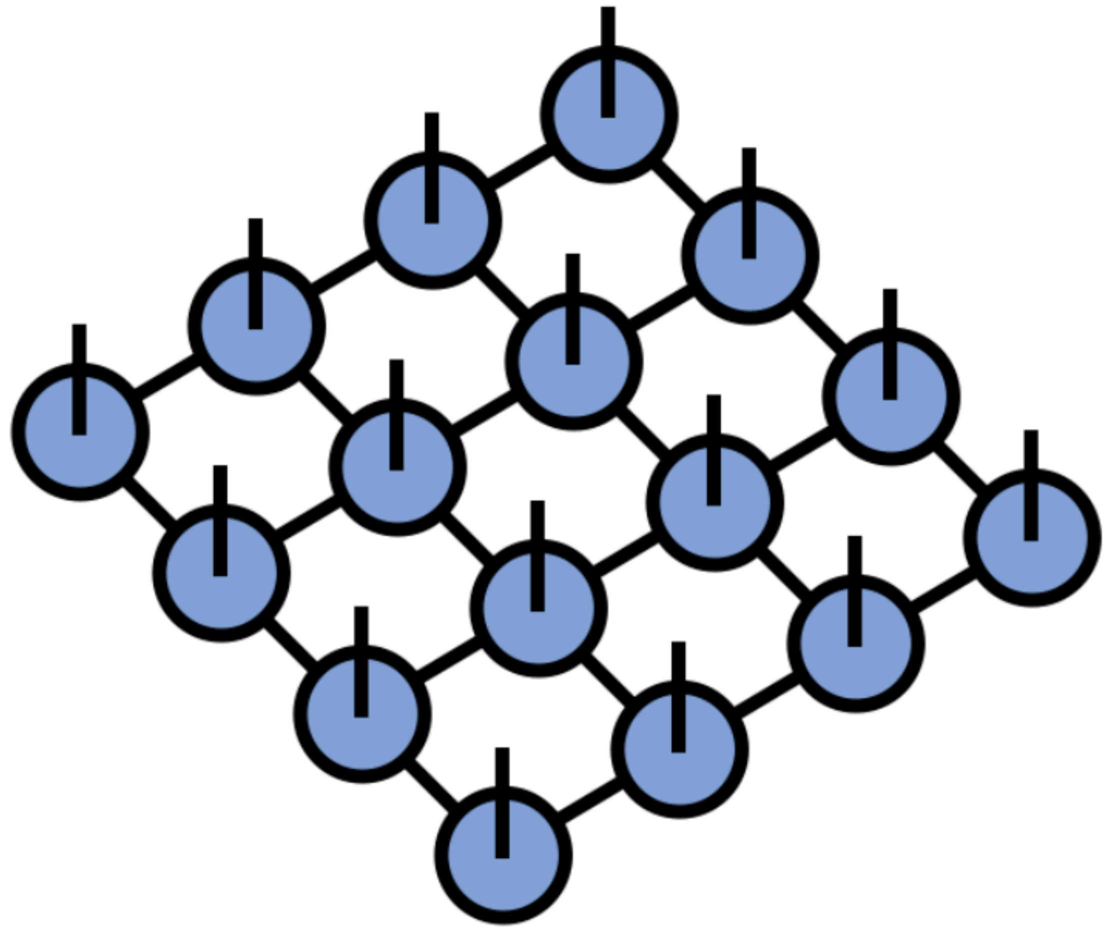
Tensor Network

Tensor network provides a good basis to construct the physical state by classical computing.

MPS



PEPS



[TensorNetwork.org]

QC_2D_2 at finite density

We numerically construct the translational and gauge invariant ground states of (1+1)-dimensional $SU(2)$ gauge theory with a single flavor at finite baryon density using tensor network method.

$$S = \int \mathrm{d}^2x \left[-\frac{1}{2}\mathrm{Tr}\left(F^{\mu\nu}F_{\mu\nu}\right) + i\bar{q}\gamma^{\mu}D_{\mu}q + \mu_q q^{\dagger}q - m_q\bar{q}q \right]$$

μ_q : quark chemical potential m_q : quark mass

*The choice $N_c = 2$ and $N_f = 1$ is made for purely technical reasons.

¡Me esforcé mucho!

Previous studies on QC_2D_2

- Two-color QCD at finite baryon density can be simulated by lattice Monte Carlo for even N_f in (3+1)-dimensions.

[Review by V. V. Braguta (2023), E. Itou (2025). Talk by E. Itou]

- The Hamiltonian formulation with a strong coupling expansion on the lattice

[H.J. Hamer (1982)]

- Matrix product state with an open boundary condition on the lattice

[M. C. Banuls, K. Cichy, J. I. Cirac, K. Jansen, S. Khun (2017), T. Hayata, Y. Hidaka, K. Nishimura (2023))]

- Grassmann tensor renormalization group approach

[K. H. Pai, S. Akiyama, S. Todo (2025)]

- Bosonization with a certain regularization at strong coupling limit

[V. Baluni (1980), P. J. Steinhardt (1980)]

Quarkyonic example? Cold and dense QCD_2 is *Tomonaga-Luttinger liquid!*

[T. Kojo (2011)] *[M. Lajer, R. M. Konik, R. D. Pisarski, A. M. Tsvelik (2021)]*

Claims of this talk

We perform first-principles numerical simulations on the lattice directly in the thermodynamic limit.

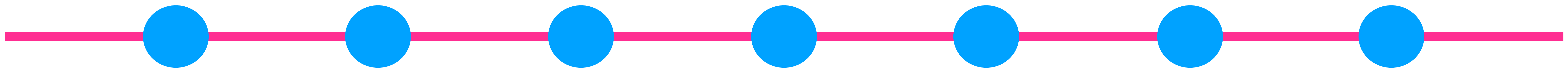
- A low-energy behavior of cold and dense $\mathrm{QC_2D_2}$ is indeed described by Tomonaga—Luttinger liquid (conformal field) theory…
- Thermodynamic quantities can be approximated by the *free quark picture* at a high baryon density.
- However, the *baryon density mode is an excitation* at any finite density.
- A central charge is $c = 1$, (not $c = N_c!$).

Kogut-Susskind Hamiltonian formulation

A spatial direction is discretized, while the temporal direction is continuous.

[Kogut, Susskind (1974)]

$|\psi\rangle =$



A gauge field lives on the link.

$$[R^a, U] = UT^a,\ \ [L^a, U] = T^a U$$

$$[R^a, R^b] = i f^{ab}_c R^c,\ \ [L^a, L^b] = -i f^{ab}_c L^c$$

L^a, R^a: Left and right electric fields

U: Wilson line in fund. rep. T^a: Generator of $SU(2)$

A matter field lives on the site.

$$\{\phi^\dagger_{c_1}(n), \phi^{c_2}(m)\} = \delta_{n,m}\delta^{c_2}_{c_1}$$

ϕ_c : (Staggered) fermion in the fundamental rep.

The calculation is mostly performed using group theory.

Hamiltonian in 1+1 dimension

The total Hamiltonian with a single flavor

$$H_{\rm tot} = H_E + H_{\rm hop} + H_{\rm mass}$$

$$H_E = \sum_n \frac{ag^2}{2} E^2(n),$$ $E^2 = (R^a)^2 = (L^a)^2$: square of electric field

$$H_{\rm hop} = \sum_n \frac{1}{2a} \left(\phi^\dagger_{c_1}(n+1) U^{c_1}_{c_2}(n) \phi^{c_2}(n) + \text{h.c.} \right),$$

Hopping term

$$H_{\rm mass} = m \sum_n (-1)^n \phi^\dagger_c(n) \phi^c(n).$$

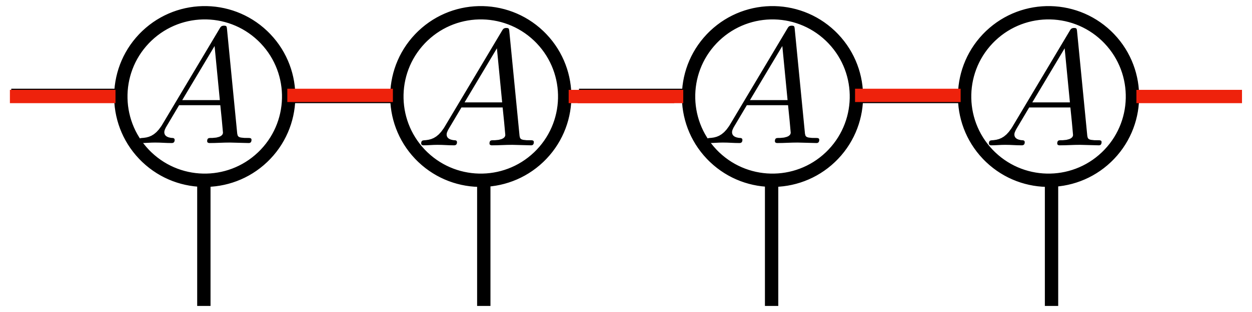
Quark mass term

a: lattice spacing, g: gauge coupling, m: quark mass

uniform Matrix Product States (uMPS)

- Assuming the translational invariance and using the truncation, uniform matrix product states reduce the number of variational parameters!

$$|\Psi\rangle = v_L^\dagger \cdots A^{s_n} A^{s_{n+1}} \cdots v_R | \cdots, s_n, s_{n+1}, \cdots \rangle = \cdots$$



$\cdots$

Number of parameters ~ D^2×(dim. of local Hilbert space)

D: bond dimension

[M. Fannes, B. Nachtergaele, R. F. Werner (1992)]

- An exact physical state is realized for $D \to \infty$.

[Verstraete, Cirac (2006)]

$$|||\Psi\rangle - |\Psi_{\mathrm{uMPS}}\rangle|| < \epsilon(D) \to 0 \ (D \to \infty)$$

- We use gauge invariant uMPS ansatz. (Details are omitted.)

¡Quiero hacer alta dimensión!

VUMPS

Variational Uniform Matrix Product State (VUMPS) algorithm

[Zauner-Stauber, Vanderstraeten, Fishman, Verstraete, Haegeman (2017)]

Energy is the target function: $E \equiv \frac{\langle \Psi_{\mathrm{uMPS}} | \hat{H} | \Psi_{\mathrm{uMPS}} \rangle}{\langle \Psi_{\mathrm{uMPS}} | \Psi_{\mathrm{uMPS}} \rangle}$

uMPS: $|\Psi\rangle = \mathrm{tr}(\cdots \underline{A^{s_n}} A^{s_{n+1}} \cdots) | \cdots, s_n, s_{n+1}, \ldots \rangle$

Variational parameters

VUMPS is based on the gradient descent method:

Find the MPS tensor minimizing energy: $\frac{\delta E}{\delta (\mathbf{A}^{s_m})^\dagger} = 0$

Cold and Dense Ground states

The ground state property is investigated by minimizing,

$$\langle \Psi_{\mathrm{uMPS}} | H_{\mathrm{tot}} - \mu_B N_B | \Psi_{\mathrm{uMPS}} \rangle$$

N_B: baryon number operator

under fixed lattice input parameters: m/g, μ_B/g, D, ga,
(The gauge coupling has mass dimension 1 in (1+1)-dim.)

with a translational and gauge invariant MPS ansatz $|\Psi_{\mathrm{uMPS}}\rangle$.

Remark: The translational invariance plays a crucial role to obtain new results.

Short Summary

Quantum field theories (Here, QC_2D_2)

↓

Lattice regularization (Kogut-Susskind Hamiltonian)

↓

Translational and gauge invariant wavefunction, $|\Psi_{\mathrm{uMPS}}\rangle$

↓

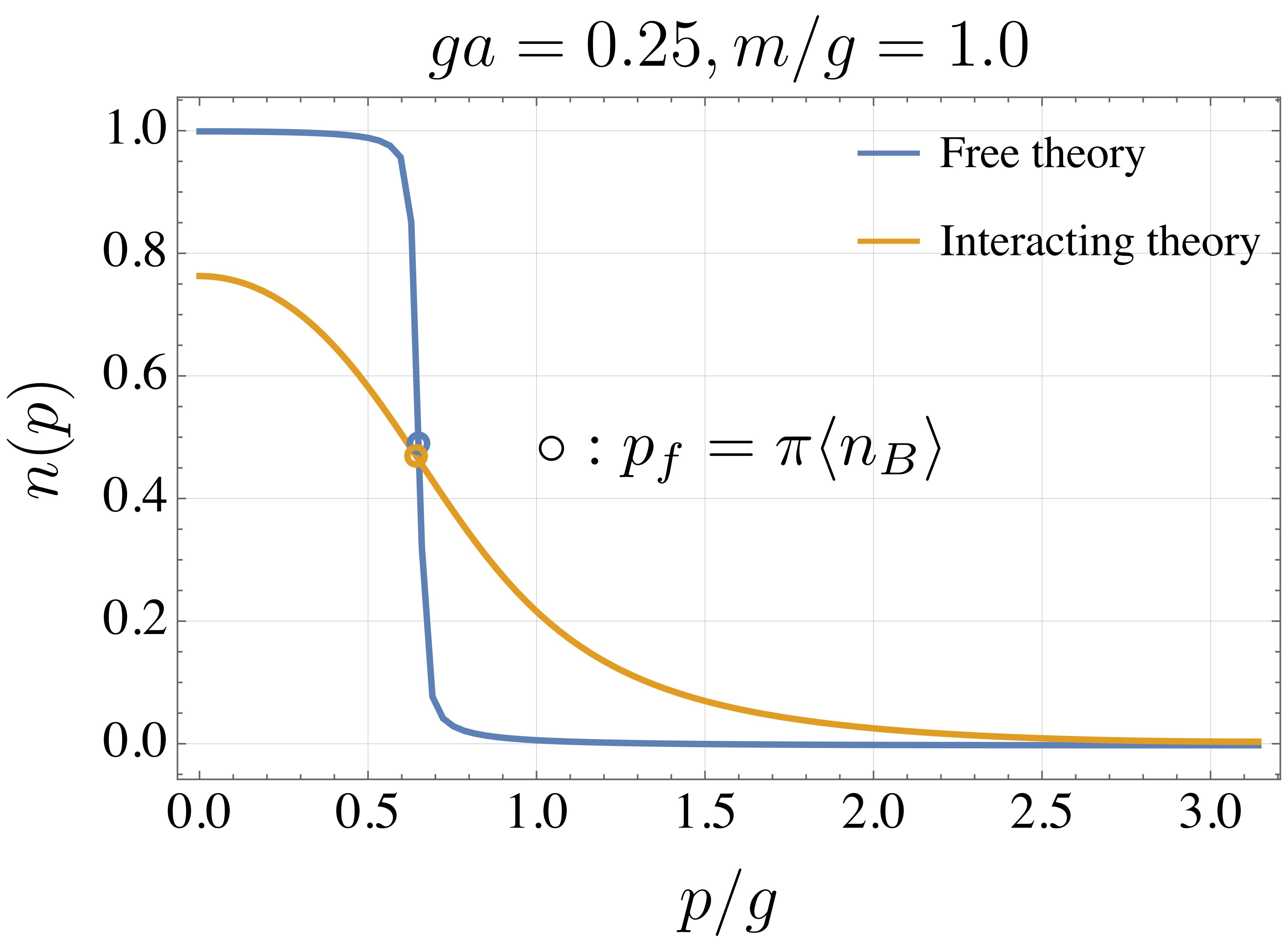
Construct the ground states (VUMPS)

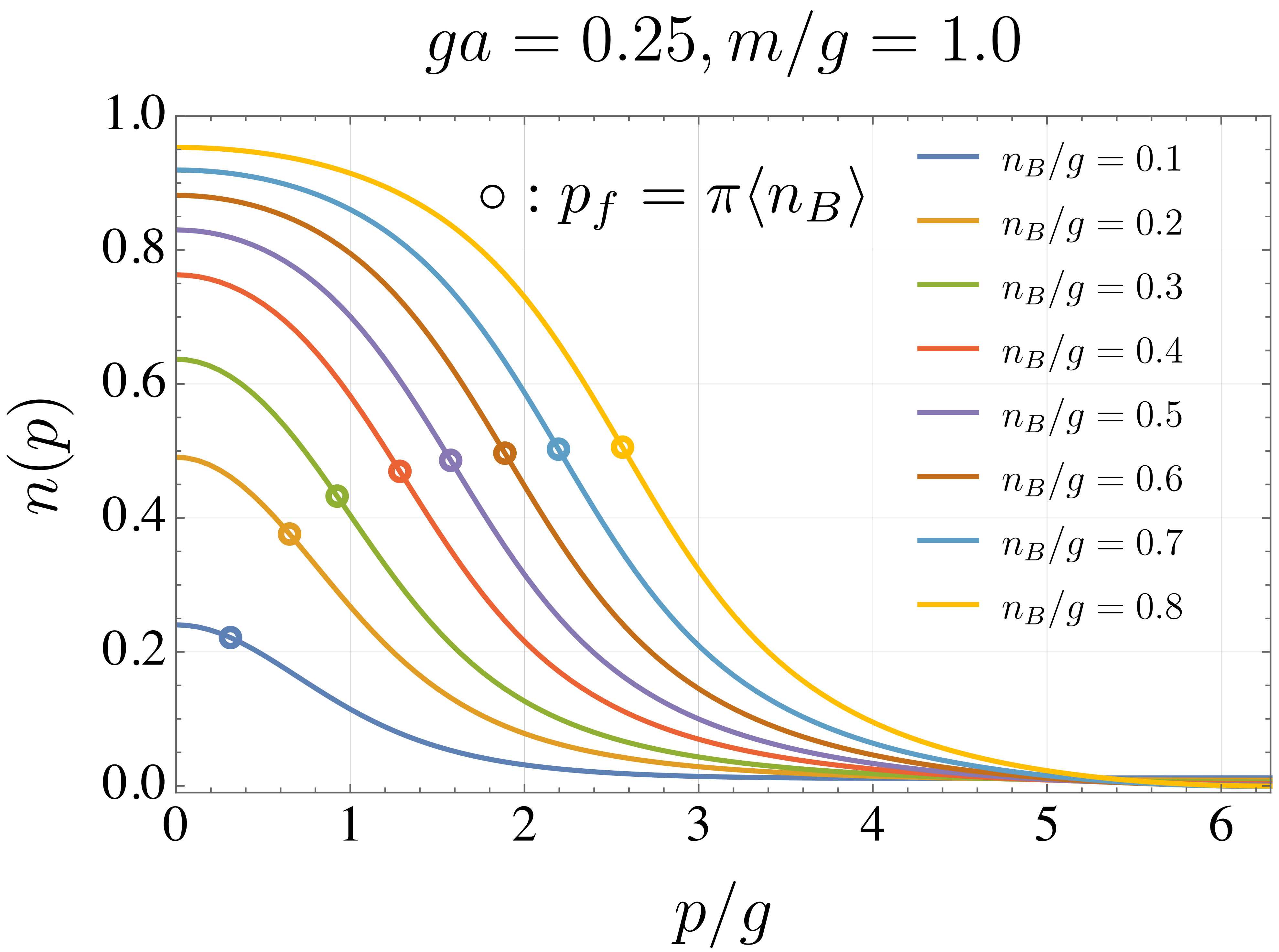
↓

Extract various physical quantities from the ground state

Quark distribution function

At high baryon density, thermodynamic quantities are well approximated by a free-quark picture.

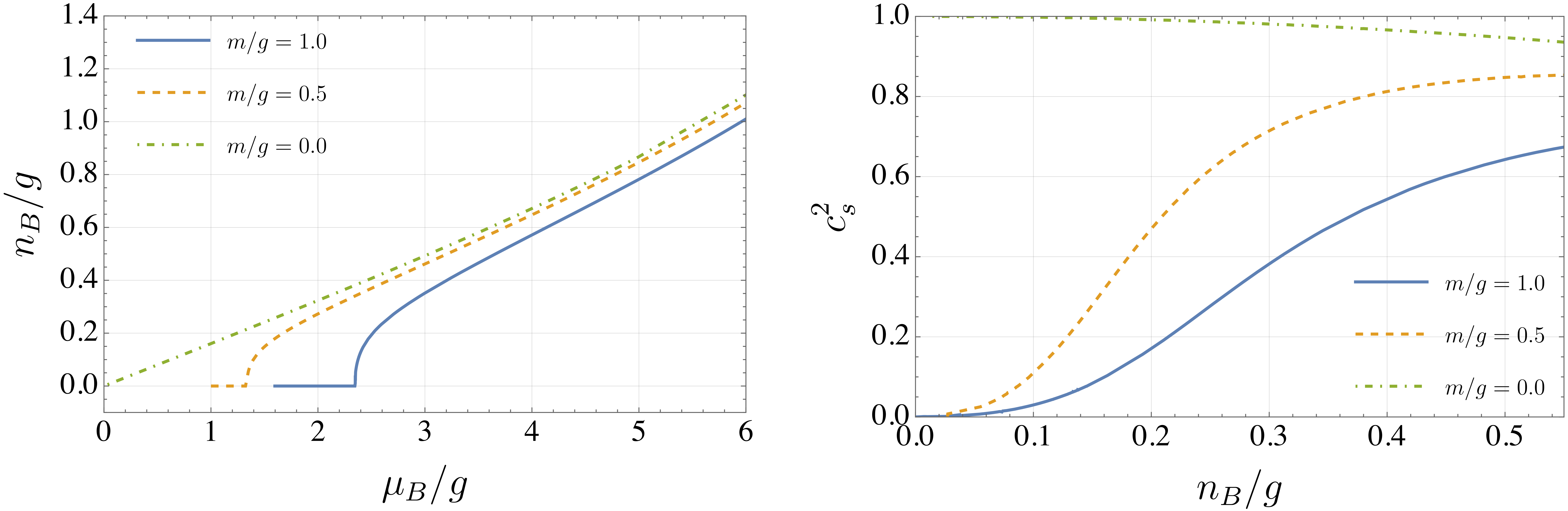




Remark: Fermi surface is destabilized by the gauge interaction.

Cold, Dense and Uniform

Thermodynamical quantities are accurately computed thanks to the translational invariance.



$\langle n_B \rangle \neq 0$ for $\mu_B \geq M_B$. (M_B: the lightest baryon mass)

Caution: No continuous global symmetry is spontaneously broken.

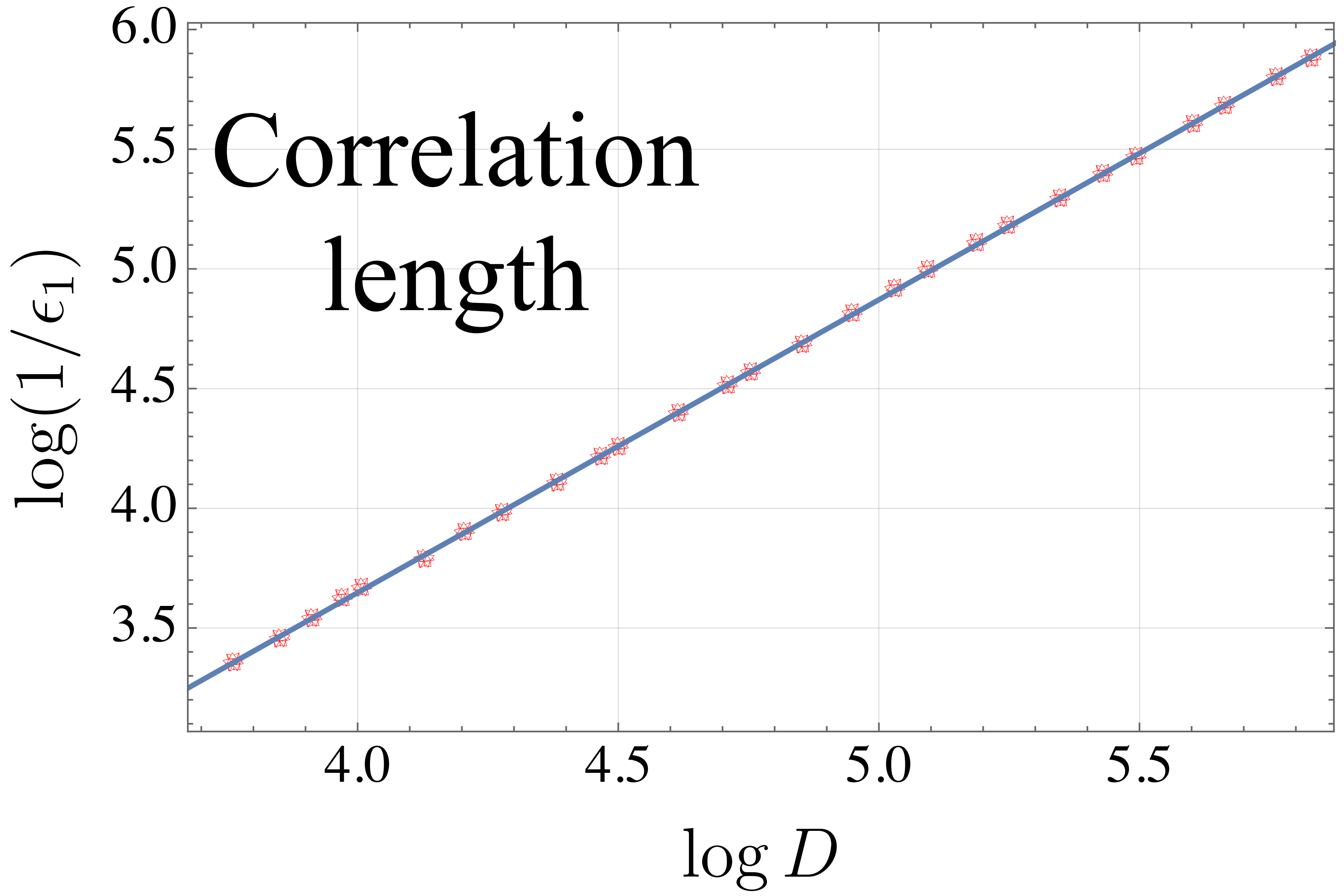
Cold, Dense, Uniform and Gapless

The system behaves as the gapless phase for $\langle n_B \rangle \neq 0$!

$$S_{\mathrm{EE}} \sim \frac{c}{6} \log(\xi/a)$$

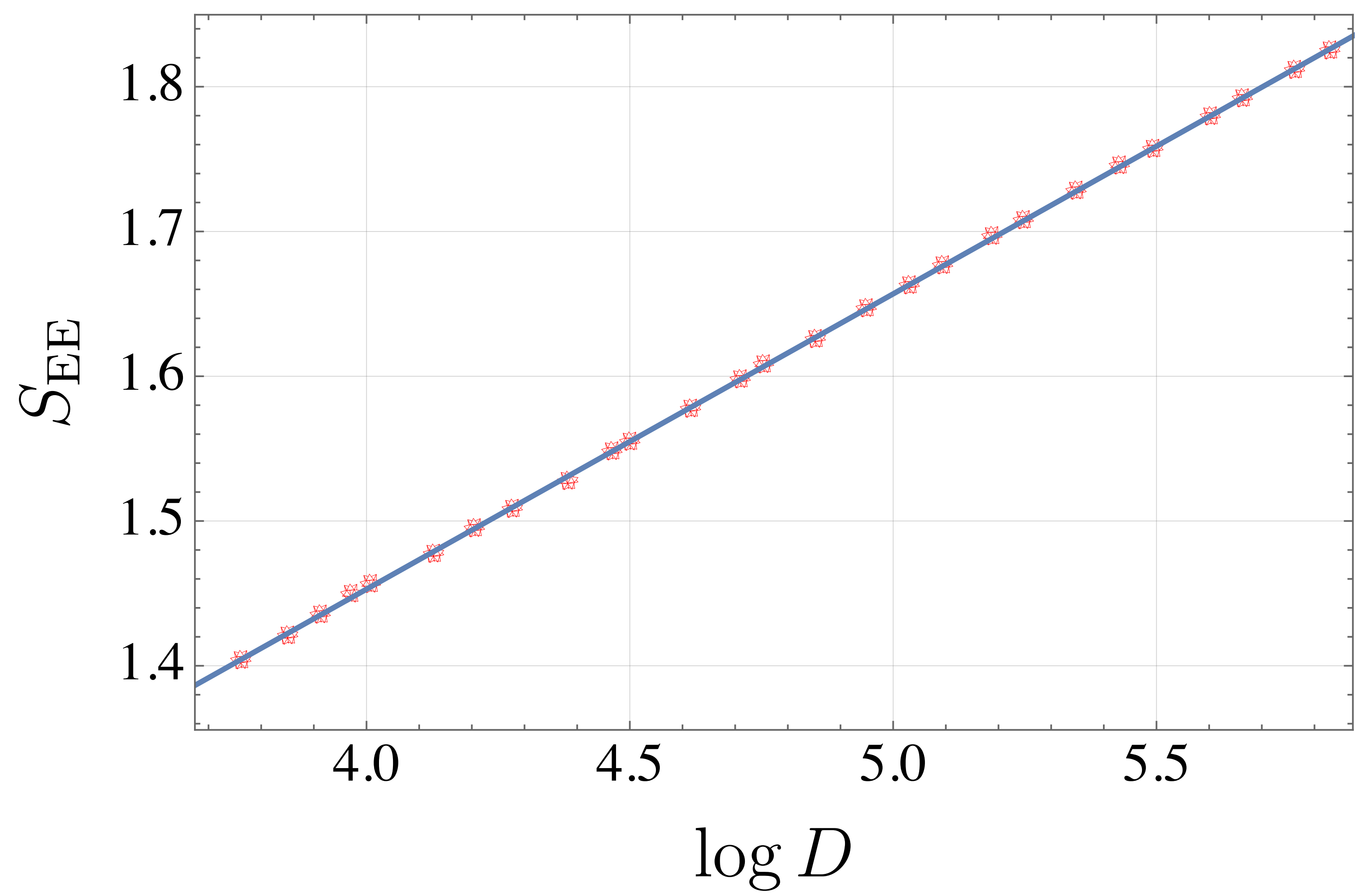
[P. Calabrese, J. Cardy (2009)]

$ga = 0.25, m/g = 1.0, \mu_B = 5.0$



Correlation length

$ga = 0.25, m/g = 1.0, \mu_B = 5.0$



The best fitted-value of the central charge is $c = 1.00083 \pm 0.005$

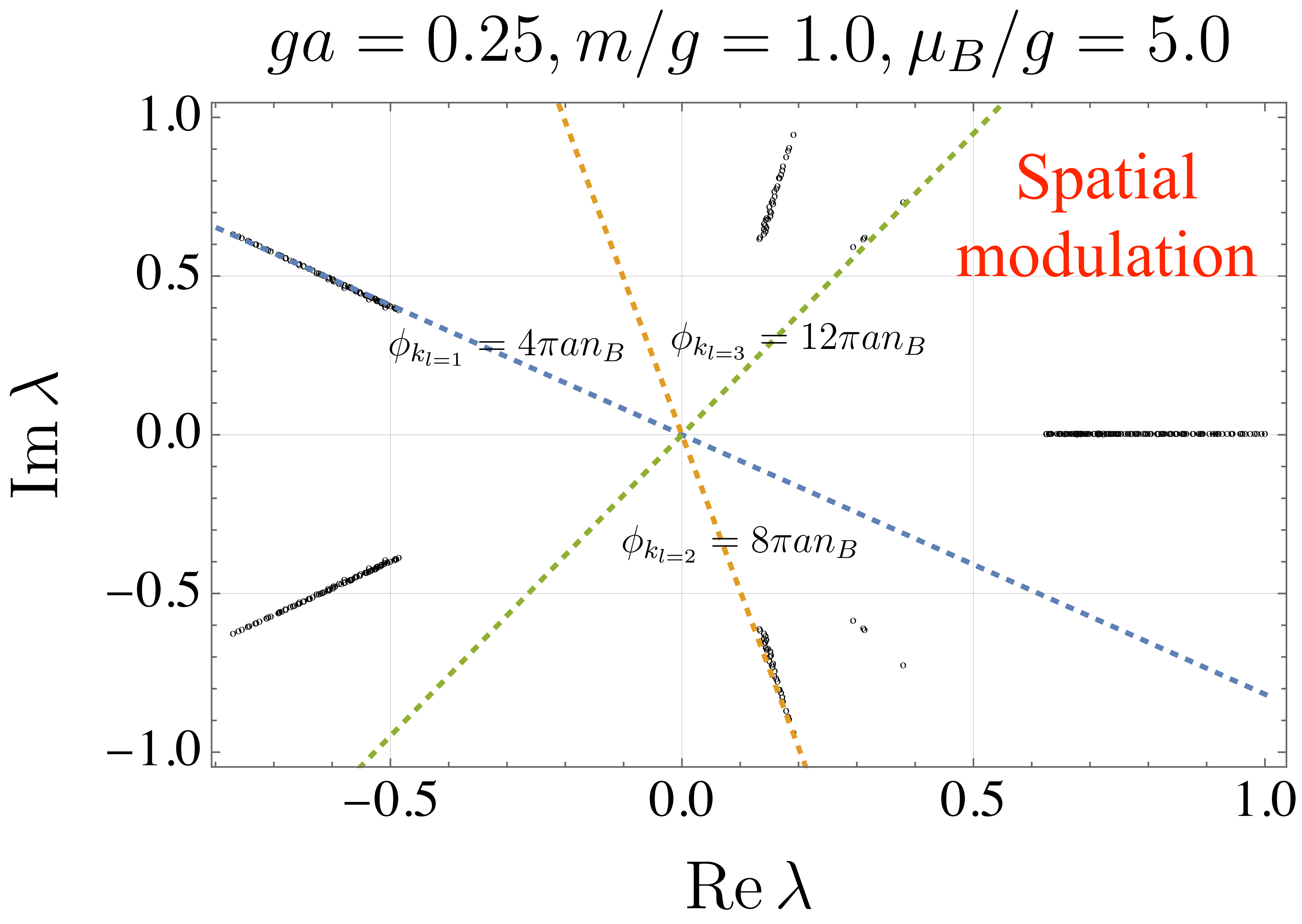
(Caution: When quark is free, $c = N_c = 2$ is observed.)

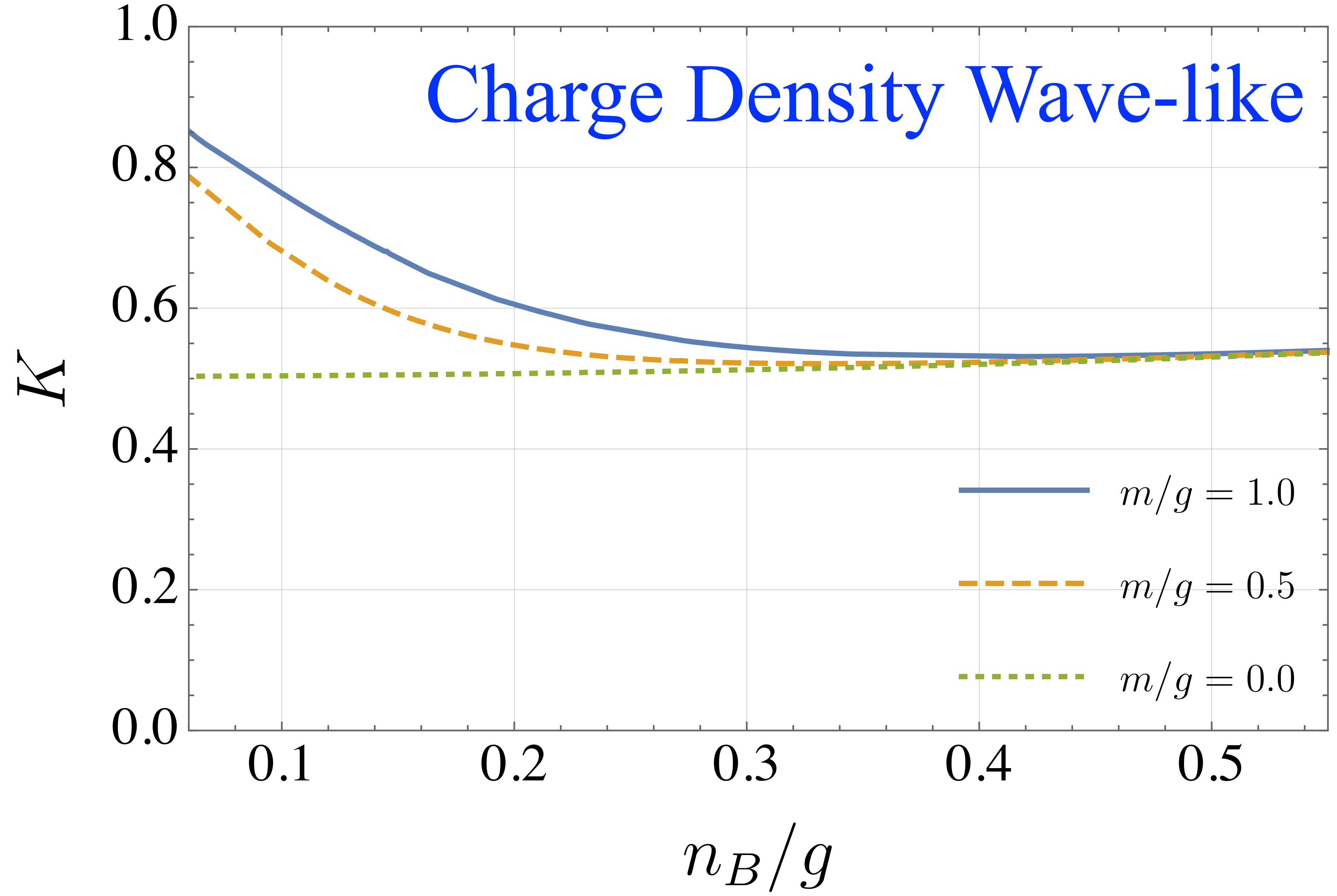
Tomonaga—Luttinger liquid theory

A low-energy behavior is described by

$$H_{\mathrm{TLL}}(\mu_B > M_B) = \int \mathrm{d}x \left[2\pi c_s K \Pi_B^2 + \frac{c_s}{8\pi K}(\partial_1 \phi_B)^2\right]$$

Density-density correlation: $\langle \delta n_B(t, x_1) \delta n_B(t, x_2) \rangle \sim \sum_{\ell=1}^{\infty} B_\ell \frac{\cos(2\pi \ell n_B |x_1 - x_2|)}{|x_1 - x_2|^{2\ell^2 K}}$



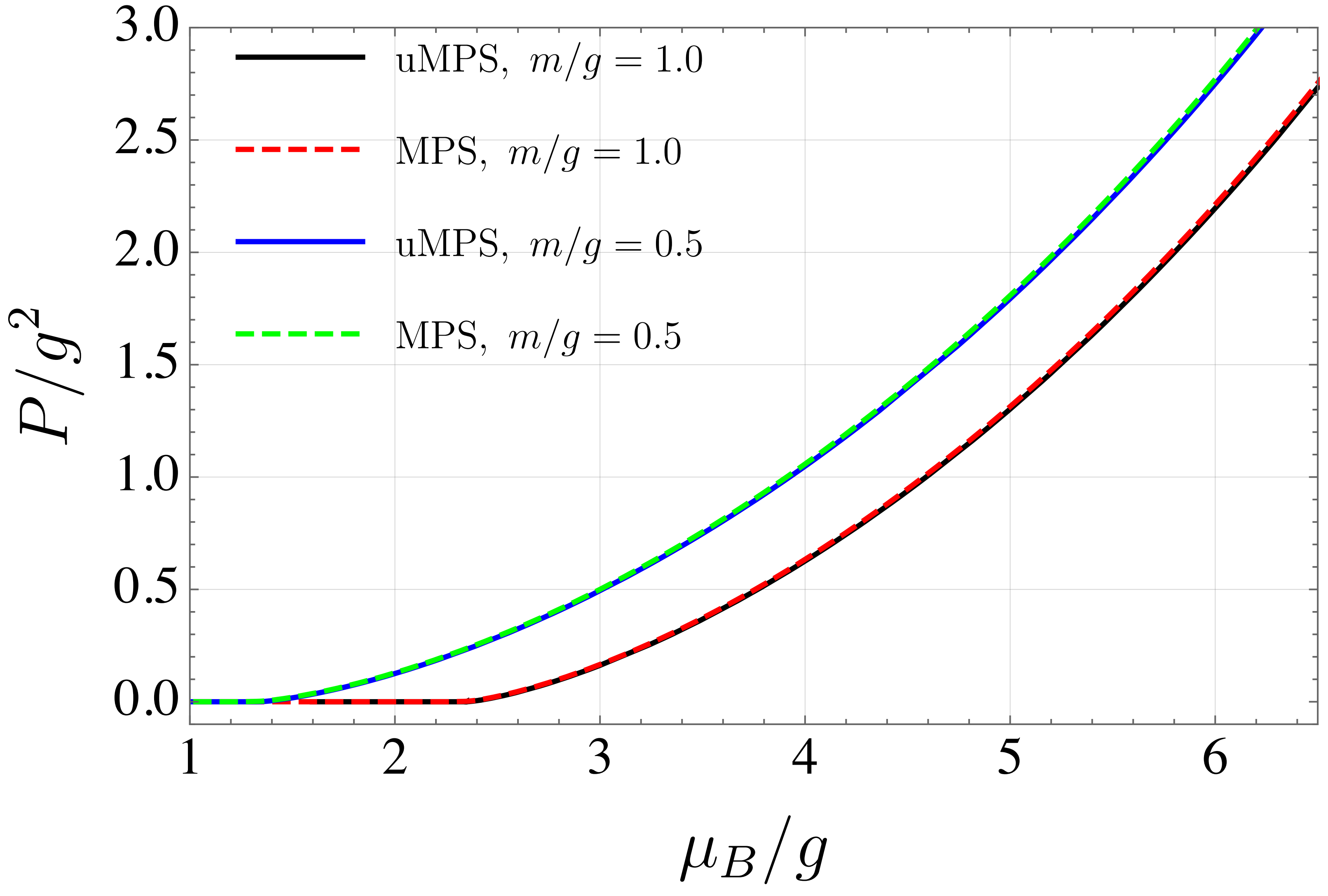


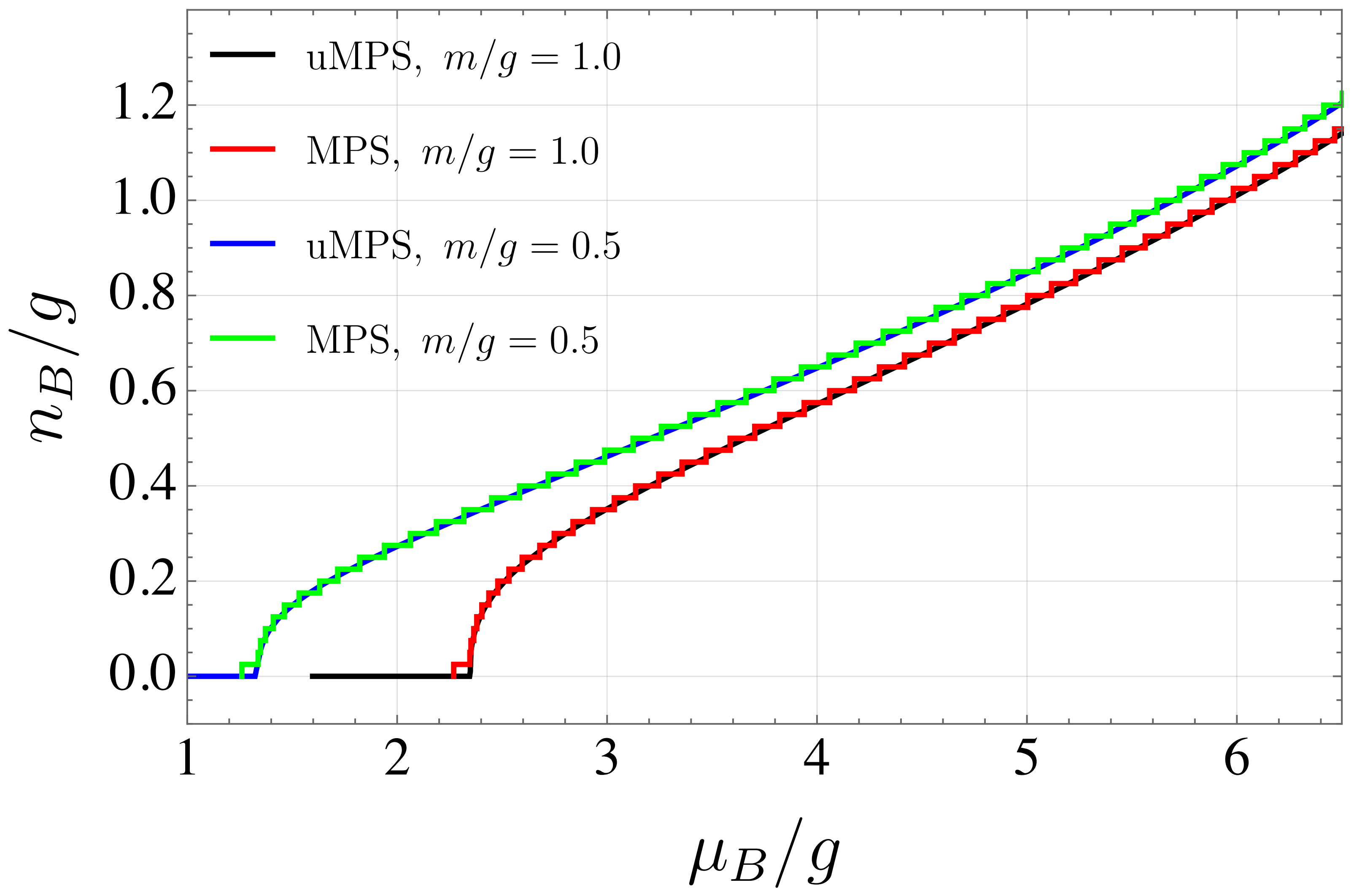
Conclusion

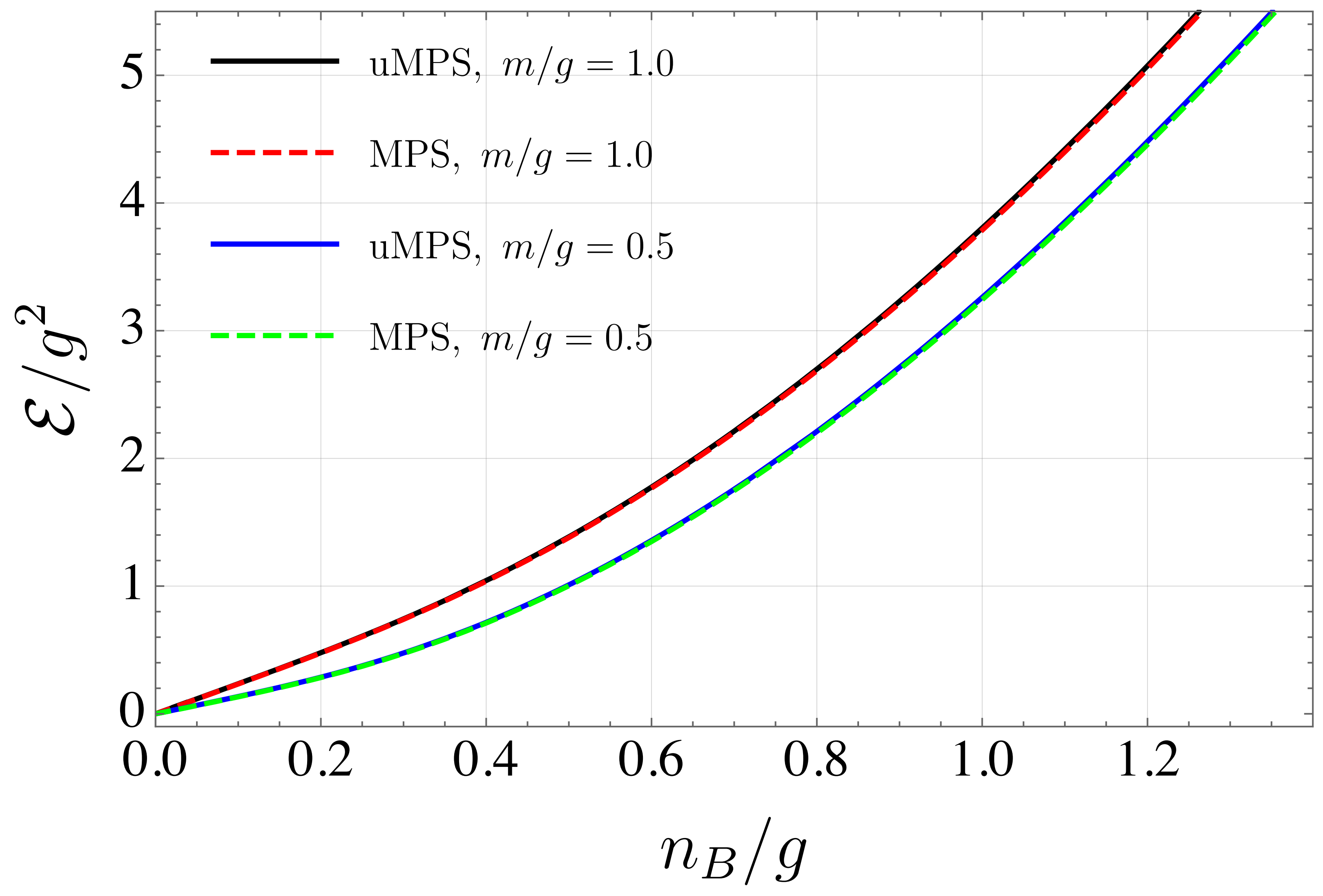
- We explore the two-color single flavor dense QCD in (1+1)-dimension with uniform matrix product states.
- Thermodynamic quantities can be approximated by the free quark picture at a high baryon density.
- The baryon density mode is a relevant mode at finite density (Tomonaga-Luttinger liquid theory).
- We compute central charge of a critical phase, and the quark distribution function.
- The Hamiltonian formulation of lattice gauge theories may offer buenas ideas for understanding various aspects of the problem.

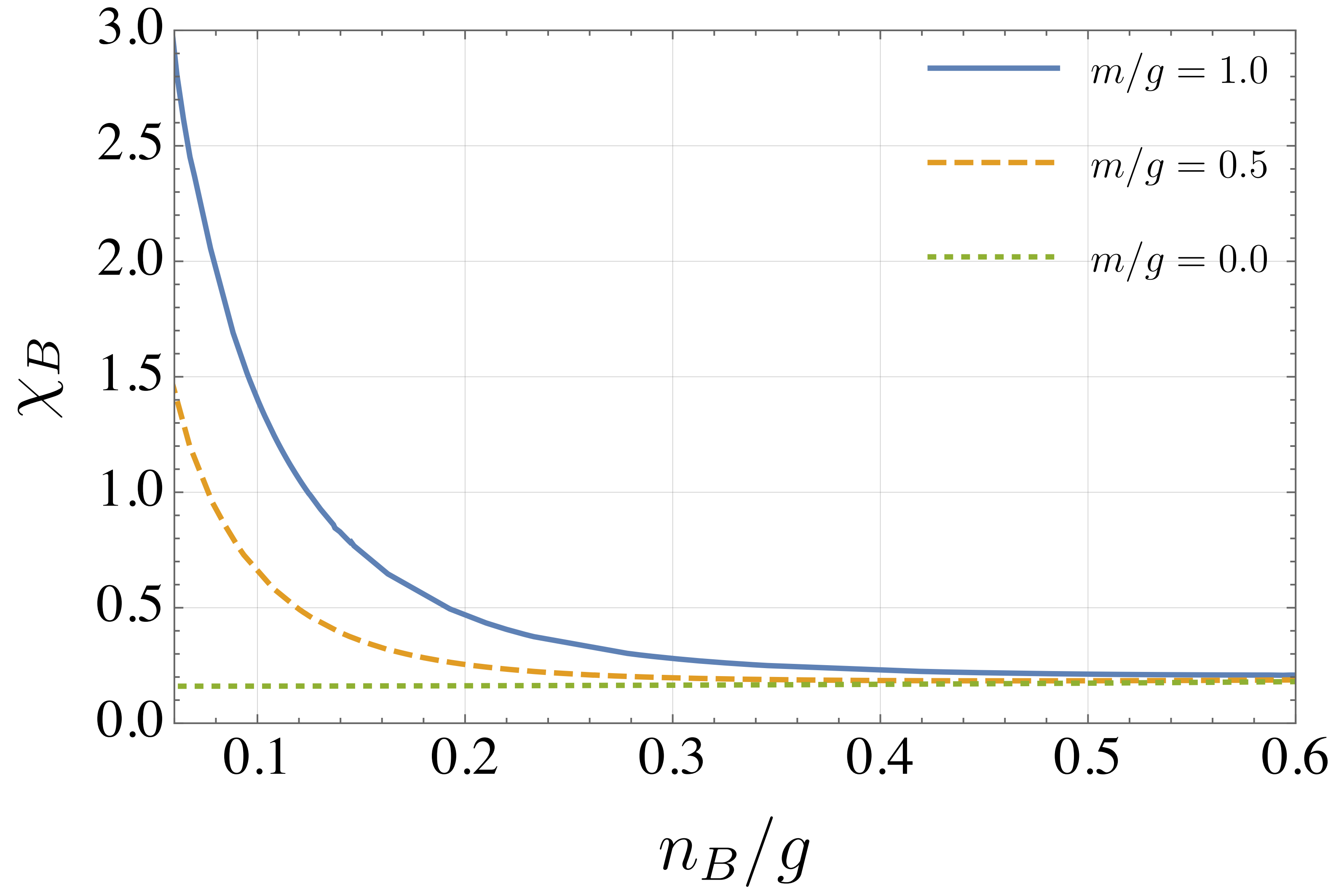
¡Gracias por escuchar!

Thermodynamic quantities









Gauss law constraint

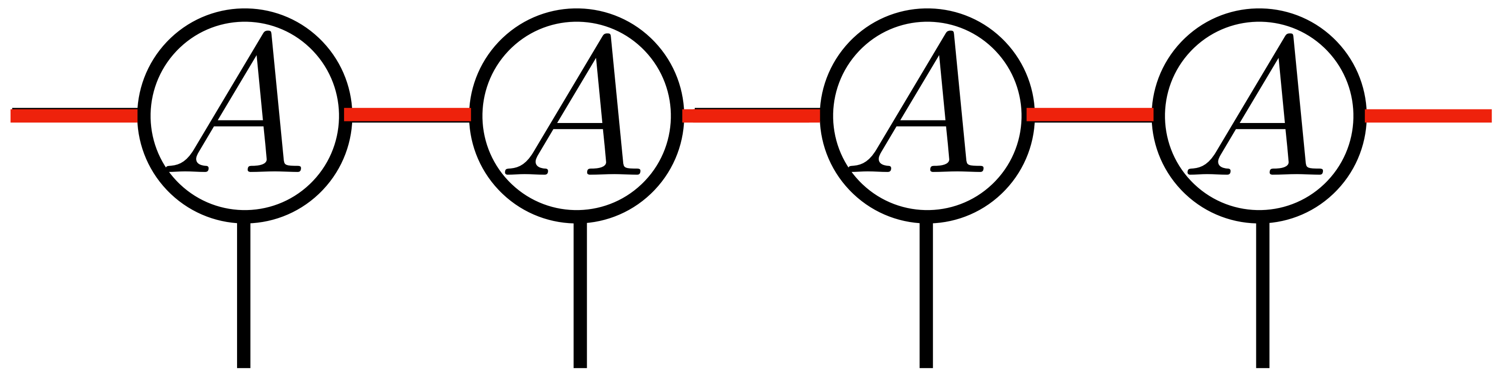
Gauss law constraint (gauge invariance) is imposed on the physical state:

$$G^a(n) = R^a(n) - L^a(n-1) + Q^a(n) \qquad Q^a(n) = \phi^\dagger(n) T^a \phi(n)$$

$$[H_{\mathrm{tot}}, G^a(n)] = 0, \quad G^a(n)|\Psi\rangle = 0 \ (\forall n)$$

Gauge invariant variational ansatz:

$$|\Psi\rangle = \quad \cdots \qquad\qquad \cdots$$



$$[A_n^{j_{L_n}, m_{L_n}, M_n, j_{R_n}, m_{R_n}}]_{J_L \alpha_{J_L}; J_R \beta_{J_R}} = \underline{b^{j_{L_n}, j_{M_n}, q_n, j_{R_n}}_{\alpha_{J_L} \beta_{J_R}}} \frac{1}{\sqrt{d_{j_{L_n}}}} C^{j_{L_n}, m_{L_n}}_{j_{R_n}, m_{R_n}, j_{M_n}, m_{M_n}} \delta^{j_{L_n}}_{J_L} \delta^{j_{R_n}}_{J_R}$$

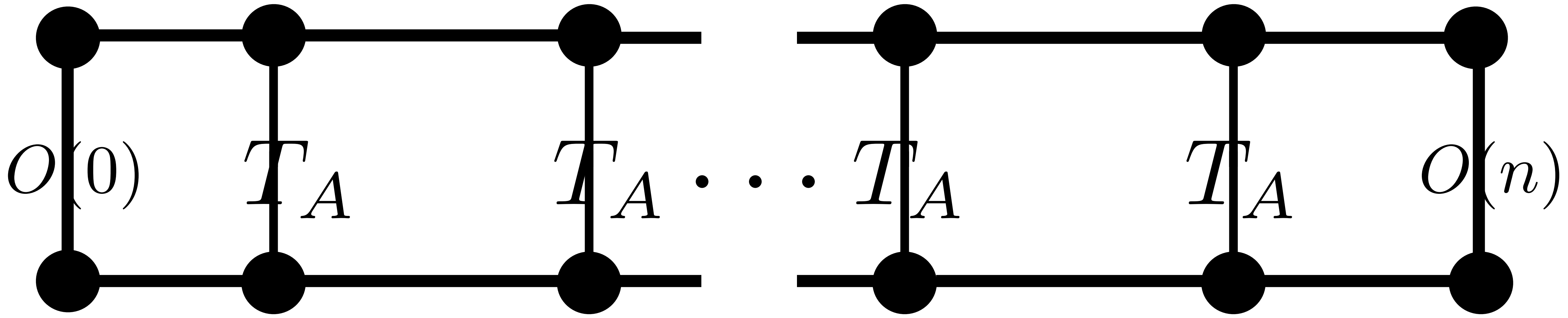
Gauge invariant variational d.o.f

Looking for transfer matrix

To see the modulation accurately, I use the translational invariance.

Two-point functions are expressed as

$$\langle O(0)O(n)\rangle =$$



If modulation exists, there should be complex phases of eigenvalues:

$$|\langle O_1(0)O_2(n)\rangle_{\mathrm{connected}}| \sim (T_A)^n \sim \sum_{j=1}^{D^2-1} e^{-(\epsilon_j - i\phi_j)n} \quad (n \gg 1)$$

Wavenumbers of modulation can be extracted without fitting.

Bosonization

My analytic analysis (with some certain assumptions) indicates the following infrared phases. (Details are omitted.)

$$S = \int d^2x \left[-\frac{1}{4}(F^{\mu\nu\, a} F^{a}_{\mu\nu}) + i\bar{q}\gamma^{\mu} D_{\mu} q - \mu_q q^{\dagger} q + m_q \bar{q} q \right] \quad \mu_B = \mu_q / N_c$$

$\mu_B \ll m_q$

Sine-Gordon model (Gapped)

$$S = \int \mathrm{d}^2x \left[\frac{1}{4\pi}(\partial\phi_B)^2 + M^2(1 - \cos(\phi_B)) \right]$$

$\mu_B \gg m_q$

Tomonaga-Luttinger liquid (Gapless)

$$S = \int \mathrm{d}^2x \, \frac{K}{4\pi} \left[-\frac{1}{v}(\partial_t \phi_B)^2 + v(\partial_x \phi_B)^2 \right]$$

$\partial_x \phi_B \sim q^{\dagger} q \quad \phi_B \sim \phi_B + 2\pi$

Ground state property at $\mu_B = 0$

From the simulation, I obtain the following behavior.

Gapless

Gapped

m

This is in agreement with the analytic study based on the bosonization.

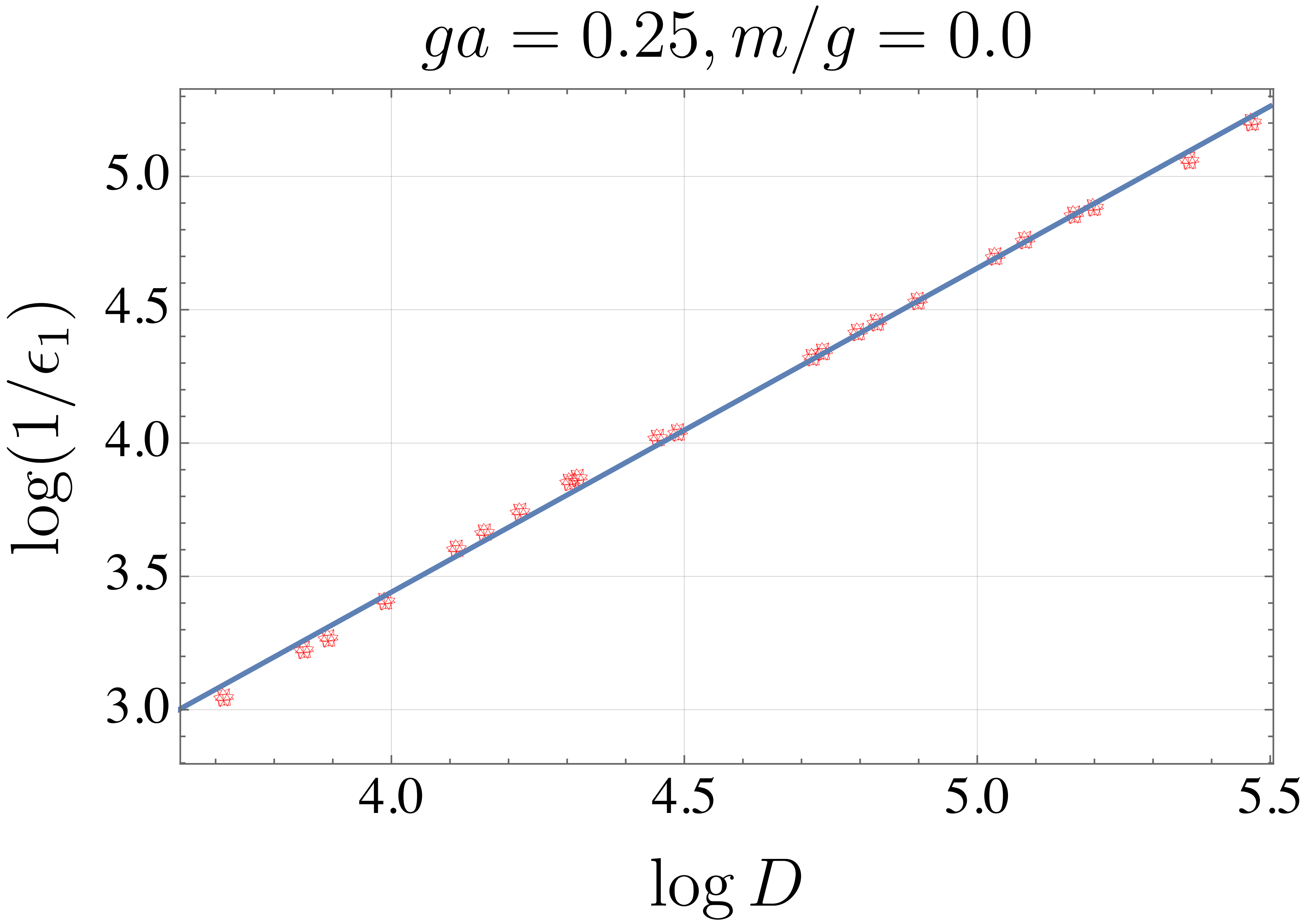
$$S = \int \mathrm{d}^2 x \left[\frac{1}{2} (\partial \phi)^2 + M \cos(\# \phi) \right] \qquad M \propto m$$

(Anti-)Baryon is (anti-)kink for $m \neq 0$ in the bosonic language.

Ground state property at $\mu_B = 0$

Baluni and Steinhardt say that there exists the critical point at $m = 0$, where the baryon becomes gapless.

Correlation length (Logarithmic scale)



Our simulation shows the critical behavior, $\xi \propto D^\kappa$. (D: bond dimension)

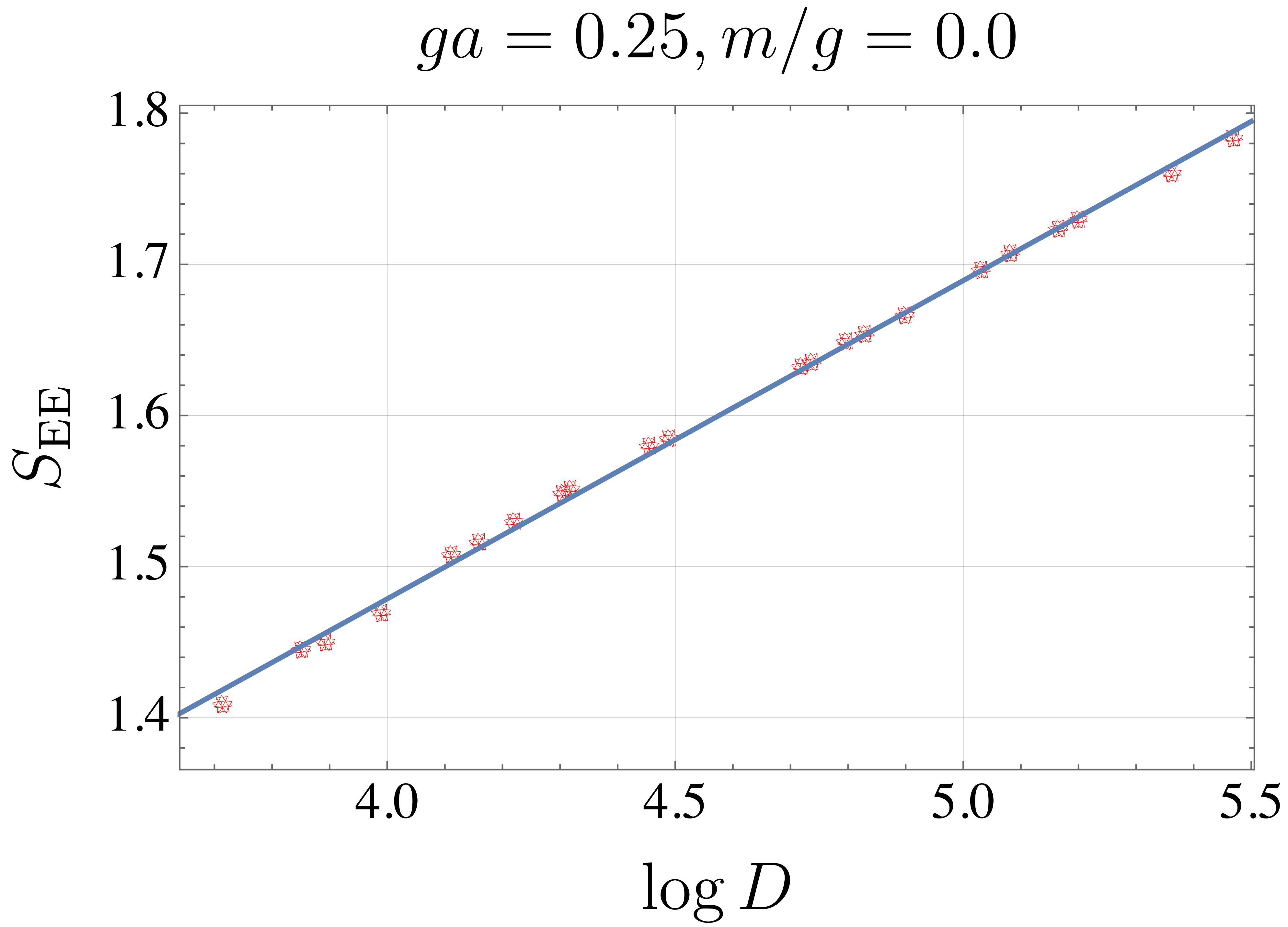
The correlation length is extracted from the transfer matrix.

Ground state property at $\mu_B = 0$

The entanglement entropy tells us the central charge of CFT.

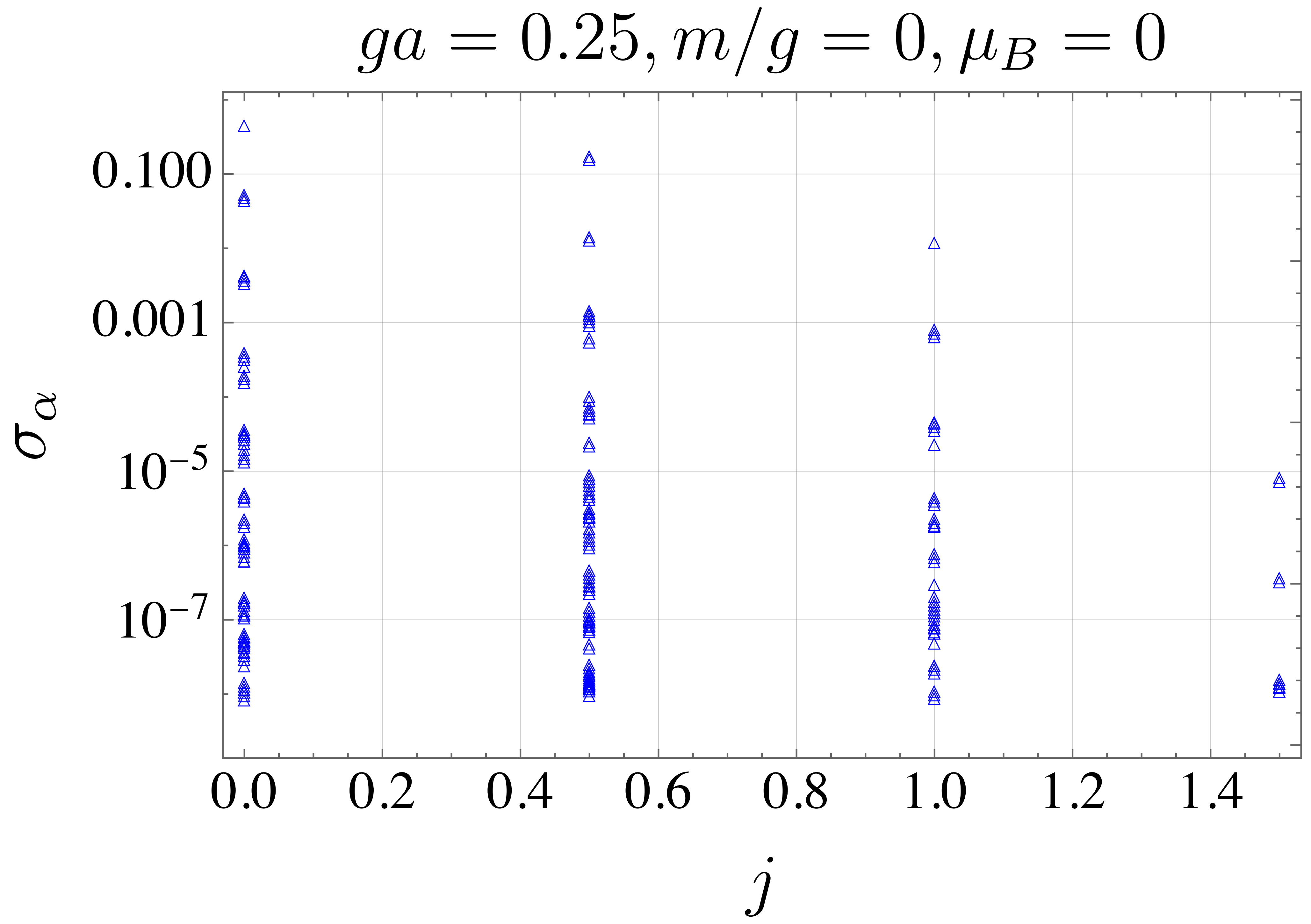
$$S_{\mathrm{EE}} \sim \frac{c}{6} \log(\xi/a)$$

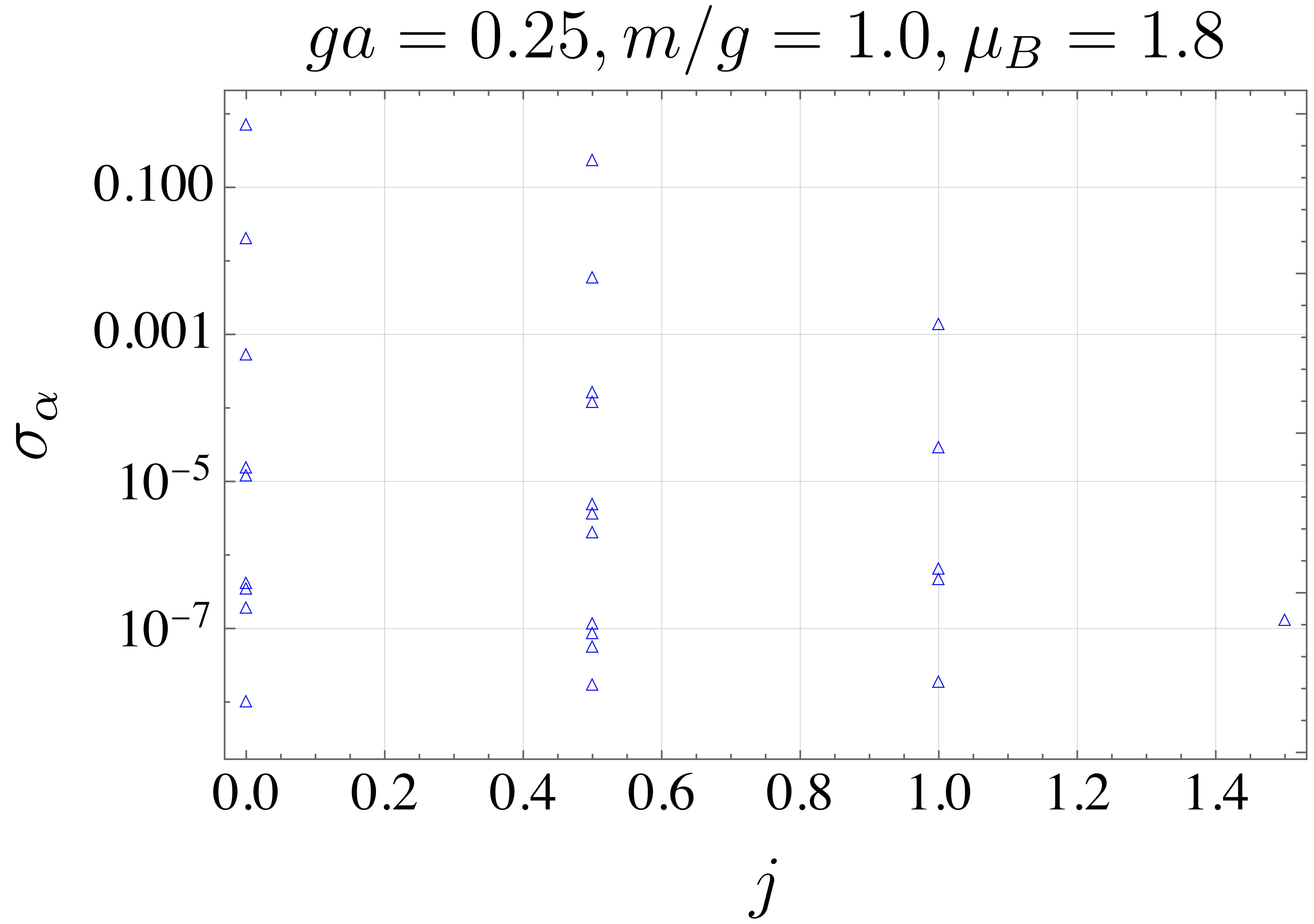
[P. Calabrese, J. Cardy (2009)]



The central charge is in agreement with the free compact boson $c = 1$.

Truncation on electric flux





In uniform MPS, gauge field remains.
We make a certain truncation on the electric flux (SU(2) spin).