# Lattice Dirac operators and K-theory

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# Dirac operator index

$$D\psi = 0 \quad D := \gamma^\mu(\partial_\mu + iA_\mu)$$

we consider
U(1) or SU(N) group

Index theorem [Atiyah & Singer 1963]

$$\overbrace{n_+ - n_-}^{\mathrm{Ind}(D)} = \frac{1}{32\pi^2}\int d^4x \epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

#sol with + chirality

#sol with - chirality

Topological charge =
winding number

Important both in physics and mathematics to understand (position space) topology of gauge fields.

But chiral symmetry and topology are both difficult on a lattice.

# A traditional solution
# = overlap Dirac operator

With the overlap Dirac operator [Neuberger 1998] (or perfect action Dirac operator [Hasenfratz et al.]) satisfying the Gingparg-Wilson relation [1982],

$$\gamma_5 D_{ov} + D_{ov}\gamma_5 = aD_{ov}\gamma_5 D_{ov}$$

a modified chiral symmetry is exact [Luescher 1998],

and the index is well-defined:

$$\mathrm{Ind} D_{ov} = \mathrm{Tr}\gamma_5 \left(1 - \frac{aD_{ov}}{2}\right)$$

[Hasenfratz et al. 1998]

but this definition is limited to even-dimensional periodic square lattice

(whose continuum limit is a flat torus).

# This work = an alternative mathematical formulation of the lattice Dirac operator index.

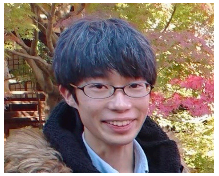
In our formulation,

- Chiral symmetry is NOT necessary : massive Wilson Dirac operator is good enough.
- K theory is used to show the convergence to the continuum Dirac index.
- Wider application than the overlap Dirac operator to the systems with (curved) boundaries and/or mod-two version of the index.

Cf.) Generalized Ginsparg-Wilson relation Clancy, Kaplan and Singh 2023.

# Phys-Math collaborators

Physicists



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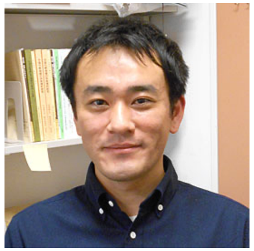

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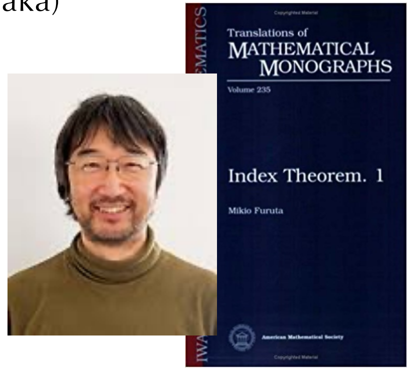

Translations of
MATHEMATICAL
MONOGRAPHS
Volume 235

Index Theorem. 1

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We also had N. Kawai, Y. Matsuki, M. Mori , K. Nakayama and M. Yamashita in the early stage of collaboration.

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2. Chiral symmetry and overlap Dirac index (review)

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7. Summary and discussion

# Continuum differential operator -> Lattice difference operator

Continuum Dirac operator

$$D\psi(x)=\gamma^\mu(\partial_\mu)\psi(x)=\int dp\gamma^\mu(ip_\mu)\tilde{\psi}(p)e^{ipx}$$

(A naïve) lattice Dirac operator

$$D\psi(x)=\gamma^\mu\frac{\psi(x+\hat{\mu}a)-\psi(x-\hat{\mu}a)}{2a}=\int dp\gamma^\mu\frac{e^{ip(x+\hat{\mu}a)}-e^{ip(x-\hat{\mu}a)}}{2a}\tilde{\psi}(p)$$

$$=\int dp\gamma^\mu i\frac{\sin p_\mu a}{a}\tilde{\psi}(p)e^{ipx}.$$

$\hat{\mu}$ : unit vector in μ direction.

$a$ :lattice spacing

which has zero points at $p_\mu = 0, \quad \frac{\pi}{a}$

(phys) Doublers appear!

(math) Ellipticity [uniqueness of zero points] is lost!

# Wilson Dirac operator

$a$ :lattice spacing

$\hat{\mu}$ : unit vector in μ direction.

The Wilson Dirac operator is commonly used in lattice gauge theory.

$$D_W = \sum_\mu \left[ \gamma^\mu \frac{\nabla^f_\mu + \nabla^b_\mu}{2} - \frac{a}{2} \nabla^f_\mu \nabla^b_\mu \right] \quad \nabla^f \psi(x) = \frac{\psi(x+\hat{\mu}a) - \psi(x)}{a}$$

$$\nabla^b \psi(x) = \frac{\psi(x) - \psi(x-\hat{\mu}a)}{a}$$

The additional term corresponds the Laplacian and the Fourier transformation

$$\sum_\mu \gamma^\mu i \frac{\sin p_\mu a}{a} + \sum_\mu \frac{(1-\cos p_\mu a)}{a}$$ = Large mass term except for $p_\mu = 0$

indicates that the doublers cannot excite (recovering ellipticity) due to heavy mass. But chiral symmetry ($Z_2$ grading) is lost instead: $\gamma_5 D_W + D_W \gamma_5 \neq 0.$

This is unavoidable by the Nielsen-Ninomiya theorem [1981]

# Overlap Dirac operator

[Neuberger 1998 Cf. Hasenfratz et al. 1998]

$$D_{ov} = \frac{1}{a}\left(1 + \gamma_5 \mathrm{sgn}(H_W)\right)$$

$$H_W = \gamma_5 (D_W - M)$$

$$M = 1/a$$

satisfies the GW relation: $\gamma_5 D_{ov} + D_{ov}\gamma_5 = a D_{ov}\gamma_5 D_{ov}$

$$\gamma_5(1 - aD_{ov}/2)\gamma_5 D_{ov} + \gamma_5 D_{ov}\gamma_5(1 - aD_{ov}/2) = 0.$$

➡ $\Gamma_5 H + H\Gamma_5 = 0.$

$$H = \gamma_5 D_{ov}, \quad \Gamma_5 = \gamma_5\left(1 - \frac{aD_{ov}}{2}\right)$$

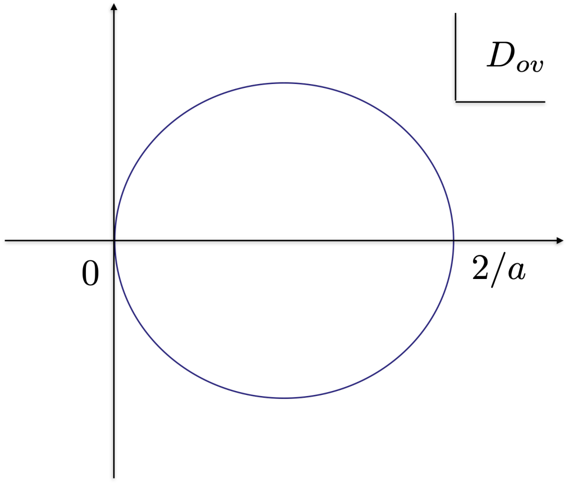
= a modified exact chiral symmetry (but $\Gamma_5^2 \neq 1.$)

[Luescher 1998]

# We can define the index ! [Hasenfratz et al. 1998]

$$\Gamma_5 = \gamma_5 \left(1 - \frac{aD_{ov}}{2}\right)$$

$a$ :lattice spacing



Overlap Dirac spectrum lies on a circle with radius 1/a

For complex eigenmodes $D_{ov}\psi_\lambda = \lambda\psi_\lambda$

$$\psi_\lambda^\dagger \Gamma_5 \psi_\lambda = 0.$$

(therefore, no contribution to the trace)

The real 2/a (doubler poles) do not contribute.

$$\mathrm{Tr}\gamma_5 \left(1 - \frac{aD_{ov}}{2}\right) = \mathop{\mathrm{Tr}}_{\text{zero-modes}} \gamma_5 = n_+ - n_-$$

# But $D_{ov}$ is defined by massive Wilson Dirac operator.

$$D_{ov} = \frac{1}{a}\left(1 + \gamma_5 \mathrm{sgn}(H_W)\right)$$

$$H_W = \gamma_5(D_W - M) \qquad M = 1/a$$

$$\mathrm{Ind} D_{ov} = \mathrm{Tr}\gamma_5 \left(1 - \frac{aD_{ov}}{2}\right) = \underbrace{\mathrm{Tr}\frac{\gamma_5}{2}}_{=0} - \frac{1}{2}\mathrm{Tr}\ \mathrm{sgn}(H_W)$$

$$= -\frac{1}{2}\mathrm{Tr}\ \mathrm{sgn}(H_W)$$

But $D_{ov}$ is defined by massive Wilson Dirac operator.

$$D_{ov} = \frac{1}{a}\left(1 + \gamma_5 \mathrm{sgn}(H_W)\right)$$

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$$= -\frac{1}{2}\mathrm{Tr}\ \mathrm{sgn}(H_W)$$

What is this ???

# η invariant of the Wilson Dirac operator

$$-\frac{1}{2}\mathrm{Tr}\,\mathrm{sgn}(H_W) = -\frac{1}{2}\sum_{\lambda_{H_W}} \mathrm{sgn}(\lambda_{H_W}) = -\frac{1}{2}\eta(H_W)$$

$$H_W = \gamma_5(D_W - M) \qquad M = 1/a$$

This quantity is known as the Atiyah-Patodi-Singer η invariant (of the massive Wilson Dirac operator).

[Atiyah, Patodi and Singer, 1975]

# The Wilson Dirac operator and K-theory

$$\mathrm{Ind}D_{ov} = -\frac{1}{2}\eta(H_W)$$

$$H_W = \gamma_5(D_W - M)$$

$$M = 1/a$$

In this talk, we try to show a deep mathematical meaning of the right-hand side, and try to convince you by K-theory [Atiyah-Hilzebruch 1959, Karoubi 1978…] that the massive Wilson Dirac operator is an equally good or even better object than $D_{ov}$ to describe the gauge field topology.

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# What is fiber bundle?

(minimum for physicists)

A united manifold of space (= base manifold) & field (fiber)

$$\phi(x) \;\to\; (x, \phi) \in \underset{\substack{\text{Spacetime}\\ \text{= base space}}}{X} \times \underset{\substack{\text{Field space}\\ \text{= fiber space}}}{F}$$

The direct product structure is realized only locally.
In general, it is “twisted” by gauge fields (connections).
In mathematics, the (isomorphism class of) total space is often denoted by $E$ .

When the fiber=vector space, we call it vector bundle.

# Vector bundles classified by $K^0(X)$ group

The element of $K^0(X)$ group is given by $[E_1, E_2]$

[ ] denotes the equivalence class (definition is given later).

Equivalently, we can consider an operator and its conjugate,

$$D_{12} : E_1 \to E_2 \qquad D_{12}^{\dagger} : E_2 \to E_1$$

* To be precise, D acts on the sections of E.

to represent the same element by $[D, \gamma]$

where

$$D = \begin{pmatrix} & D_{12} \\ D_{12}^{\dagger} & \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

which act on

$$E = E_1 \oplus E_2,$$

* $K^0$ group describes classification of Dirac operator which anti-commutes with chirality operator.

# K-theory pushforward

When we are interested in global structure only,
we can forget about details of the base manifold X by taking
the so-called K-theory pushforward ( ~ “integration over X” ) :

$$G : K^0(X) \to K^0(\text{point})$$
$$[D_E, \gamma_E] \mapsto [D_{\text{Matrix}}, \gamma_{\text{Matrix}}]$$

The map just forgets vector bundle E.

Cf. In lattice QCD code, we have a similar map :
LinearAlgebra.Matrix(Dirac) .

A lot of information is lost but
one (the Dirac operator index) remains.

# Suspension isomorphism

The "point" can be suspended to an interval:

There is an isomorphism between

$$K^0(\text{point}) \cong K^1(I, \partial I)$$

One-parameter deformation of
Dirac operator

$$[D, \gamma] \leftrightarrow [\{D_t\}_{-1 \le t \le 1}]$$

$$I = [-1, +1]$$

superscript "1" reflects removal of the chirality operator.
* The Dirac operator must become one-to-one (no zero $\partial I$ mode) at the two endpoints :

Physical meaning of isomorphism will be given soon later .

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# Atiyah-Singer index (in continuum theory)

Index theorem

$$\overbrace{n_+ - n_-}^{\mathrm{Ind}(D)} = \frac{1}{32\pi^2}\int d^4x\,\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

#sol with + chirality

#sol with - chirality

In the standard formulation, we need a massless Dirac operator and its zero modes with definite chirality : $[D,\gamma] \in K^0(\mathrm{point})$
But we will show that it is isomorphic to

$$[\{\gamma(D-ms)\}|_{s\in[-1,1]}] \in K^1(I,\partial I) \qquad I := [-1,1]$$

# Eigenvalues of continuum massive Dirac operator

$H_s = \gamma_5(D - ms)$ on a Euclidean even-dimensional manifold.

For $D\phi = 0,$ $$H_s\phi = -\gamma_5 ms\phi = \underbrace{\pm}_{\text{chirality}} ms\phi.$$

For $D\phi \neq 0,$ $\{H_s, D\} = 0.$

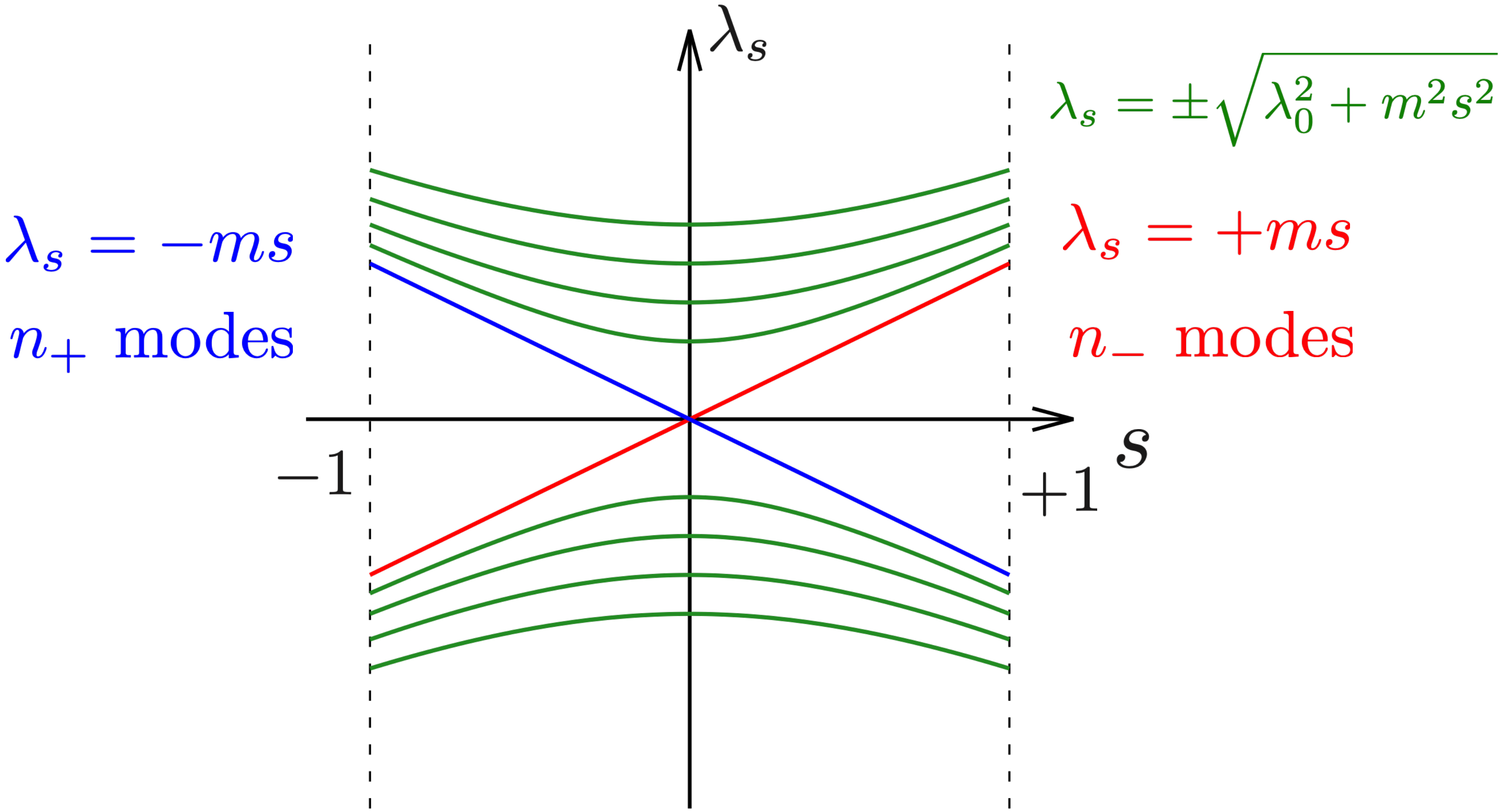
The eigenvalues are paired: for $H_s\phi_{\lambda_s} = \lambda_s\phi_{\lambda_s}$

$$H_s D\phi_{\lambda_s} = -\lambda_s D\phi_{\lambda_s}$$

As $H_s^2 = -D^2 + (ms)^2$ , we can write them

$$\lambda_s = \pm\sqrt{\lambda_0^2 + (ms)^2}$$

# Spectrum of $H_s = \gamma_5(D - ms) \quad s \in [-1, 1]$



# Dirac index = Spectral flow = η invariant

$n_+$ = # of zero-crossing eigenvalues from - to +

$n_-$ = # of zero-crossing eigenvalues from + to -

$n_+ - n_-$ =: spectral flow of $H_s = \gamma_5(D - ms)$
$s \in [-1, 1]$

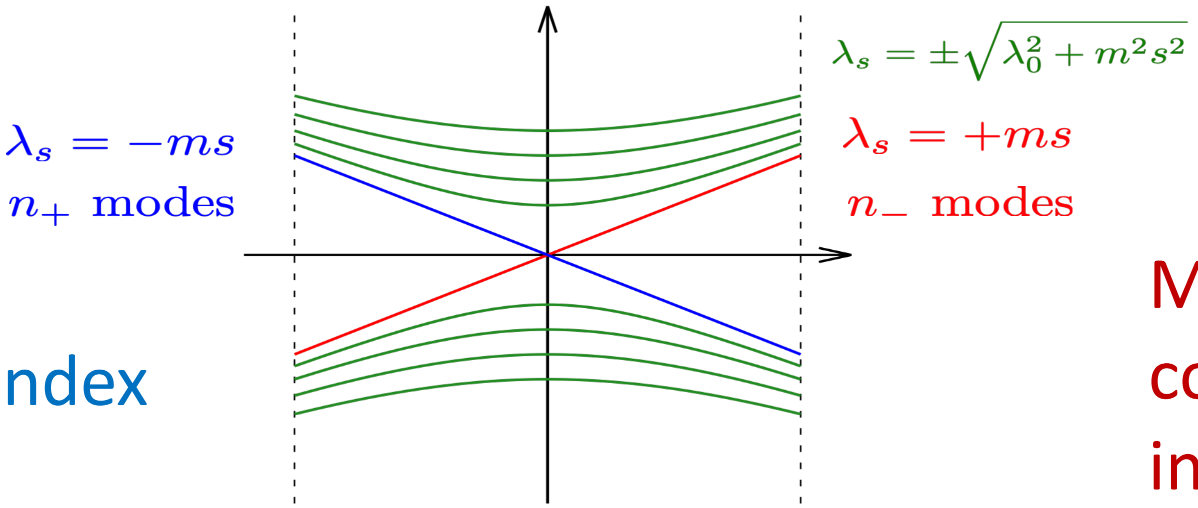
Equivalent to the eta invariant: whenever an eigenvalue crosses zero,

$\eta(H_s)$ jumps by two. $\eta(H) = \sum_{\lambda \geq 0}^{reg} - \sum_{\lambda < 0}^{reg}$

$$-\frac{1}{2}\eta(H_{+1}) + \frac{1}{2}\eta(H_{-1}) = n_+ - n_-.$$

Pauli-Villars subtraction

# Suspension isomorphism in K-theory



Massless:
counting index
by points

Massive:
counting
index by lines

$$K^0(\text{point}) \cong K^1(I, \partial I)$$

point

with chirality operator

line=interva

without chirality operator

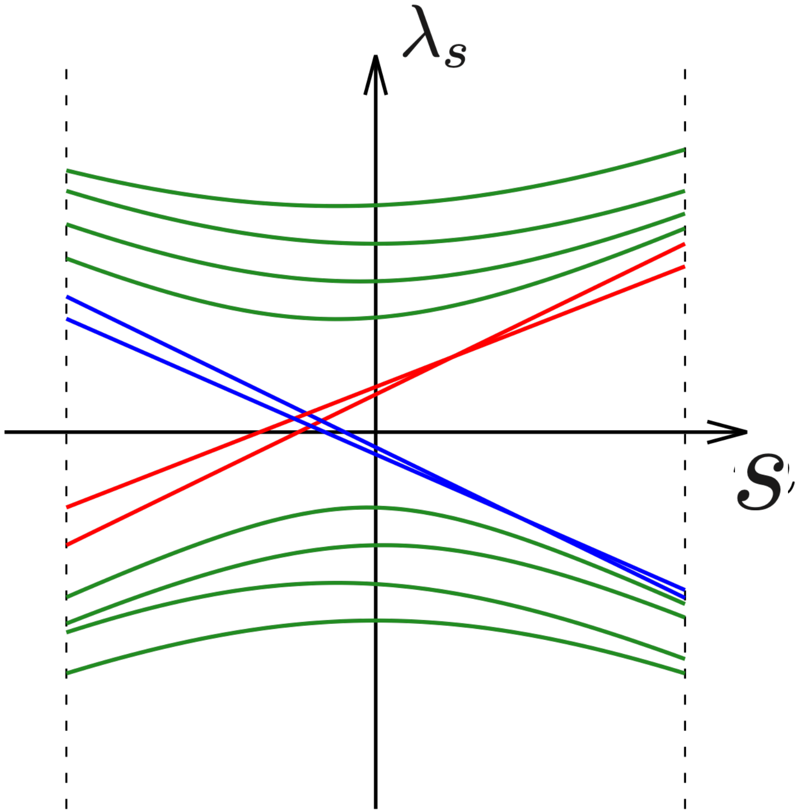
⇨ The two definitions of the index agree.

## With chiral symmetry breaking regularization (on a lattice), counting points (massless) is difficult but counting lines (massive) still works.

Standard massless definition:

Where is m=0?

What are zero modes?



Spectral flow of massive Dirac:

We can still count the crossing lines

Note) this fact was known even before overlap Dirac by Itoh-Iwasaki-Yoshie 1982 and other literature, but its K-theoretic meaning was not discussed. [See also Adams, Kikukawa-Yamada, Luescher, Fujikawa, and Suzuki]

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Counting lines (massive, $K^1$) is easier than counting points (massless, $K^0$).

5. Generalization (in continuum)

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# Real Dirac operators and the mod-two index

In previous slides, we considered general complex Dirac operators, and we have shown

$$\mathrm{Ind}D = \mathrm{sf}[\{\gamma_5(D-ms)\}_{s\in[-1,1]}] = -\frac{1}{2}\eta^{\mathrm{reg.}}(\gamma_5(D-m))$$

$$\in K^1(I,\partial I)$$

For real Dirac operators, in SU(2) gauge theory in 5D (origin of Witten anomaly), for example, K theory knows that

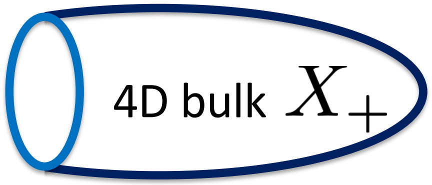
$KO^0(I,\partial I)$ characterized by the mod-two spectral flow describes the mod-two index (in odd dimensions).

# Atiyah-Patodi-Singer (APS) index

4D example:

[Atiyah-Patodi-Singer 1975]

$$\mathrm{Ind}_{\mathrm{APS}}[D|_{X_+}] = \frac{1}{16\pi^2}\int FF - \frac{1}{2}\eta(iD^{\mathrm{3D}})$$



LHS is defined by # of chiral zero modes of a massless Dirac operator on $X_+$ with a non-local (causality-breaking) boundary condition (APS condition allowing only positive/negative eigenmode components of boundary Dirac operators), which is unlikely to be realized in physics.

In continuum theory [F, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita 2019], we relate this to the domain-wall fermion Dirac operator.

# Theorem 1 [F-Furuta-Matsuo-Onogi-Yamaguchi-Yamashita 2019]

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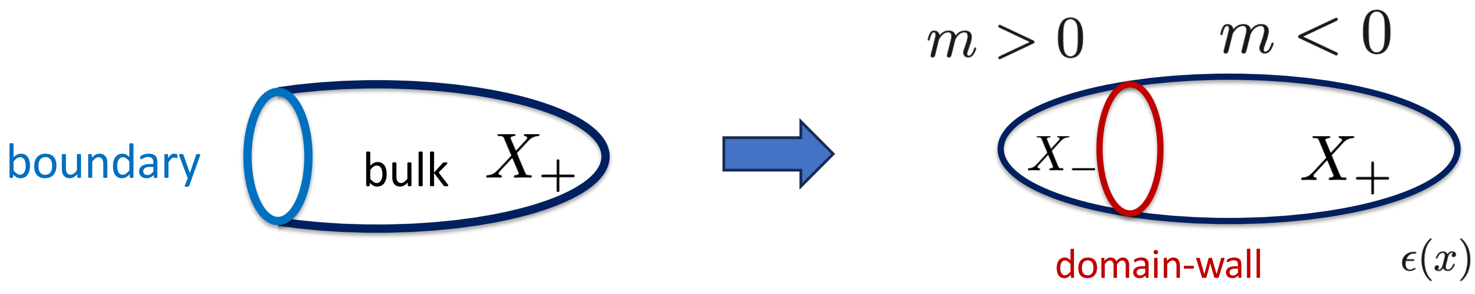


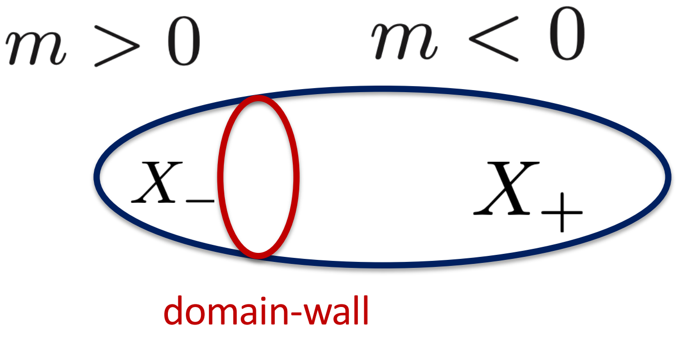
$$\epsilon(x) = \begin{cases} +1 & \text{for } x \in X_+ \\ -1 & \text{for } x \in X_- \end{cases}$$

For any APS index of a massless Dirac operator on an even-dimensional Riemannian manifold $X_+$ with boundary, there exists a massive Dirac operator on a closed manifold $X = X_+ \cup X_-$ with domain-wall at the location of the original boundary and its spectral flow (or η invariant) is equal to the original index.

$$\mathrm{Ind}_{\mathrm{APS}}[D|_{X_+}] = \mathrm{sf}\left[\gamma\left(D_X + m\kappa(x,s)\right)\right] = -\frac{1}{2}\eta^{\mathrm{reg.}}(\gamma(D_X - m\epsilon))$$

$$\kappa(x,s) = \begin{cases} -s & \text{for } x \in X_+ \\ +1 & \text{otherwise} \end{cases}$$

# K-theory tells that the massive expression unifies various types of the index formulas.



- Atiyah-Singer and Atiyah-Patodi-Singer indices are unified on a closed manifold. We do not need any nonlocal boundary conditions.
- Application to the mod-two version is straightforward (counting # of pairs of zero-crossing eigenmodes).
- Available both in odd and even dimensions.
- Chiral symmetry is NOT necessary: application to the lattice gauge theory is straightforward (NEXT).

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✓ 5. Generalization

Massive fermion Dirac operator unifies various types of the index.

6. Mathematical theorem on the lattice

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# Dirac operator in continuum theory

E : Complex vector bundle

Base manifold M: 2n-dimensional flat torus $T^{2n}$

Fiber F : vector space of rank r with a Hermitian metric

Connection : Parallel transport with gauge field $A_i$

D : Dirac operator on sections of E

$$D_{\mathrm{cont.}} = \gamma_i(\partial_i + A_i)$$

Chirality ($Z_2$ grading) operator: $\gamma = i^n \prod_i \gamma_i$

$$\{\gamma, D\} = 0, \{\gamma, \gamma_i\} = 0.$$

# Wilson Dirac operator on a lattice

We regularize $T^{2n}$ is by a square lattice with lattice spacing $a$

link variables : $$U_k(\boldsymbol{x}) = P\exp\left[i\int_0^a A_k(\boldsymbol{x}')dl\right],$$

$$D_W = \sum_i \left[\gamma^i \frac{\nabla_i^f + \nabla_i^b}{2} - \frac{a}{2}\nabla_i^f \nabla_i^b\right]$$

Wilson term

* In mathematics, the Wilson term is important in that it guarantees the ellipticity.

$$a\nabla_i^f \psi(\boldsymbol{x}) = U_i(\boldsymbol{x})\psi(\boldsymbol{x}+\boldsymbol{e}_i) - \psi(\boldsymbol{x})$$

$$a\nabla_i^b \psi(\boldsymbol{x}) = \psi(\boldsymbol{x}) - U_i^\dagger(\boldsymbol{x}-\boldsymbol{e}_i)\psi(\boldsymbol{x}-\boldsymbol{e}_i)$$

# Definition of $K^1(I, \partial I)$ group

The group element is given by equivalence class:

$[D_s]$ having the same spectral flow.

Note: $K^1$ group does NOT require chirality operator and does NOT distinguish the continuum and lattice operators.

* Precise definition : a vector bundle (Hilbert bundle) with

Base space $I$ = range of the parameter $s \in I = [-1, 1]$

boundary $\partial I$ = $\pm$1 points

Fiber space $\mathcal{H}$ = Hilbert space to which D acts

$D_s$ : one-parameter family labeled by s.

We assume that $D_{\pm 1}$ has no zero mode.

# Definition of $K^1(I,\partial I)$ group

Group operation: $[D_s^1]\pm[D_s^2]=[\begin{pmatrix} D_s^1 & \\ & \pm D_s^2 \end{pmatrix}]$

Identity element: $[D_s]|_{\mathrm{Spec.flow}=0}$

We compare the continuum and Wilson Dirac operator, by considering the "combined" Dirac operator

$$\hat{D}_s=\begin{pmatrix} \gamma(D_{\mathrm{cont.}}+m\kappa(x,s)) & tf_a \\ tf_a^* & -\gamma(D_W+m\kappa(x,s)) \end{pmatrix} \quad \kappa(x,s)=\begin{cases} -s & \text{for } x\in X_+ \\ +1 & \text{otherwise} \end{cases}$$

where $f_a^*$ and $f_a$ are "mixing mass term" with some "nice" mathematical properties:

$$f_a^*f_a=\mathrm{Identity}_{\mathcal{H}^{\mathrm{lat.}}}+O(a),\quad f_af_a^* \quad =\mathrm{Identity}_{\mathcal{H}^{\mathrm{cont.}}}+O(a).$$

If its spectral flow=0, then, the two Dirac operators have the same index.

# Theorem 2

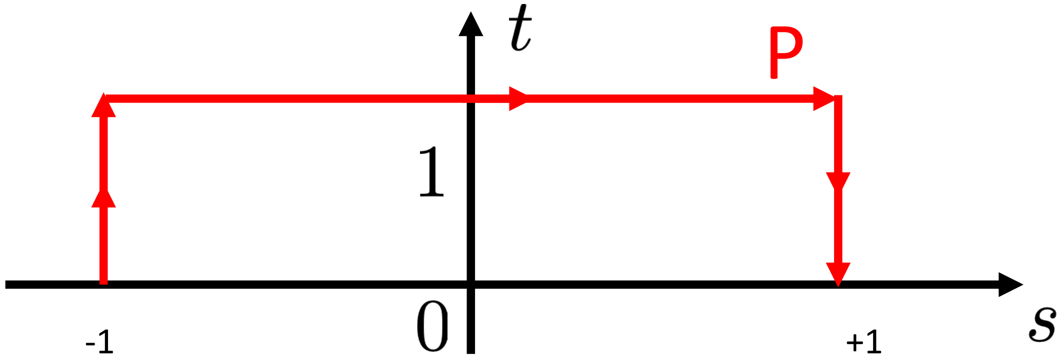
[Aoki, Fujita,F, Furuta, Matsuo, Onogi, Yamaguchi 2026]

Consider a continuum-lattice combined Dirac operator

$$\hat{D}_{s,t} = \begin{pmatrix} \gamma(D_{\mathrm{cont.}} + m\kappa(x,s)) & tf_a \\ tf_a^* & -\gamma(D_W + m\kappa(x,s)) \end{pmatrix}$$

$$\kappa(x,s) = \begin{cases} -s & \text{for } x \in X_+ \\ +1 & \text{otherwise} \end{cases}$$

on the path P :



# Theorem 2 (continued)

There exists a finite lattice spacing $a_0$ s.t. for any $a < a_0$

$$\hat{D}_{s,t} = \begin{pmatrix} \gamma(D_{\text{cont.}} + m\kappa(x,s)) & tf_a \\ tf_a^* & -\gamma(D_W + m\kappa(x,s)) \end{pmatrix}$$

is invertible (has no zero mode) on the staple-like path P [which is a sufficient condition for Spec.flow=0]

⇨ $\{\gamma(D_{\text{cont.}} + m\kappa(x,s))\}$ and $\{\gamma(D_W + m\kappa(x,s))\}$

have the same spec.flow

$$\frac{1}{2}\eta(\gamma(D - m\varepsilon(x)))^{\text{PV reg.}} = \frac{1}{2}\eta(\gamma(D_W - m\varepsilon(x)))$$

In our work, the proof is given by contradiction.

The continuum and lattice indices agree.

* Application to the mod-two case is straightforward.

# Numerical evaluation (simple recipe)

You can forget all about K-theory, suspension isomorphism, η invariant, and so on.

Just compute the near-zero Wilson Dirac eigenvalue spectrum changing mass (with/without domain-wall) from -1 to 1.

Then, the number of eigenvalues (pairs) crossing zero gives you

Atiyah-Singer index
Atiyah-Patodi-Singer index with boundaries
Their mod-two versions

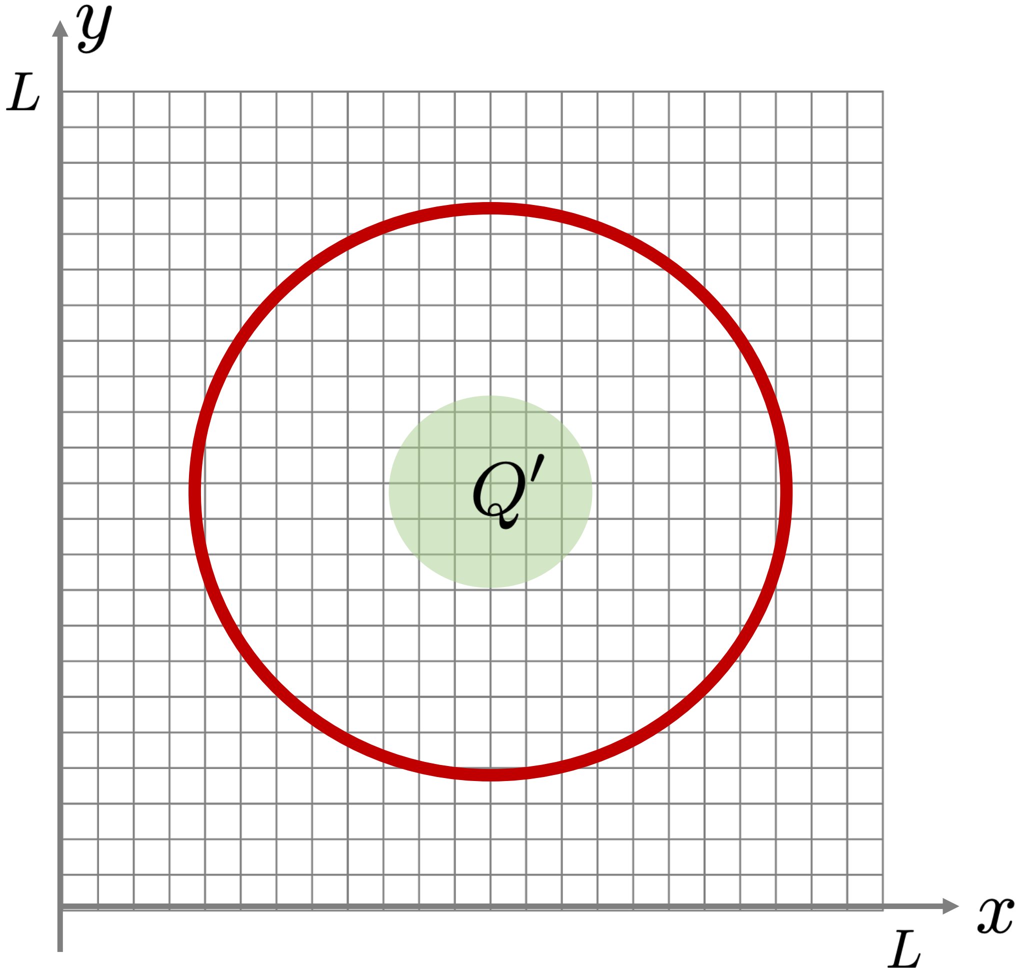
# Example1. APS index on a 2D disk

We put a circular curved domain-wall : m=-s/a inside, m=+1/a outside and change s from -1 to 1.
We put U(1) flux Q' and numerically check if the APS index theorem holds or not.

$$-\frac{1}{2}\eta(\gamma_5 D_{DW}) = \underbrace{\frac{1}{2\pi}\int F}_{=Q'} - \frac{1}{2}\eta(iD^{1D})$$

L=33, DW radius=10, flux radius=6.

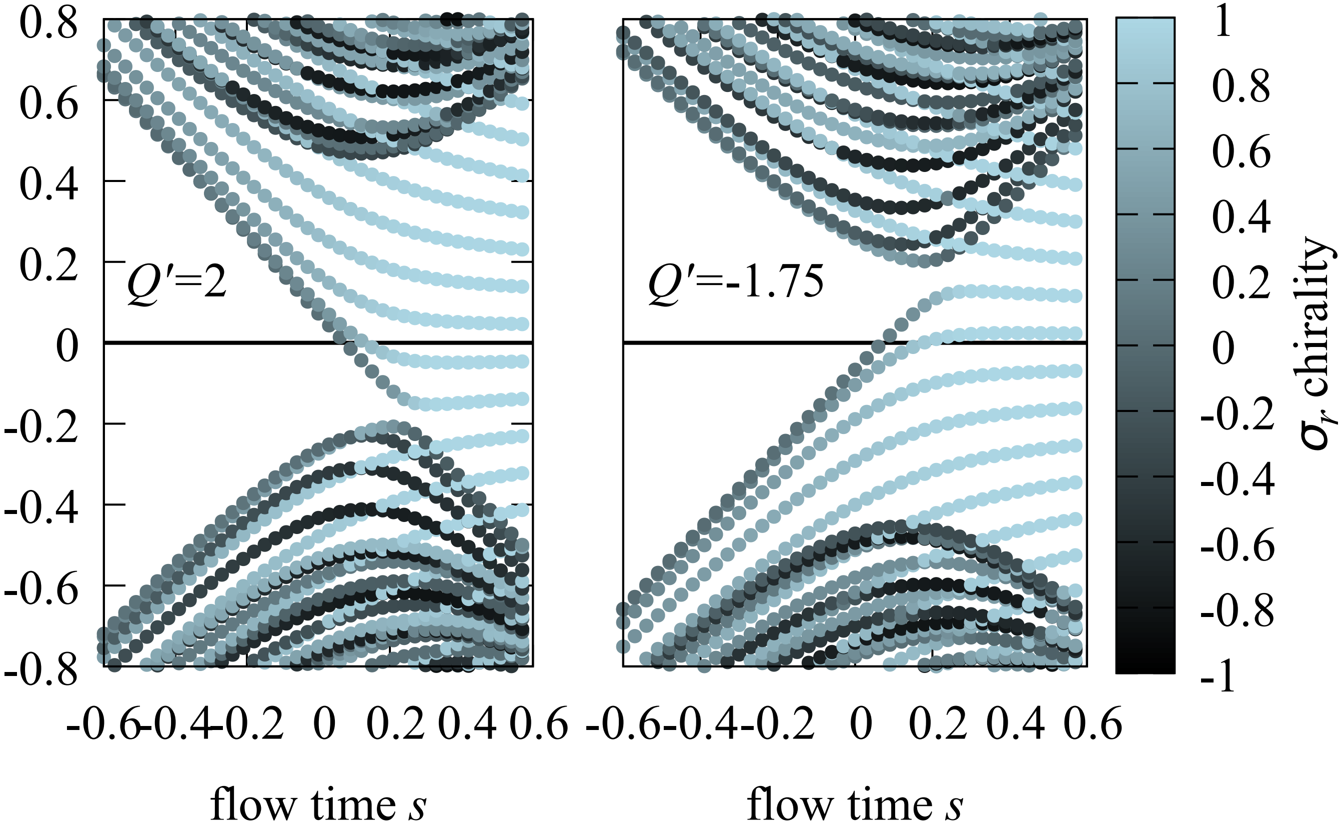
[Cf. Aoki-F, 2022,2023]



# Dirac spectrum on a 2D disk

$$-\frac{1}{2}\eta(\gamma_5 D_{DW}) = \underbrace{\frac{1}{2\pi}\int F}_{=Q'} - \frac{1}{2}\eta(iD^{1D})$$

$$\eta(H) = \sum_{\lambda\geq 0}^{reg} - \sum_{\lambda<0}^{reg}$$



$$-\frac{1}{2}\eta(iD^{1D}) = 0. \qquad -\frac{1}{2}\eta(iD^{1D}) = -0.25$$

Edge-localized chiral :

$$\sigma_r = (\sigma_1 x + \sigma_2 y)/r \sim 1$$

modes appear on the 1-dimensional circle domain-wall = the source of boundary eta invariant.

Consistent with the APS Index theorem.

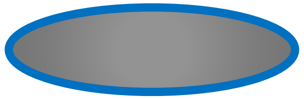
# Numerical test for Majorana $S^1$ domain-wall fermion

Free Wilson Dirac operator is real:

$$iH_m = \sigma_1 \partial_x + \sigma_3 \partial_y + i\sigma_2 (W + M(x))$$

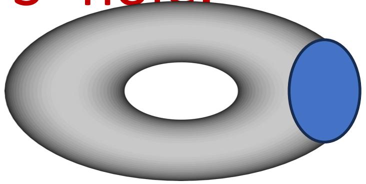
Mass change inside the domain-wall =disk

$M(x) = -ms \;\; \text{for } x < r_0$

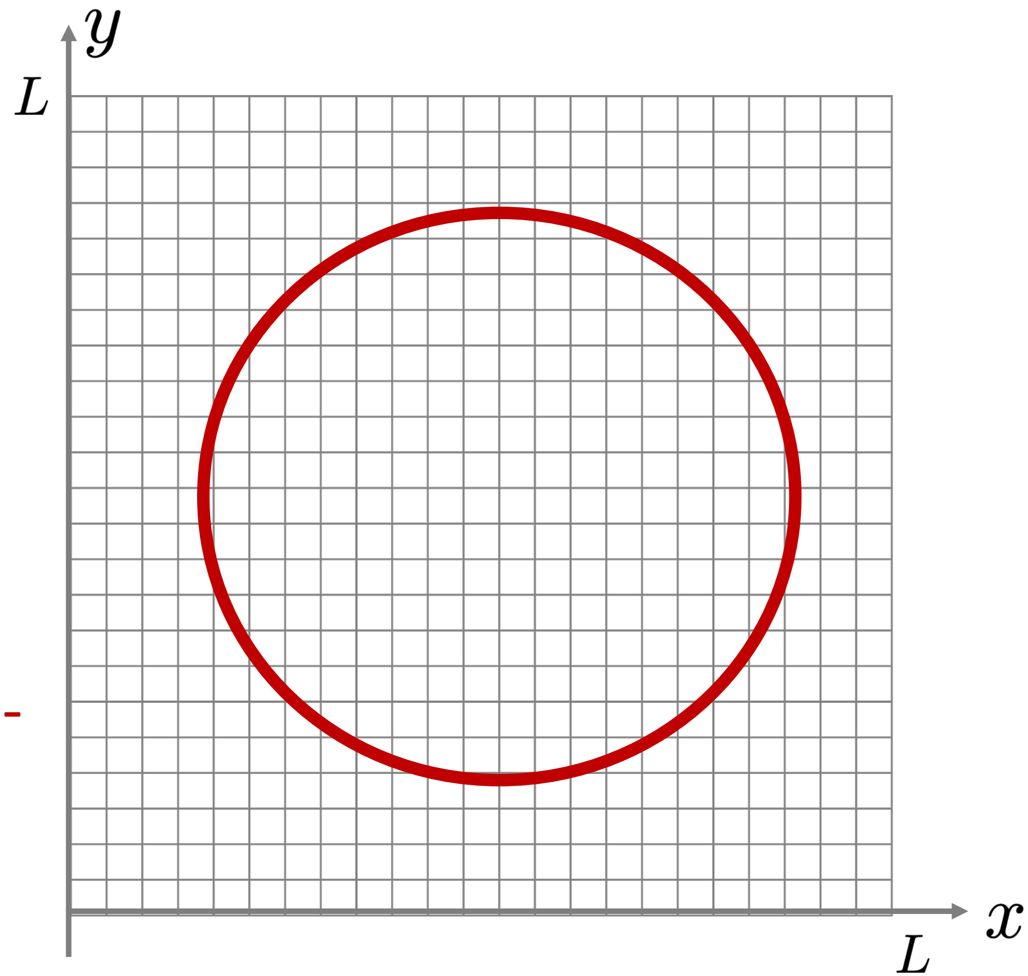


Mass change outside the domain-wall= torus with a $S^1$ hole.

$M(x) = -ms \;\; \text{for } x > r_0$

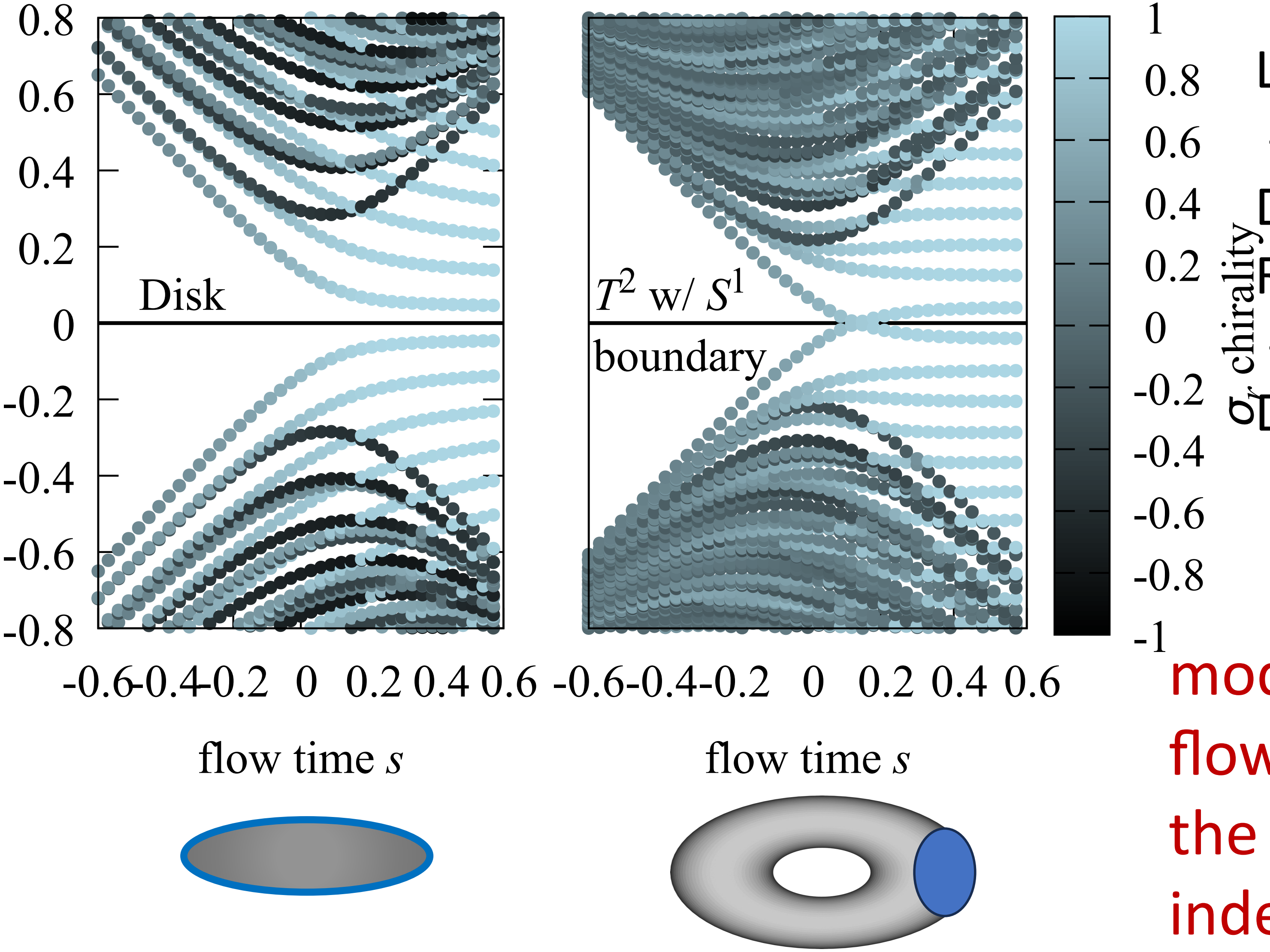


The continuum mod-two APS index = 0 and 1 respectively.



# Majorana Dirac spectrum

$$iH_m = \sigma_1\partial_x + \sigma_3\partial_y + i\sigma_2 m(s,r)$$



Left panel :

-sM inside $S^1$ DW

Right :

-sM outside $S^1$ DW

mod-two spectral flow agrees with the mod-two APS index.

# Contents

✓ 1. Introduction
We consider the lattice index in terms of K-theory.

✓ 2. Lattice chiral symmetry and the overlap Dirac index (review)
great but equivalent to the eta invariant of the massive Wilson Dirac op.

✓ 3. K-theory
classifies the vector bundles. $K^1(I, \partial I)$ is important in this work.

✓ 4. Massless Dirac ($K^0$ group) vs. massive Dirac ($K^1$ group) in continuum
Counting lines (massive, $K^1$) is easier than counting points (massless, $K^0$).

✓ 5. Generalization
Massive fermion Dirac operator unifies various types of the index.

✓ 6. Mathematical theorem on the lattice
Massive fermion Dirac operator unifies various types of the lattice index.

7. Summary and discussion

# Summary

The massive Wilson Dirac operator can be identified as a mathematical object in K-theory and the associated spectral flows describe various index formulas.

In our formulation,

- Chiral symmetry (GW relation) is NOT necessary.
  ( still agrees with the overlap index on periodic lattices)
- Boundaries can be introduced by domain-walls.
- Domain-walls can be flat/curved (with induced gravitational background).
- Formulated in arbitrary dimensions,
- Standard/mod-two versions treated in a unified way.

# Outlook

More nontrivial invariants ($Z_8$, $Z_{16}$, η…).
[Araki, F, Onogi, Yamaguchi 2025]. So far VERY GOOD numerical results for $Z_8$ invariant on 2-dimensional lattice but rigorous mathematical proof is missing.

Lattice version is limited to the case flat bulk + curved domain-wall.
-> curved bulk and curved domain-wall may be realized by higher co-dimensional junctions of domain-walls

Admissibility condition for lattice-only formulation?