Monopole Scattering, Unitarity Puzzle, and Symmetric Mass Generation

Nat. Phys '26 arXiv:2510.26780

arXiv:2606.23684

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Today's main message

1. The (3+1)d Monopole scattering is a paradox on the chiral fermion scattering. Unitarity puzzle is its generalization.

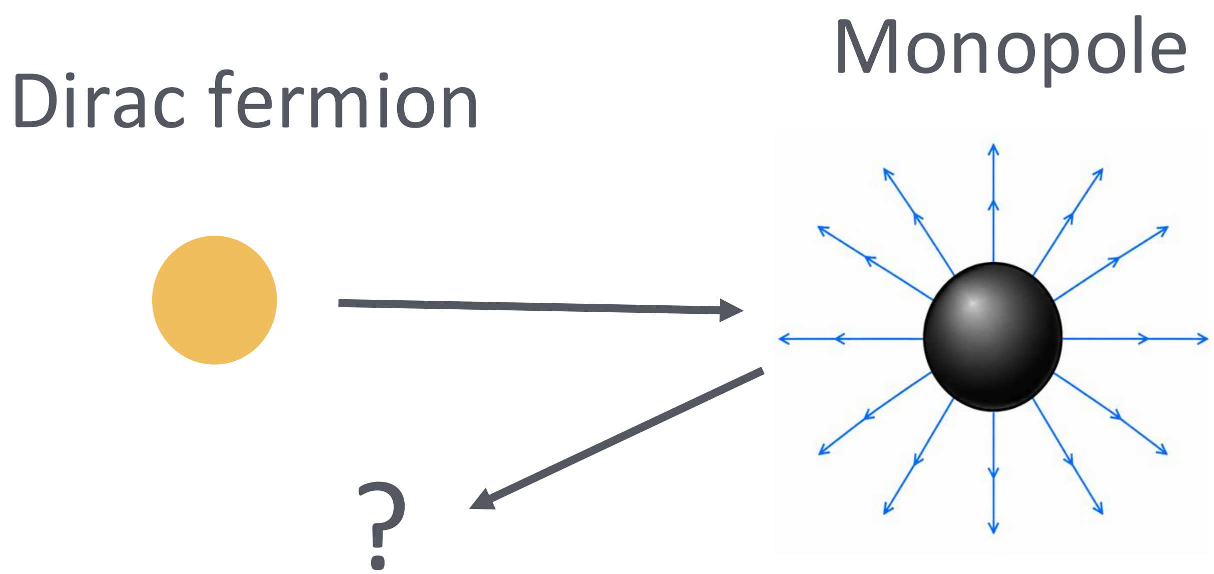
2. The unitarity puzzle occurs in the scattering problem off topological objects with non-trivial quantum dimensions.

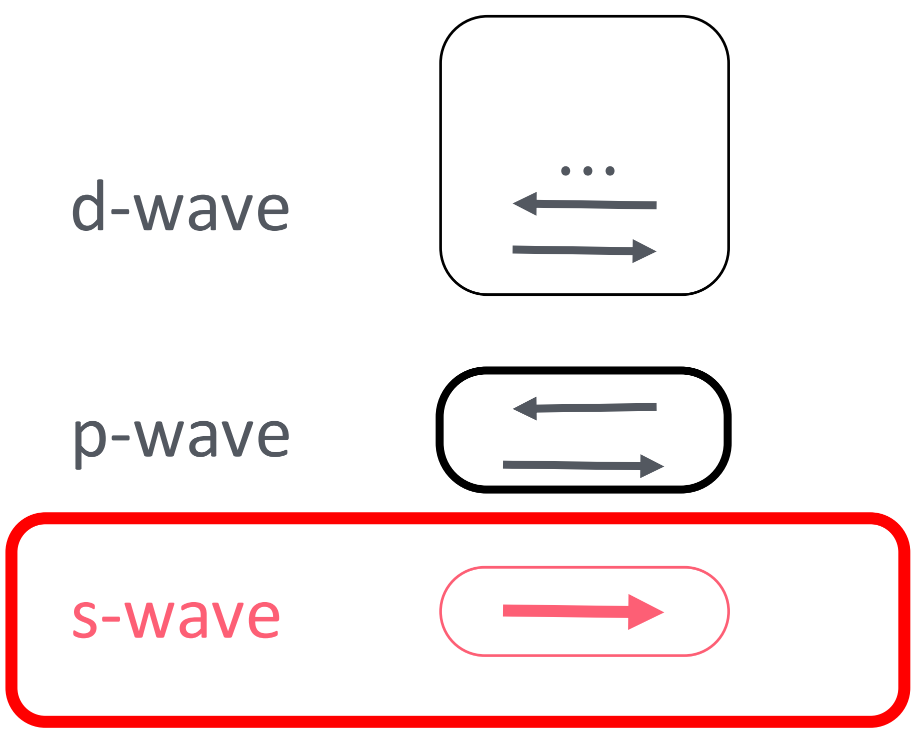
3. The outgoing particles becomes **nonlocal** with a topological string.

4. On the lattice, these models are systematically constructed by duality transformations on a half chain. The topological impurity naturally appears as a duality defect as a virtual degrees of freedom. (No CFT required!)

5. Folding this model yields a (partial/simple) realization of **the monopole scattering on the lattice**.

Monopole paradox





[Callan (1982), Rubakov(1982)]

Symmetry-allowed process

$$e_R \longrightarrow \frac{1}{2}\left(\bar{u}_R^1 + \bar{u}_R^2 + \bar{q}_L^3 + l_L\right)$$

Semiton? Proton decay?

No electron can scatter back from the monopole

Unitarity of S-matrix

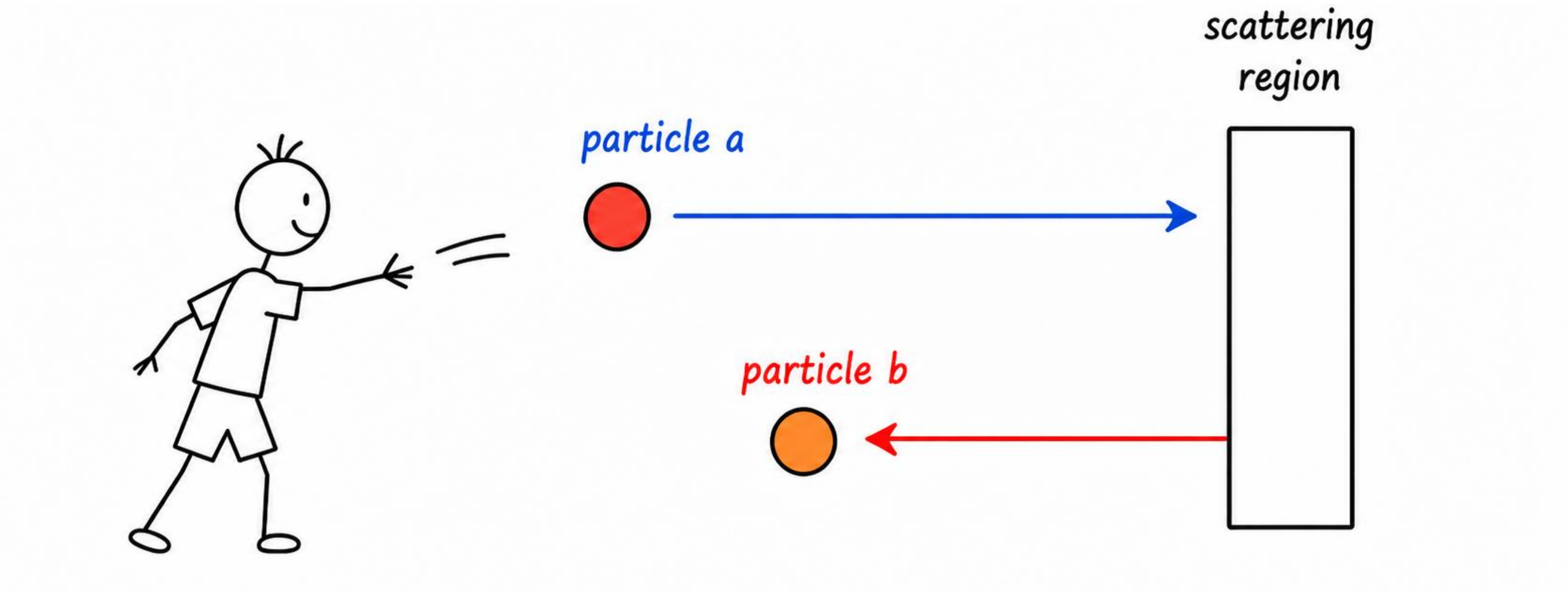
This is a paradox because S-matrix must be **unitary**.

$$\sum_b |S_{ba}|^2 = 1$$

$$S_{ba} = \langle b, \mathrm{out} | a, \mathrm{in} \rangle$$

$$P(a \to b) = |S_{ba}|^2$$

No probability disappears during scattering.

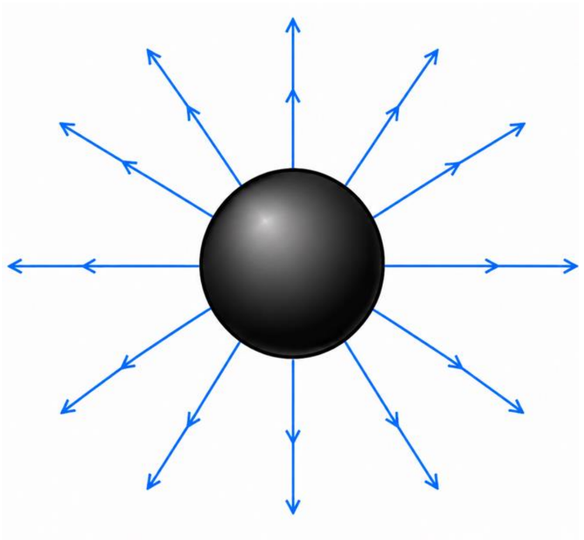


Unitarity puzzle

There are wide class of scattering problems where the S-matrix appears non-unitary if one only consider a naïve Hilbert space.

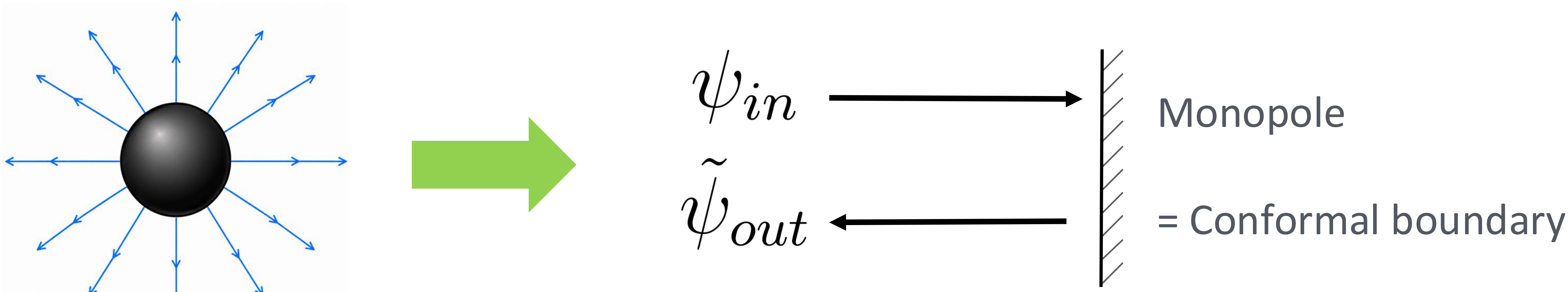
$$S^{\dagger}S \neq 1$$

Monopole scattering seems non-unitary in the S-matrix of **one-particle sector** .



How to think about Monopole Paradox

The s-wave scattering reduces to a one-dimensional scattering problem, where the monopole serves as **a conformal boundary**.



[Affleck (1994), Maldacena (1995)]

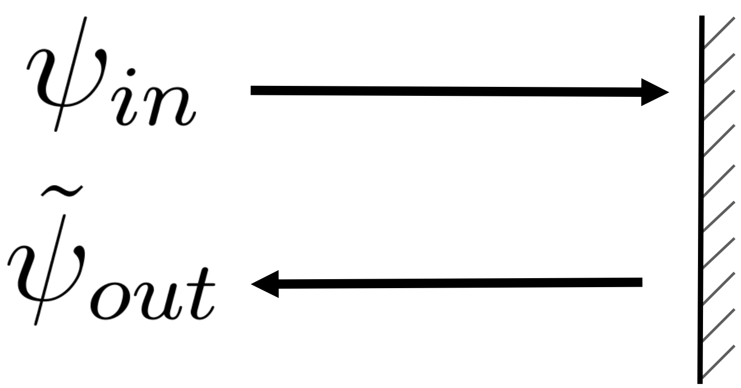
Conformal boundary can create non-trivial states. Thus, nothing seems to be scattered in a single particle sector.

$$S^{1\,\text{particle}}_{\psi_{in},\psi_{out}} = 0$$

Recent solution

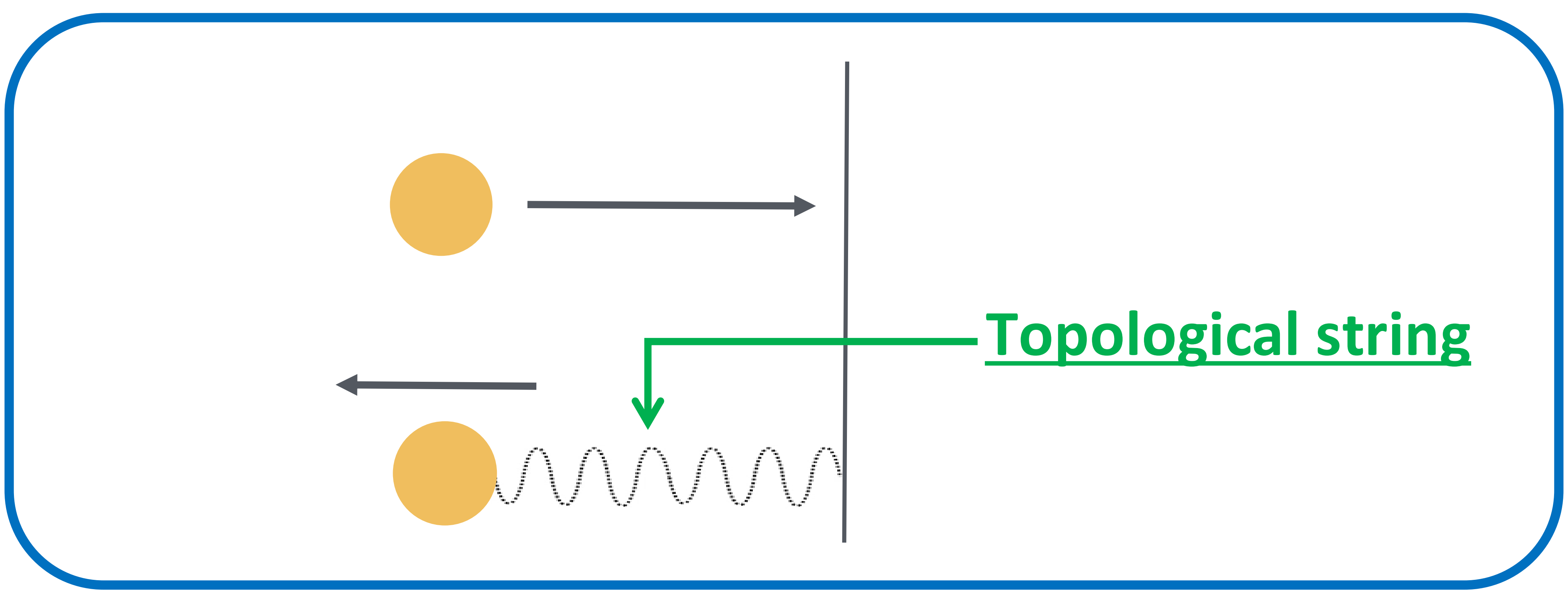
[van Beest (2023), Loladze (2024)]

The scattered state is an electron in the twisted sector!



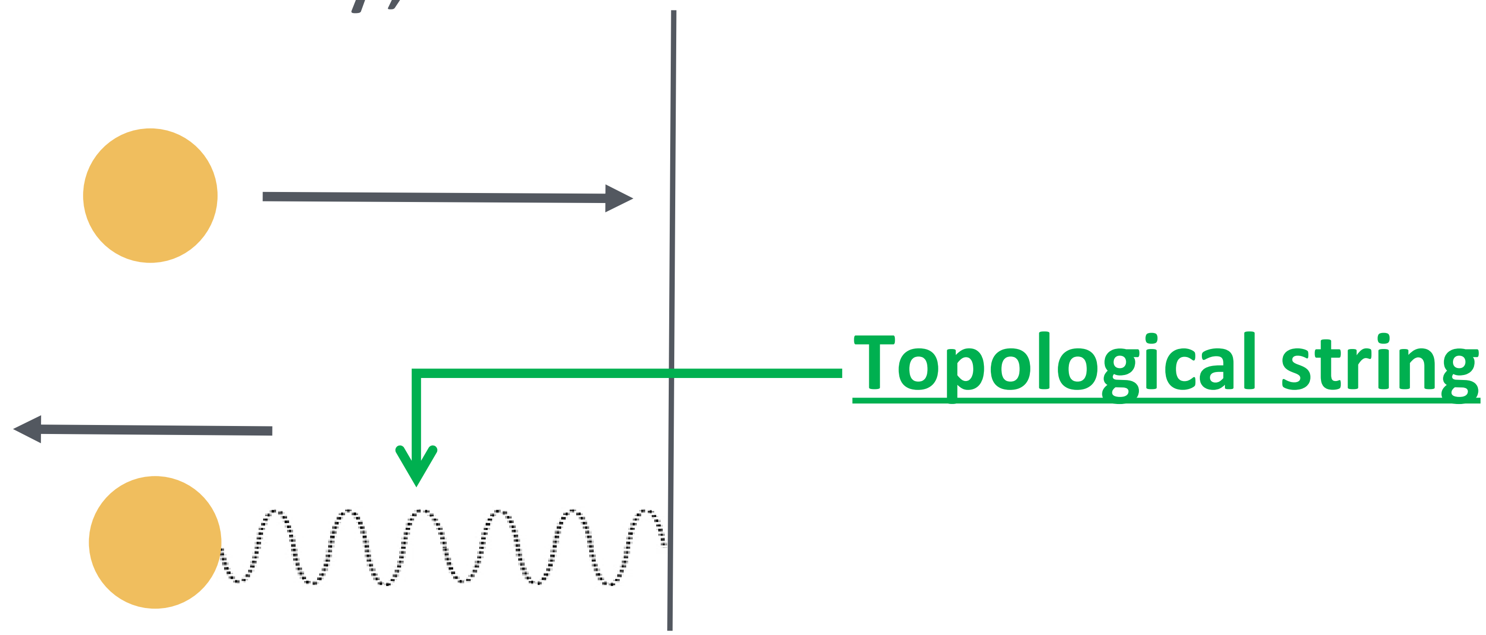
$\tilde{\psi}_{out}$: Electron in **twisted sector**

The branch cut attaches a topological string to the electron



Unitarity with extended Hilbert space

To restore the unitarity, one need to consider boundary to be dynamical.



Roughly roughly speaking,

$$\langle \psi_{in} \psi_{out} \rangle = 0 \qquad \langle \psi_{in} \hat{O}_{bd} \psi_{out} \rangle \neq 0$$

The unitary scattering theory needs extended Hilber space

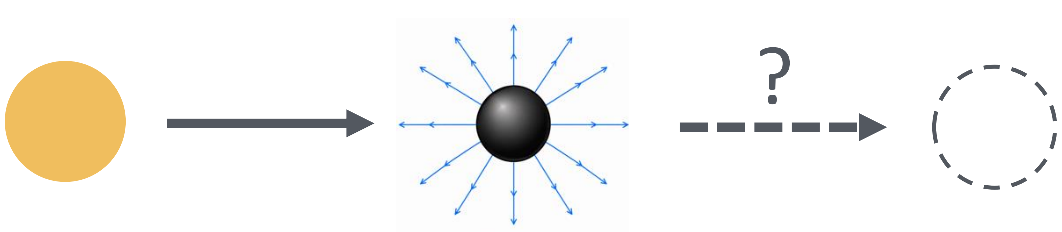
$$\mathcal{H}_{full} = \mathcal{H}_{original} + \mathcal{H}_{boundary}$$

Recent solution (Summary)

Monopole paradox

$$S_{\psi_{in},\psi_{out}} = 0$$

No electron comes out
→ **Non-unitary**

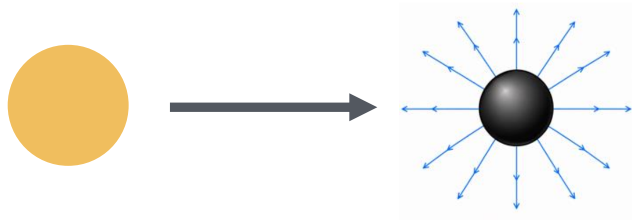


Solution

$$S_{\psi_{in},\tilde{\psi}_{out}} = 1$$

Electron + twisted sector
→ **Restored unitarity**

Original Hilbert space



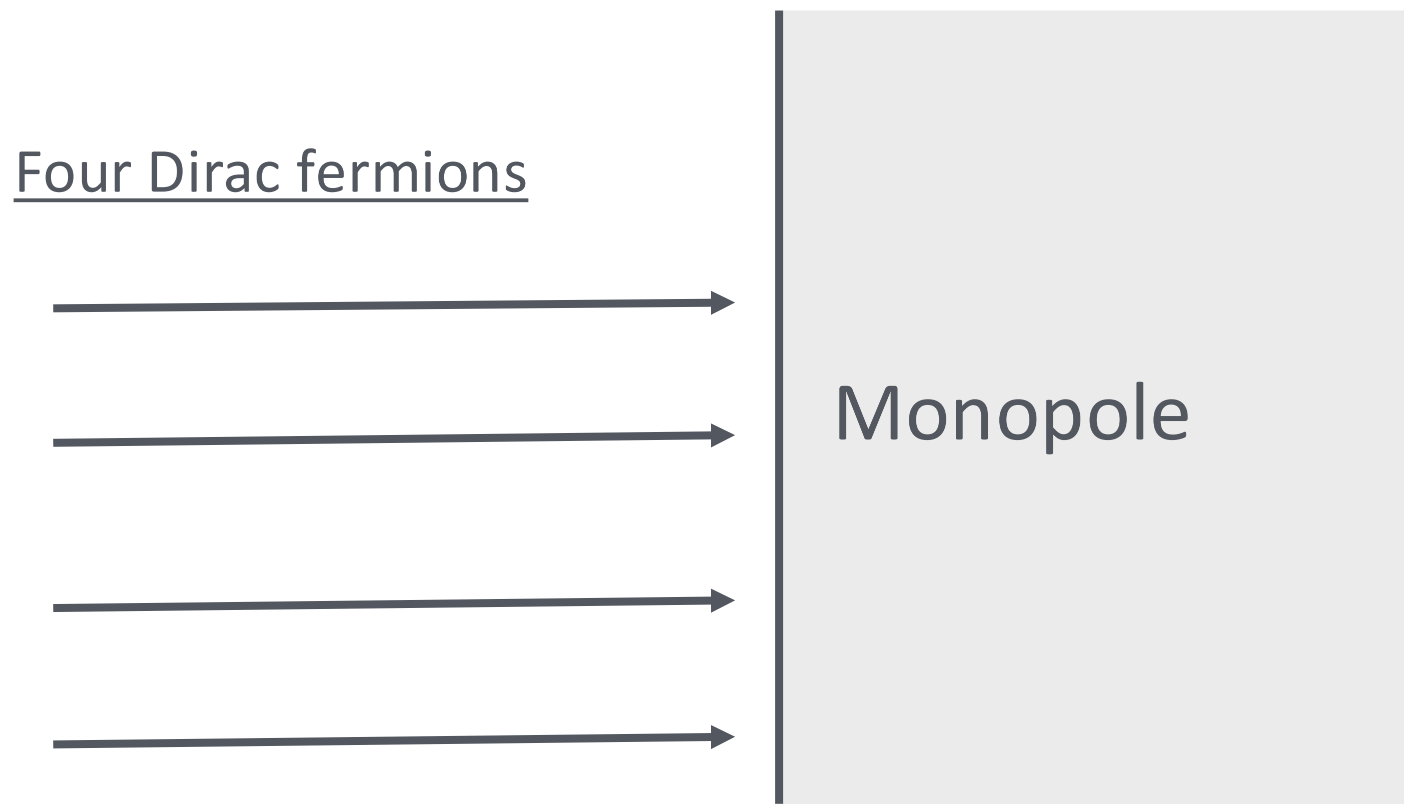
Twisted sector

Maldacena-Ludwig wall

[Maldacena Ludwig(1995)]

Monopole acts as a conformal boundary to s-waves

$$H = \frac{v_F}{2\pi} \sum_{\alpha=1}^{4} \int dr \left[\psi^\dagger_{L,\alpha}(r)\, i\partial_r \psi_{L,\alpha}(r) - \psi^\dagger_{R,\alpha}(r)\, i\partial_r \psi_{R,\alpha}(r) \right]$$



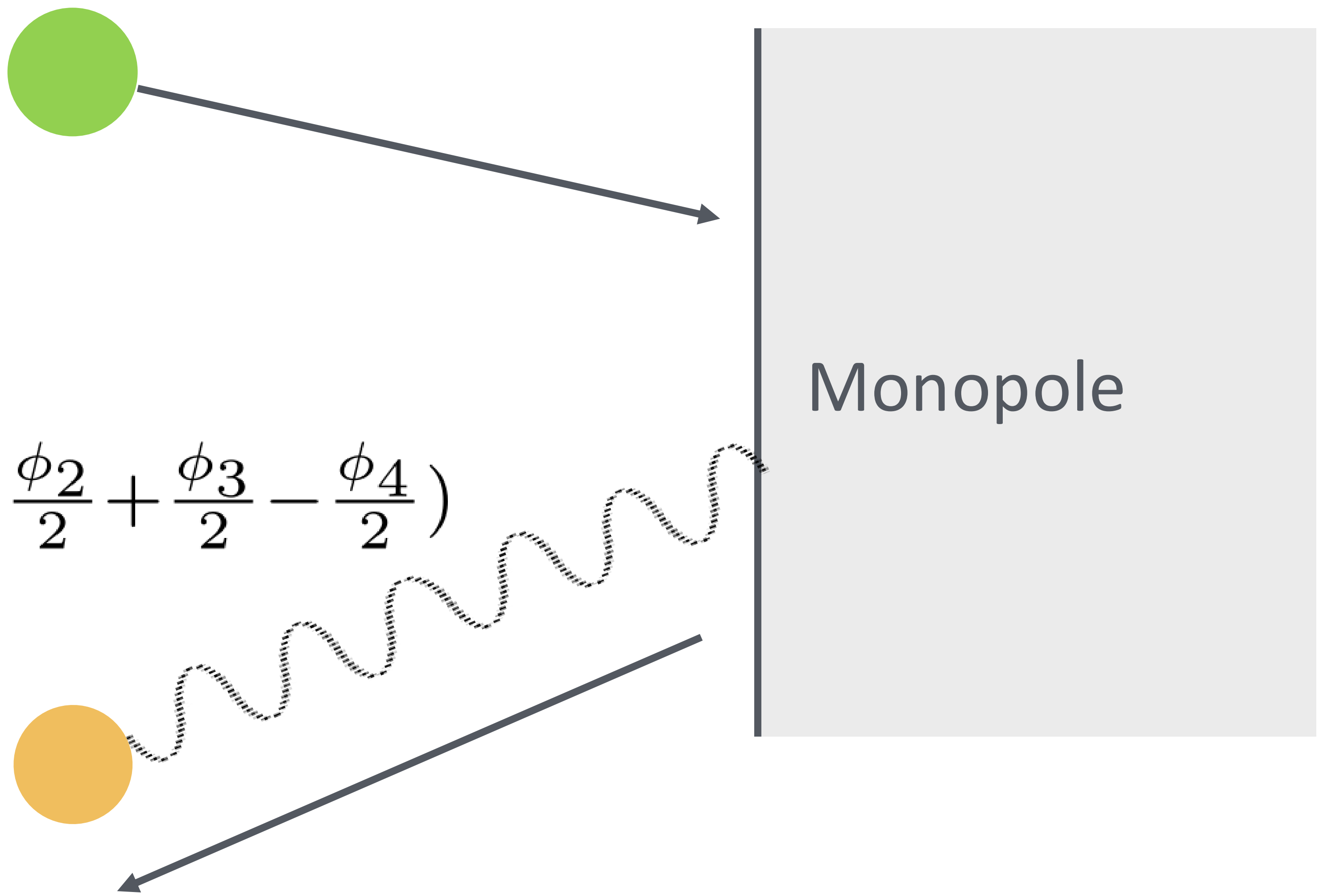
Boundary CFT

[Maldacena Ludwig(1995)]

The electron is scattered to a nonlocal twist field.

$$\psi_{in} = e^{i\phi_1}$$

$$\psi_{out} = e^{i(\frac{\phi_1}{2}+\frac{\phi_2}{2}+\frac{\phi_3}{2}-\frac{\phi_4}{2})}$$



The monopole serves as a conformal boundary with the Affleck-Ludwig boundary entropy $g = \sqrt{2}$.

Question:

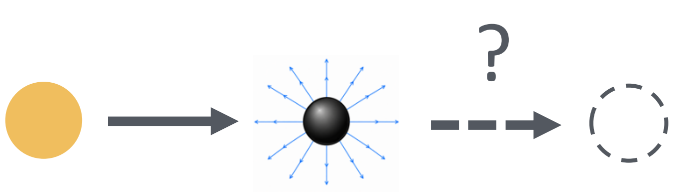
How do we understand particles that *disappear*?

1. Can we realize the monopole unitarity puzzle in **a simple lattice model**?

2. Can the missing probability be restored by enlarging the Hilbert space?

3. What is the nature of this **extended Hilbert space**?

"Particles disappear"



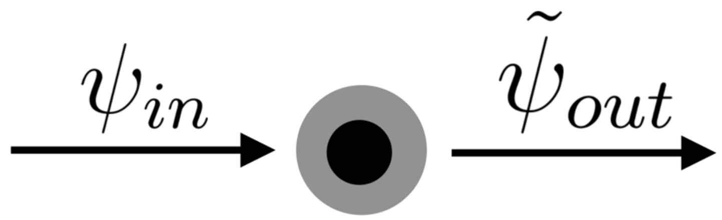
Non-unitary S-matrix

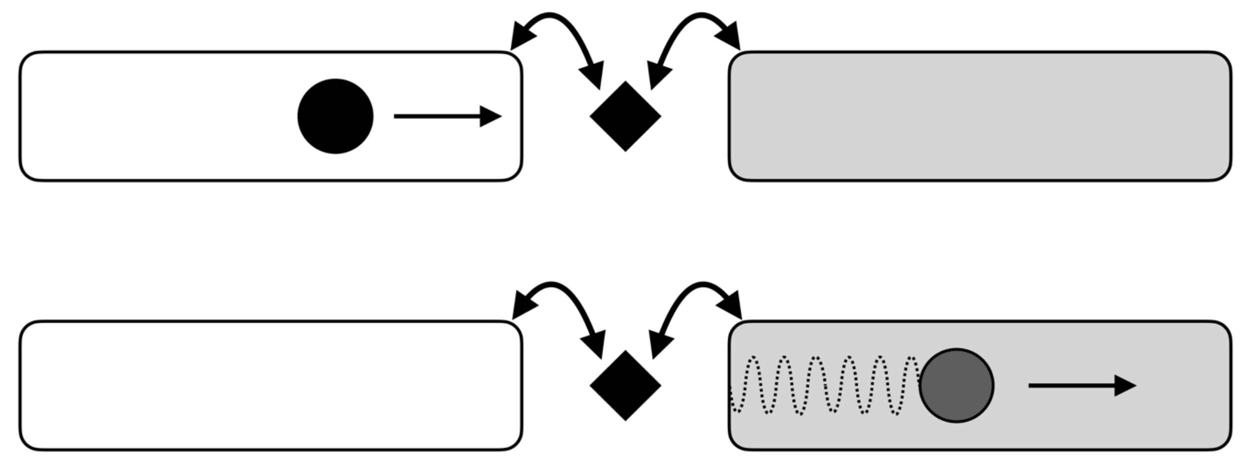
Restored unitary with **extended Hilbert space**

Now, the lattice model.

Lattice impurity model

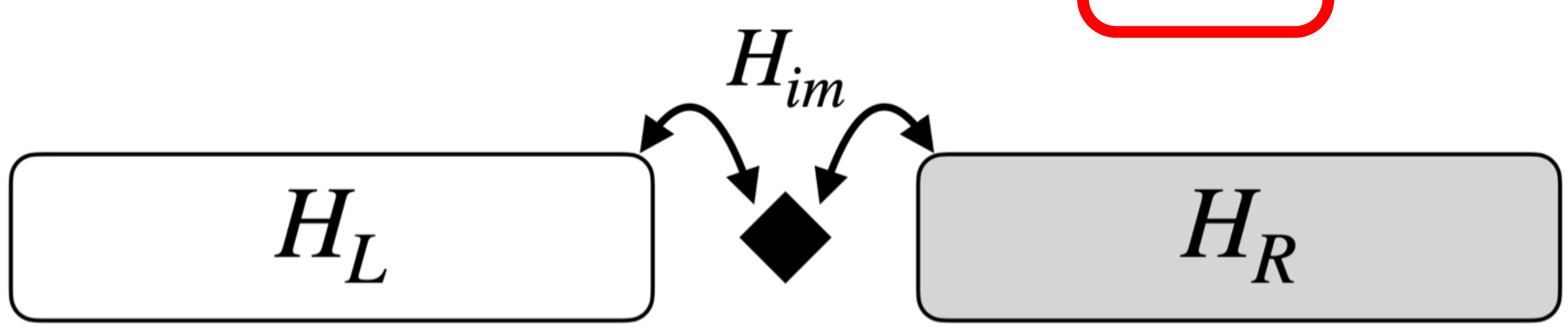
We consider **unfolded** scattering model.





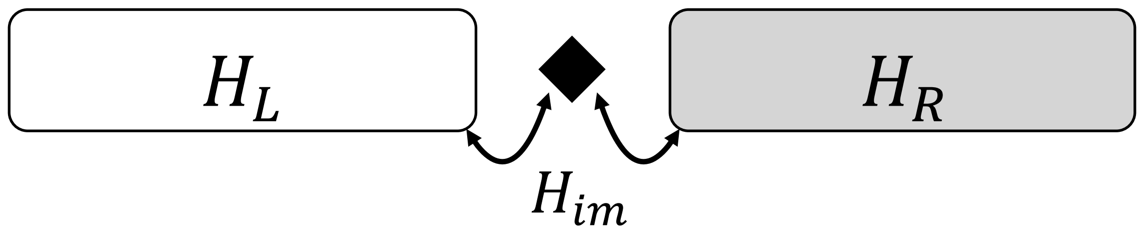
The scattering process is governed by an impurity interaction

$$H = H_L + H_R + H_{im}$$



Dual Ising-impurity model

We prepare the Ising model of different phases as H_L and H_R



H_{L} : Symmetric

$$H_L = -\sum_{i=1}^{I-2} X_i X_{i+1} - g_L \sum_{i=1}^{I-1} Z_i$$

H_{R} : SSB phase

$$H_R = -\sum_{i=I+1}^{L-1} X_i - g_R \sum_{i=I+1}^{L-1} Z_i Z_{i+1}$$

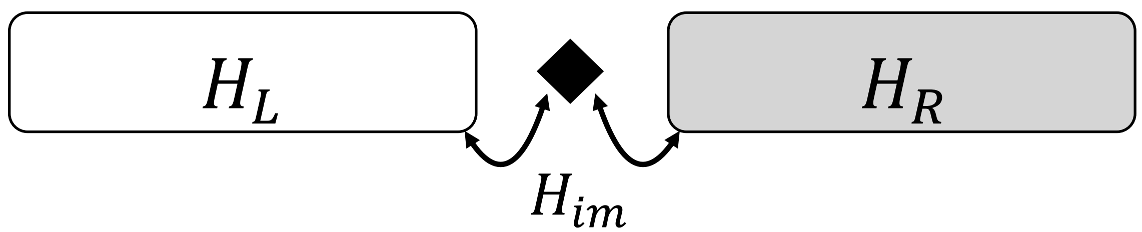
Both systems are polarized in the z-direction. $g_L > 1,\ g_R > 1$

Two systems interact through the impurity site.

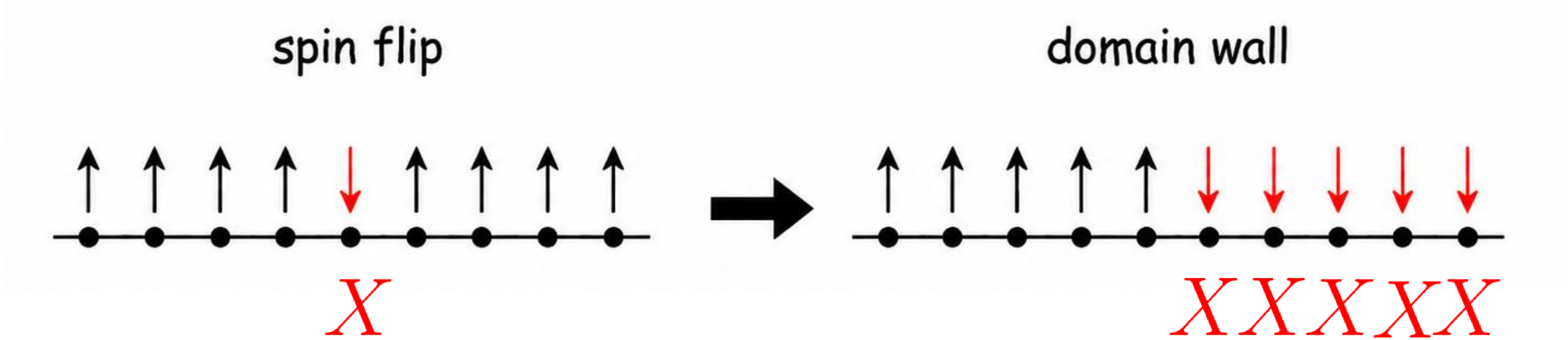
$$H_{im} = -X_{I-1} X_I - g_R Z_I Z_{I+1}$$

Elementary excitations

The elementary excitations are different between H_L and H_R



H_L : Symmetric phase $\qquad$ H_R : SSB phase



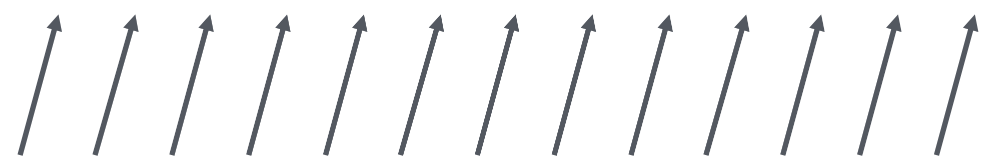
The spin flip in H_R and the domain wall in H_L
cost high energy.

Numerical setup

The wavepacket is created on top the ground state.

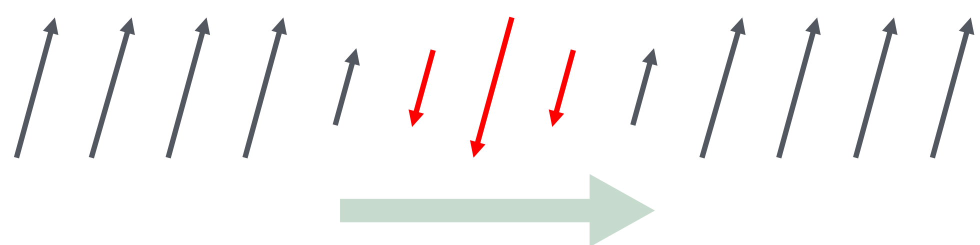
$$\hat{W} = \sum_{j \in H_L} e^{-\frac{(x-x_0)^2}{2\sigma^2}} e^{ikj} X_j \qquad |\psi(t=0)\rangle = \hat{W}|\psi_{GS}\rangle$$

$$|\psi_{GS}\rangle =$$



<u>The state is time-evolved with the original Hamiltonian.</u>

$$\hat{W}|\psi_{GS}\rangle =$$

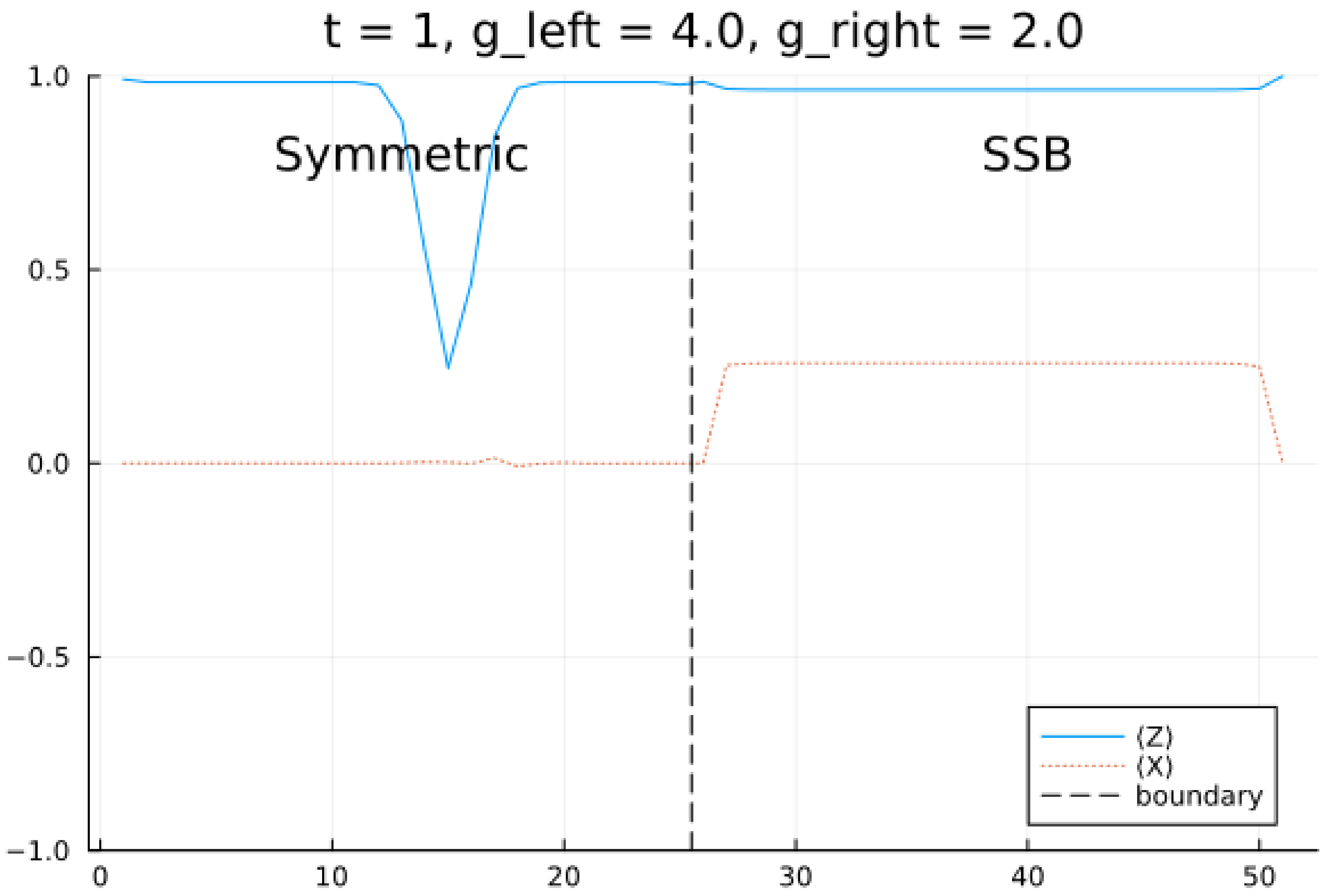


$$|\psi(t)\rangle = e^{itH}\hat{W}|\psi_{GS}\rangle$$

Q: Can a spin-flip wavepacket propagate into the SSB phase through H_{im}?

Generic case: Reflection $g_L = 4, g_R = 2$

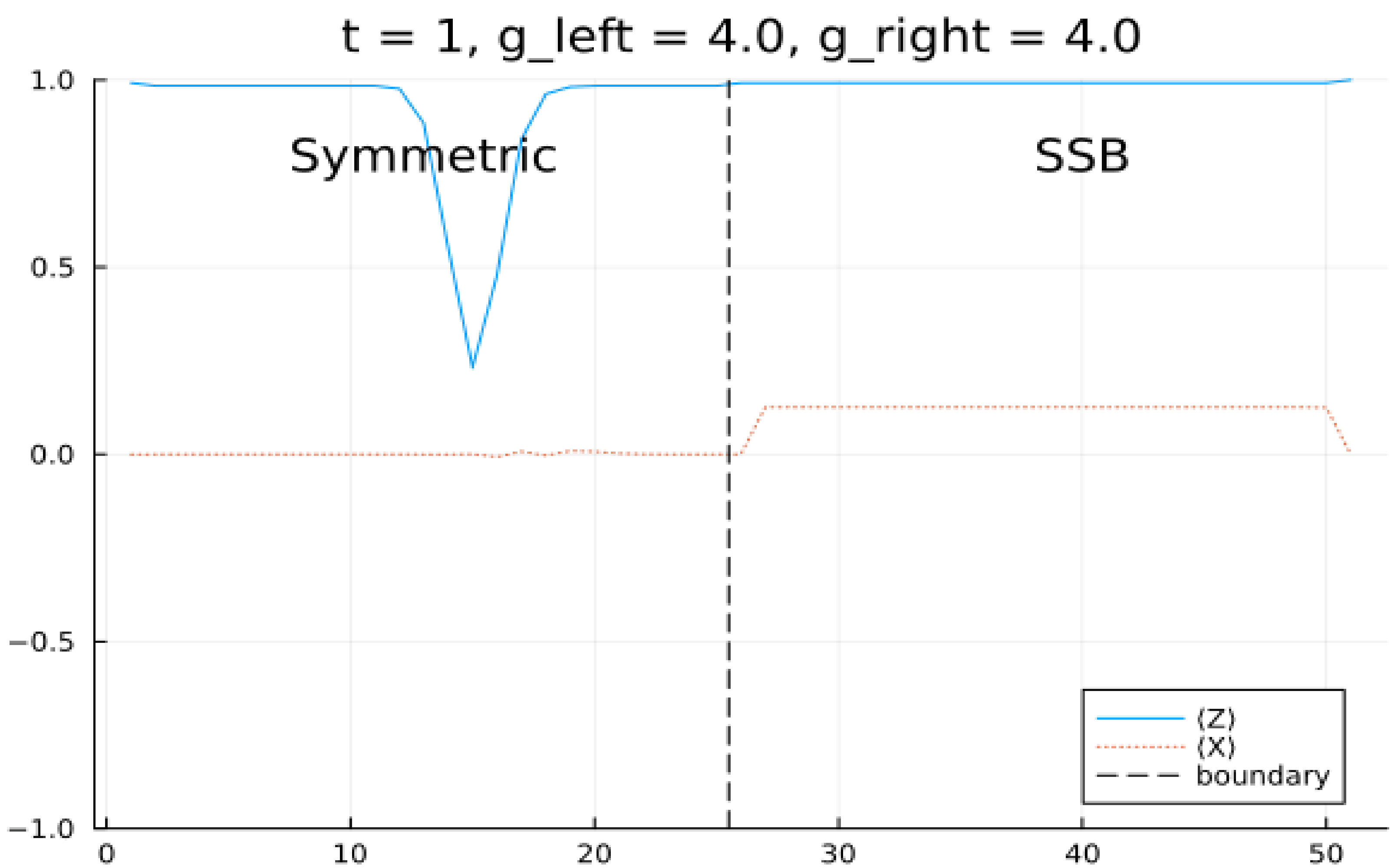
The spin flip excitation cannot enter the SSB phase.

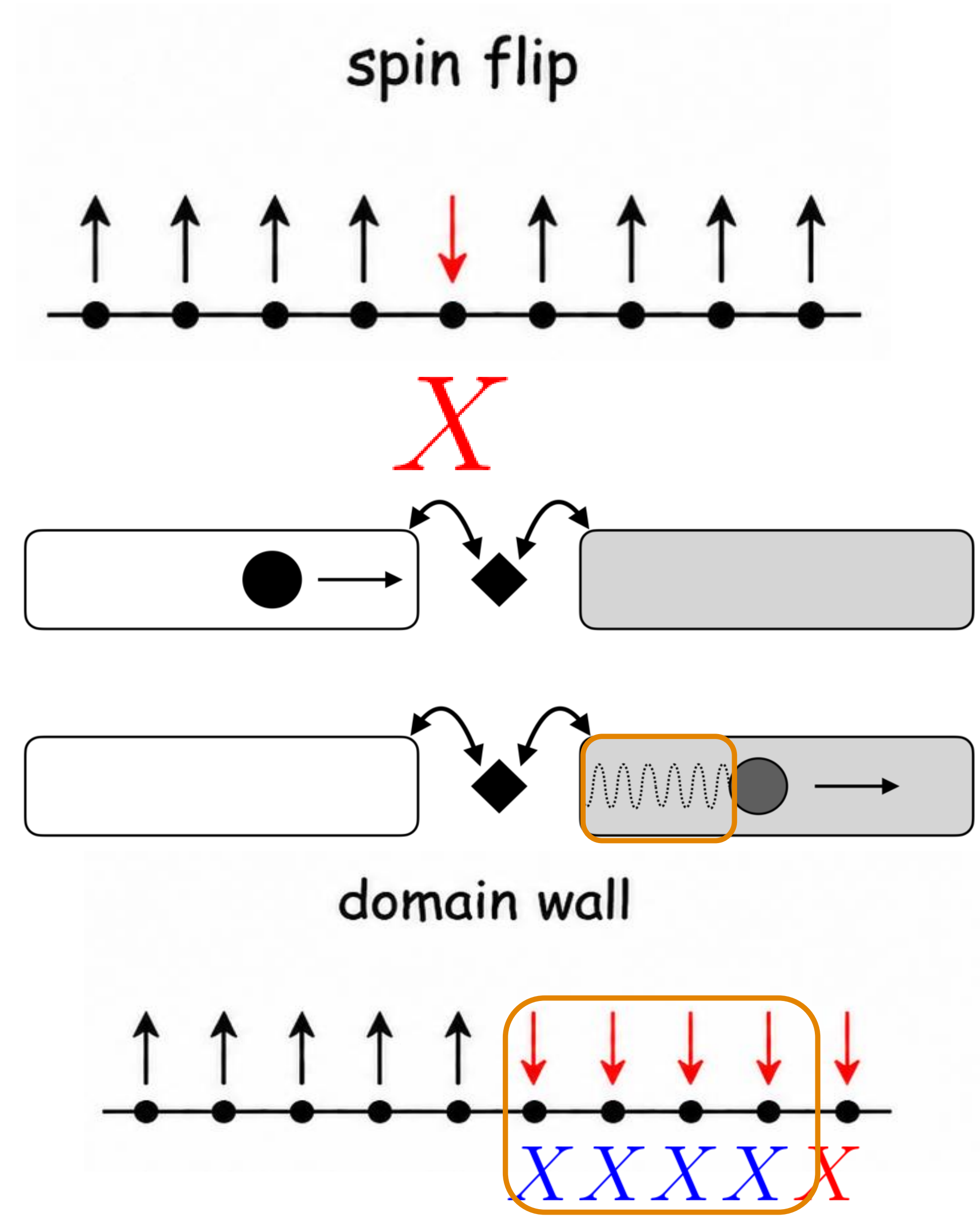


The wavepacket is reflected back to the right.

Case 2: Perfect Transmission $g_L = g_R = 4$

The spin flip excitation enter the SSB phase as a domain wall.

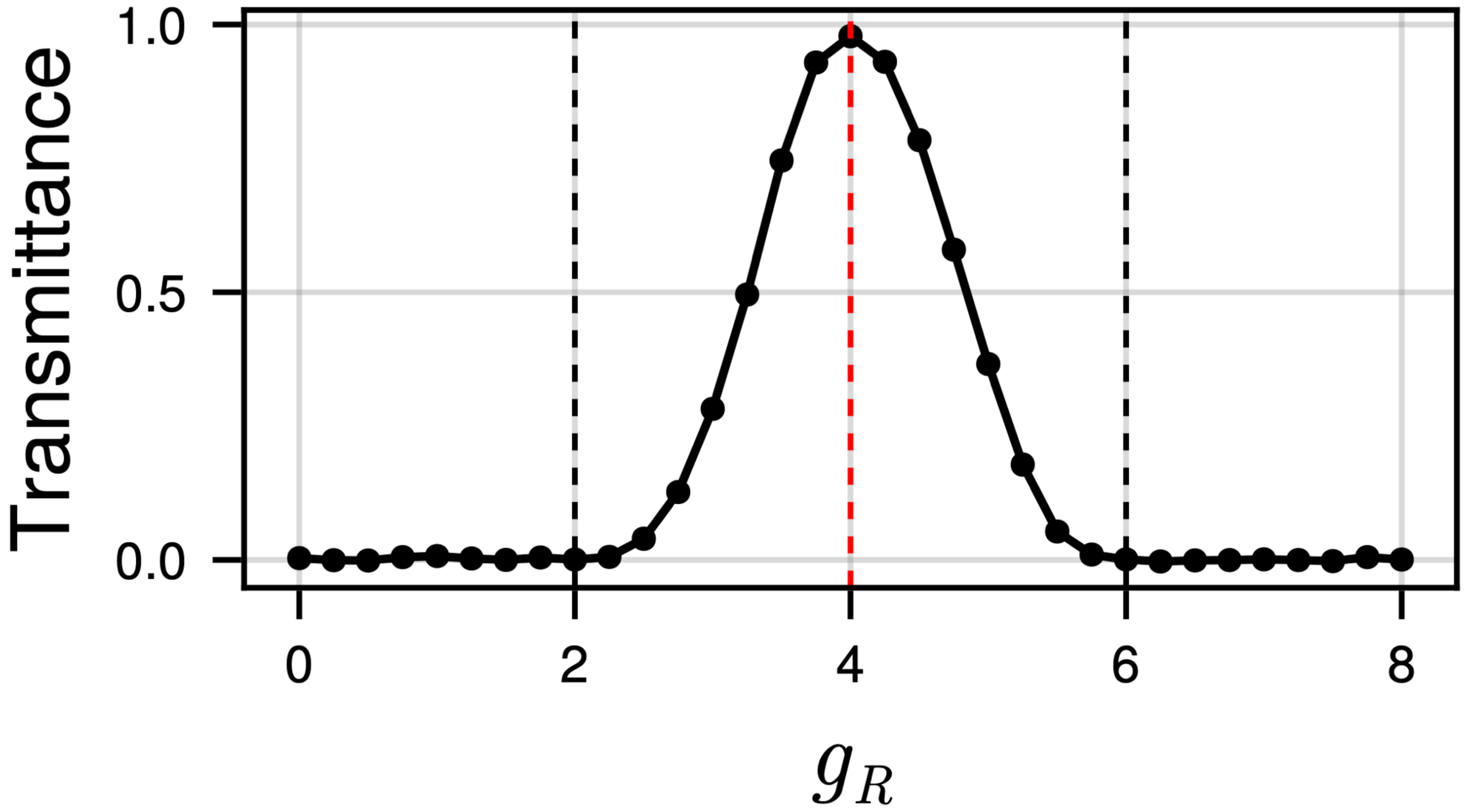




The domain wall can be considered as a spin flip with a Z2 topological string.

Magic point

Transmittance is 100% at the Magic point $g_L = g_R$



Why?

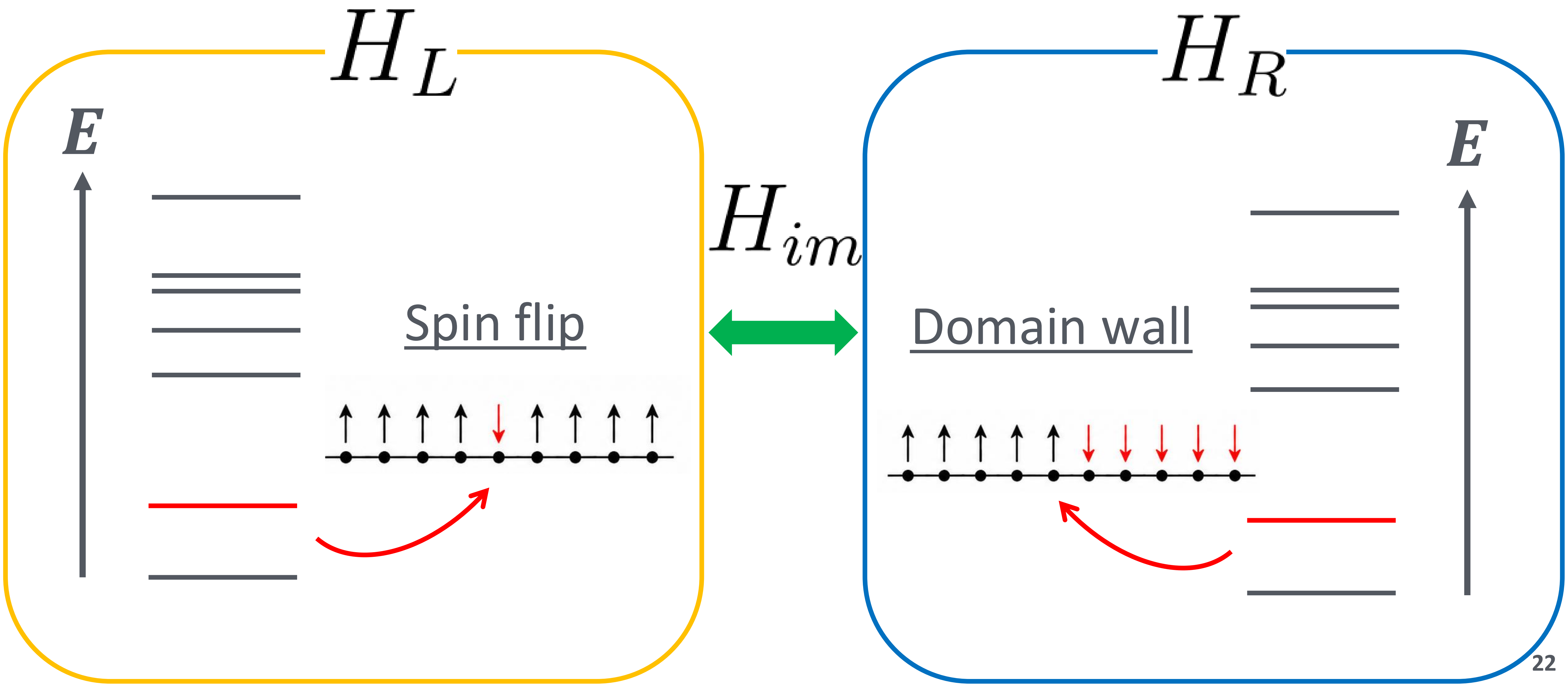
Mechanism:

Topological interface and Duality

Energy resonance = Duality

Each excitation has **one-to-one spectral matching**.

The impurity is just a **translator**.

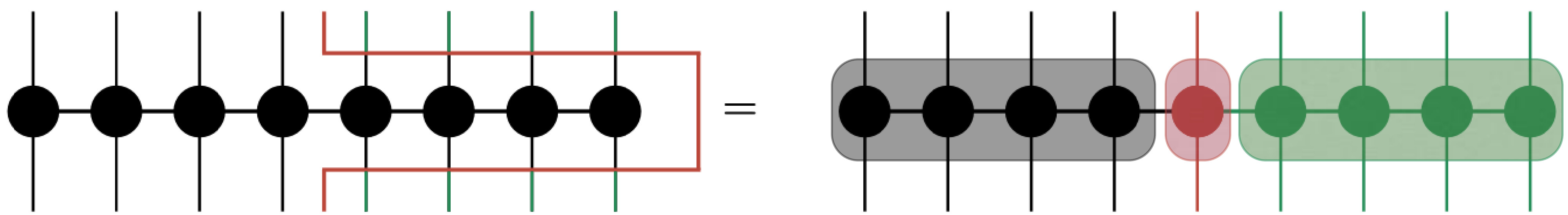


Main result: half unitary transformation

Impurity model is constructed by applying the unitary duality transformation only to the right half of the chain.

$$H = U_{KW}^{\dagger}(\tilde{H} \otimes \mathbb{1}_{L+1})U_{KW}$$

In a tensor network language, this is written as



$U_{KW} =$ 

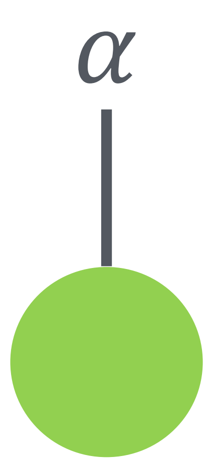
What's this???

Don't worry, I explain!

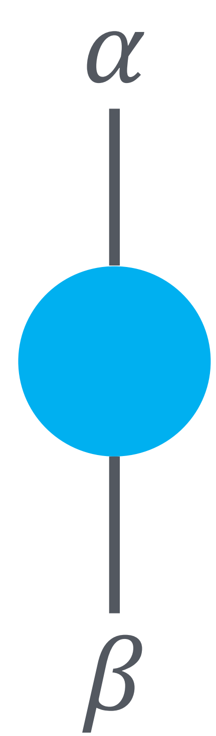
Tensor networks 101

Tensor networks are representation theory of linear algebra.

$$V = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \qquad M = \begin{pmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

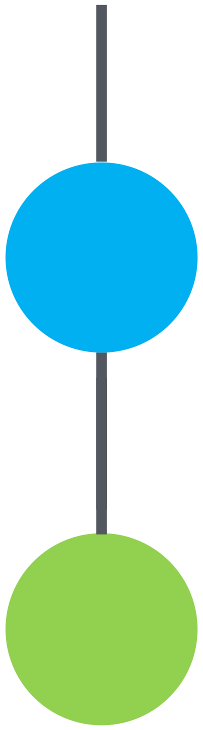


$= V_\alpha$



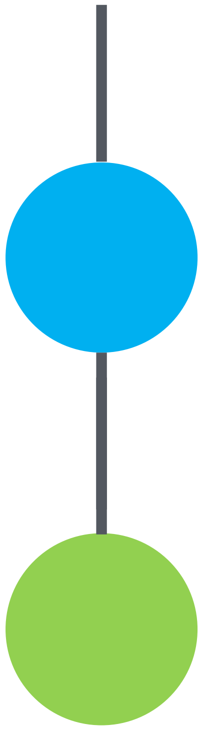
$= M_{\alpha\beta}$

Tensor multiplication is contraction



$= \mathrm{MV}$

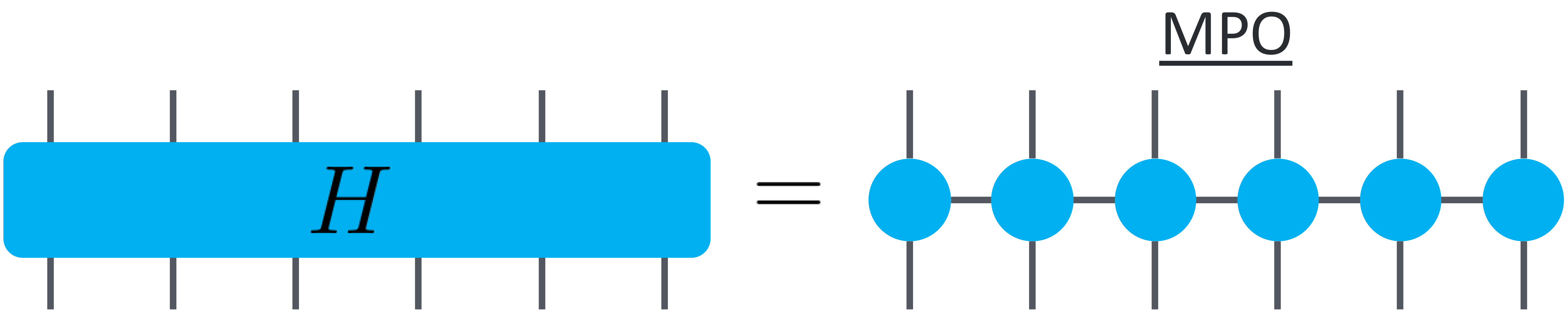
Tensor multiplication is contraction



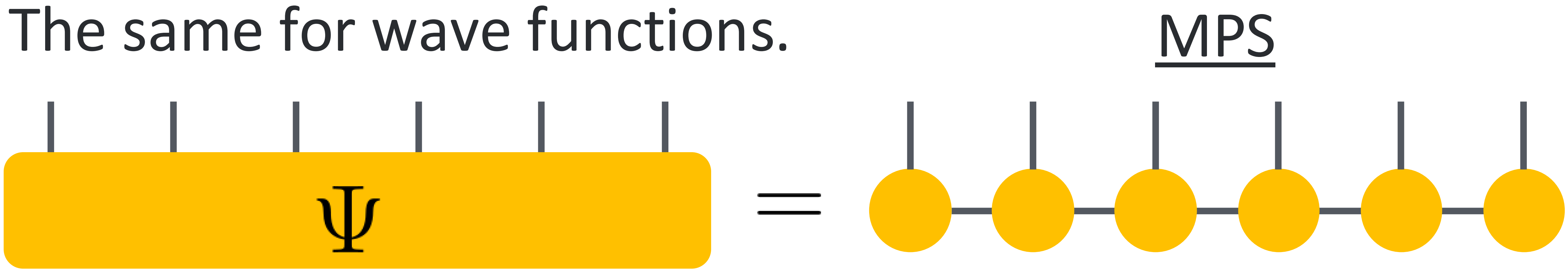
$$= \mathrm{MV}$$

Tensor networks 103

Hamiltonians and unitary circuits can be decomposed into pieces to make the translational invariance manifest.



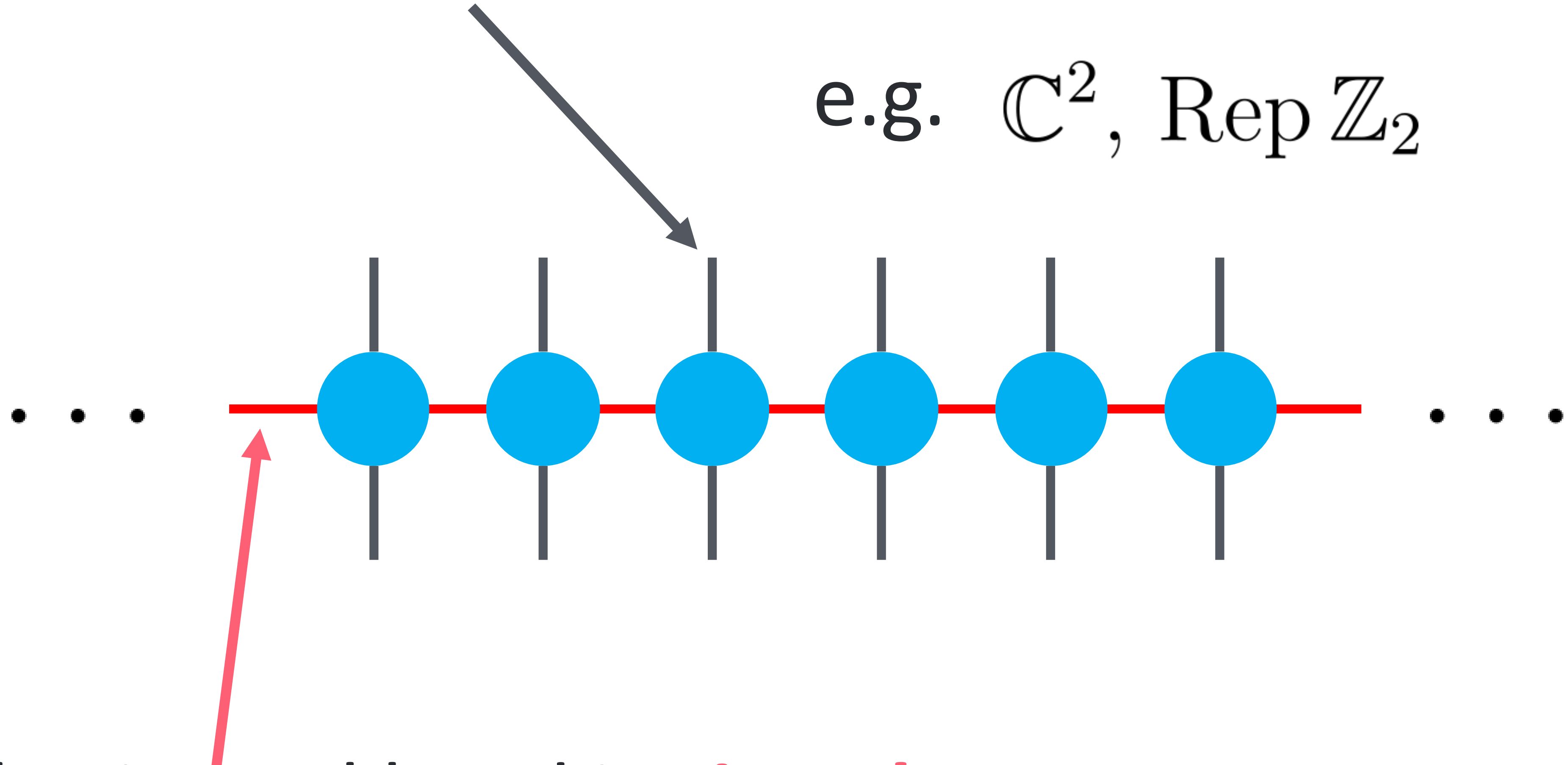
The same for wave functions.



Tensor networks 104

The vertical bond is **physical space**

e.g. $\mathbb{C}^2, \ \mathrm{Rep}\,\mathbb{Z}_2$



The horizontal bond is **virtual space**

Virtual space represents entanglement degrees of freedom.

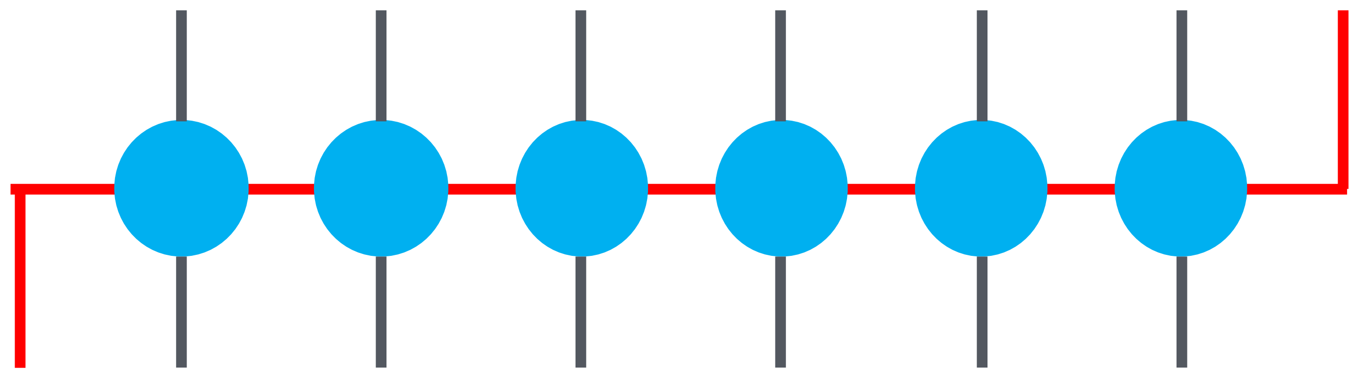
e.g. needed for bookkeeping the global symmetry locally.

Tensor networks 105 (Last, non-trivial)

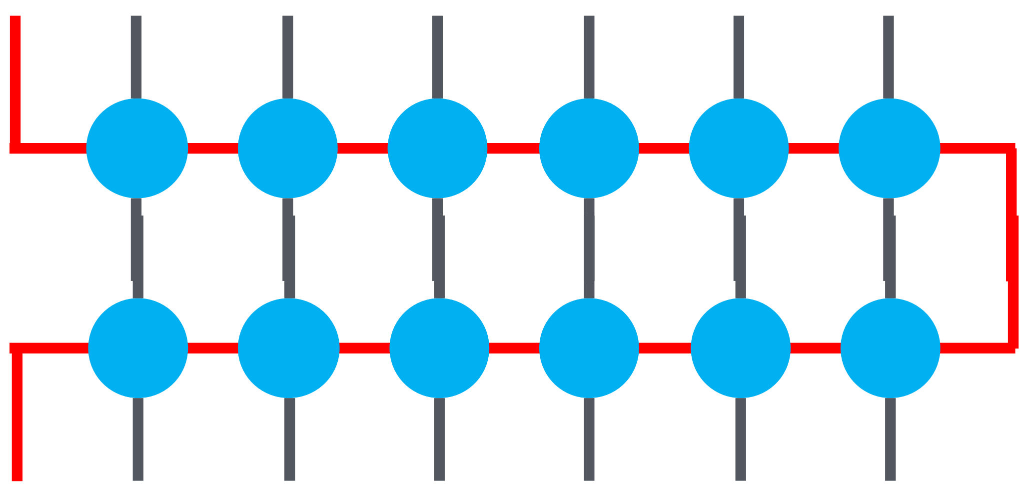
[Lootens (2025)]

Duality transformation can be implemented as a unitary by making the virtual bond physical

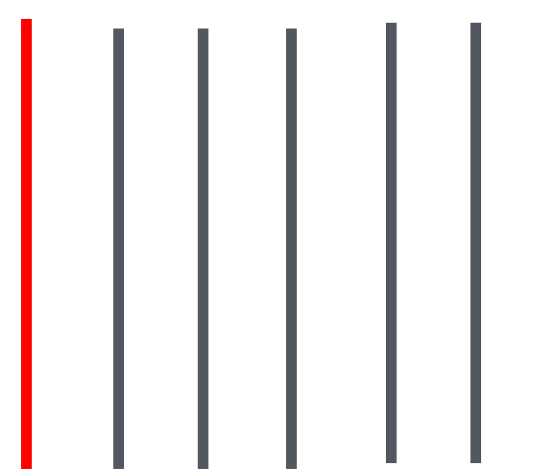
$U_{duality} =$



$UU^{\dagger} =$



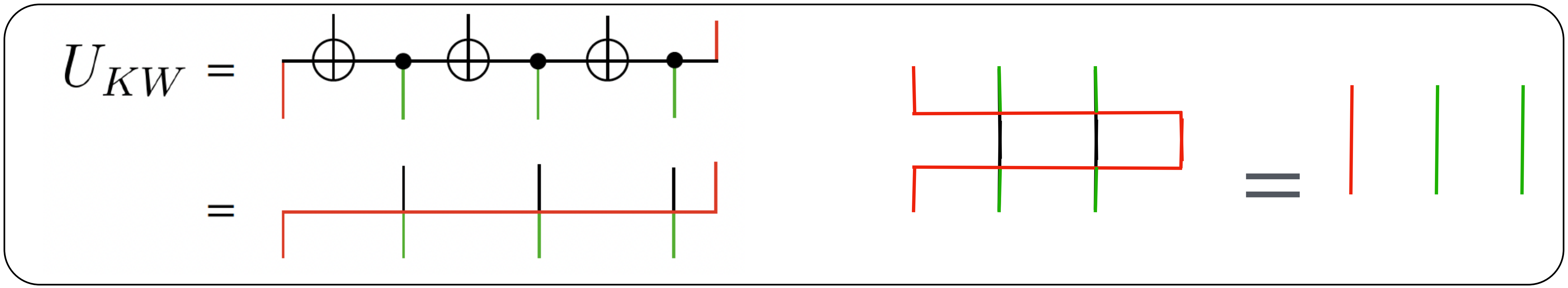
$=$

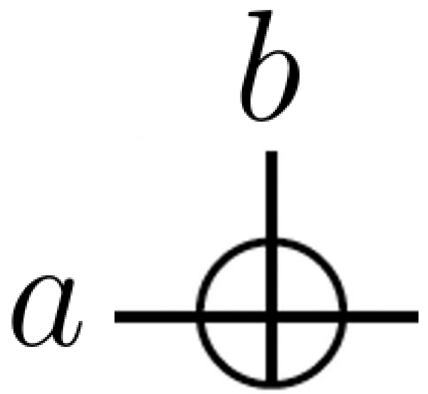


$= 1$

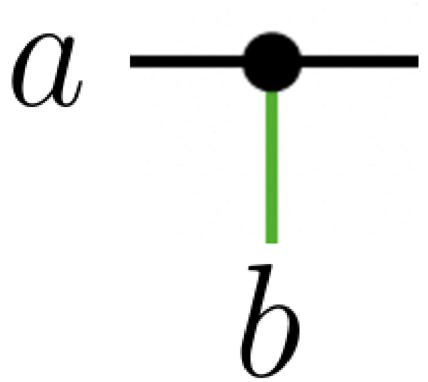
Krammers-Wannier Duality as a unitary circuit

H_L and H_R are related by KW duality at $g_L = g_R$ as $H_R = U_{KW}^{\dagger} H_L U_{KW}$





$= \delta_{a+b,c(\mathrm{mod}2)}$ $\quad a, b, c \in \{0, 1\}$



$= \delta_{a,b,c}$

Krammers-Wannier Duality as a unitary circuit

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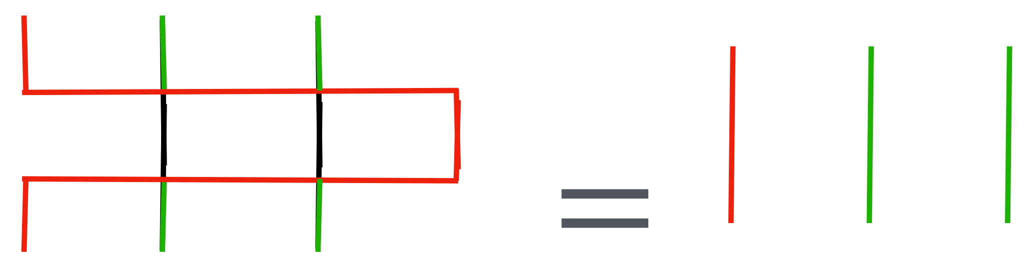
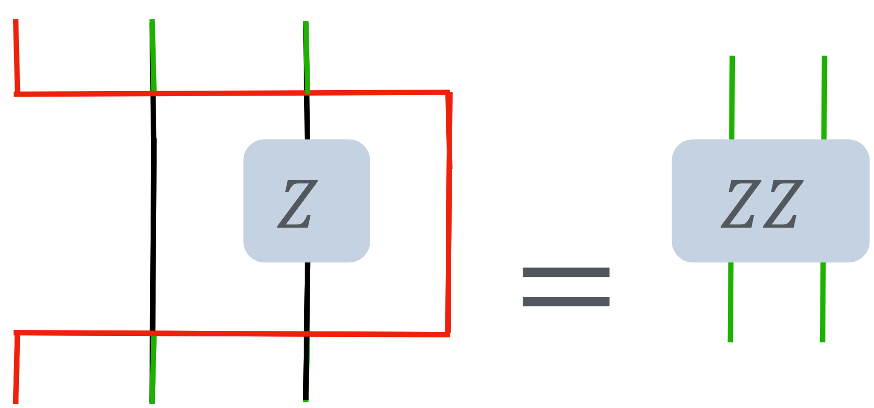
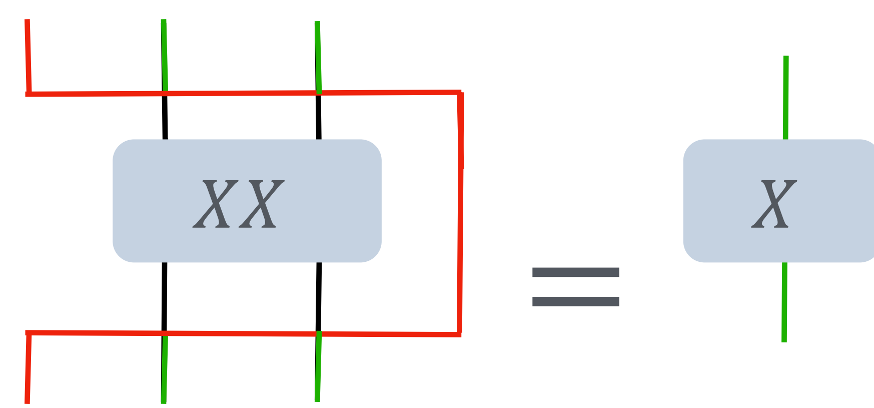




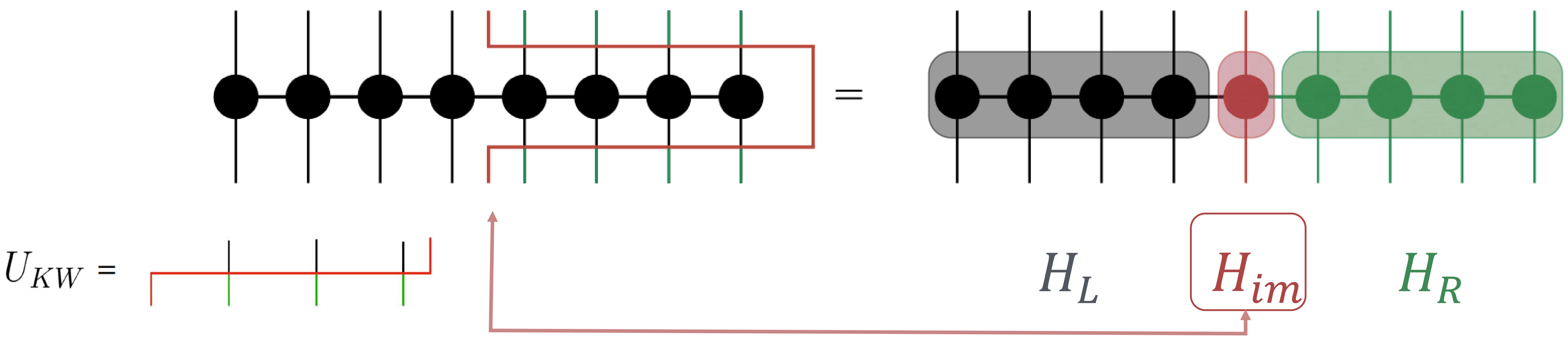


$$H_L = -\sum_{i=1}^{I-2} X_i X_{i+1} - g_L \sum_{i=1}^{I-1} Z_i \qquad H_R = -\sum_{i=I+1}^{L-1} X_i - g_R \sum_{i=I+1}^{L-1} Z_i Z_{i+1}$$

Constructing the impurity from a half-duality

Impurity model is made by unitary transformation of a half chain of the uniform Hamiltonian made by HL

$$H = U_{KW}^{\dagger}(\tilde{H} \otimes \mathbb{1}_{L+1})U_{KW}$$



Impurity = Virtual leg of the MPO

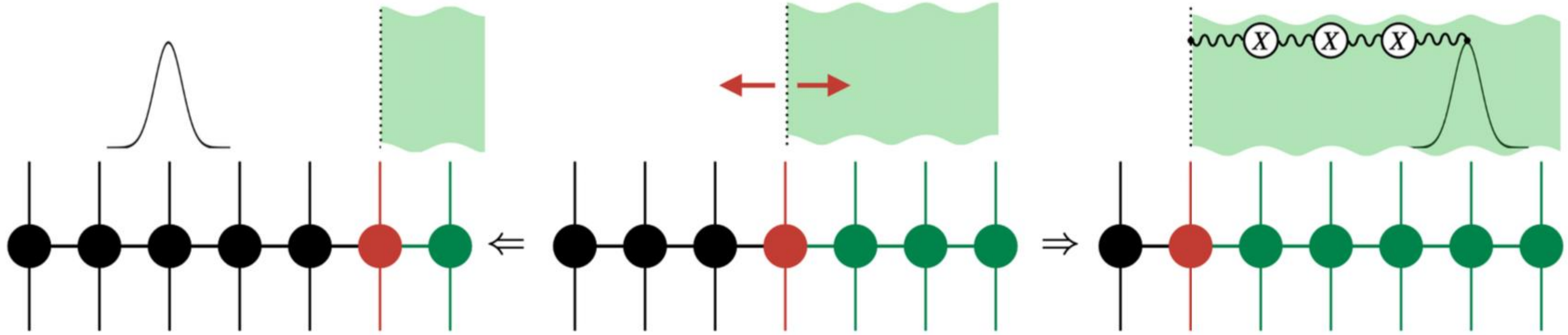
The Hilbert space recovers unitarity by including this virtual degree of freedom

Impurity is topological

Length of string **does not alter the energy** = topological defect

$$H = U_{KW}^{\dagger}(\tilde{H} \otimes \mathbb{1}_{L+1})U_{KW}$$

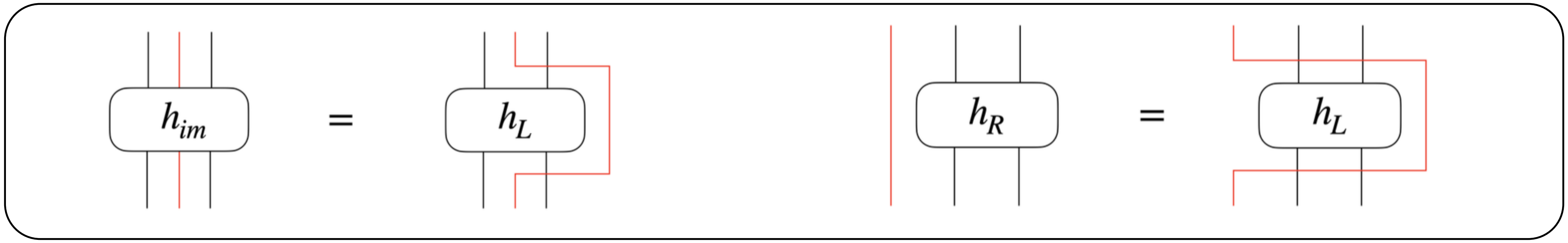
You can freely change N_R



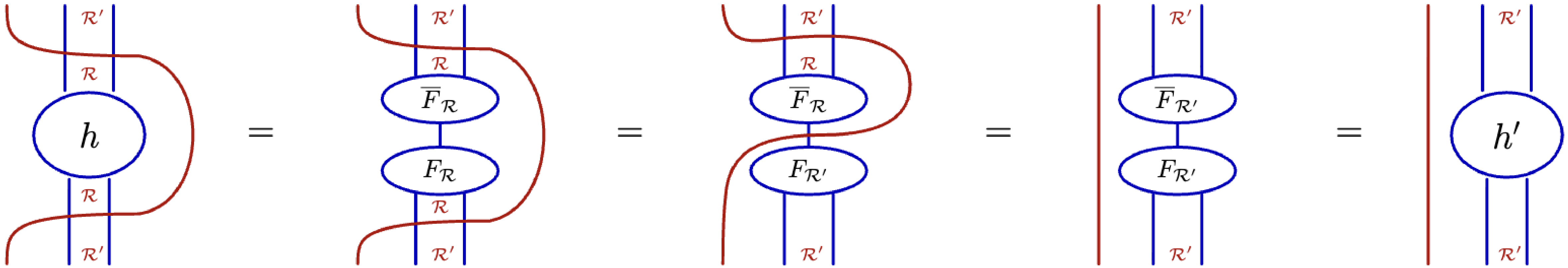
The impurity can move around freely.

Impurity Hamiltonian is unique

The impurity and dual Hamiltonians are determined uniquely once the duality transformation is fixed.

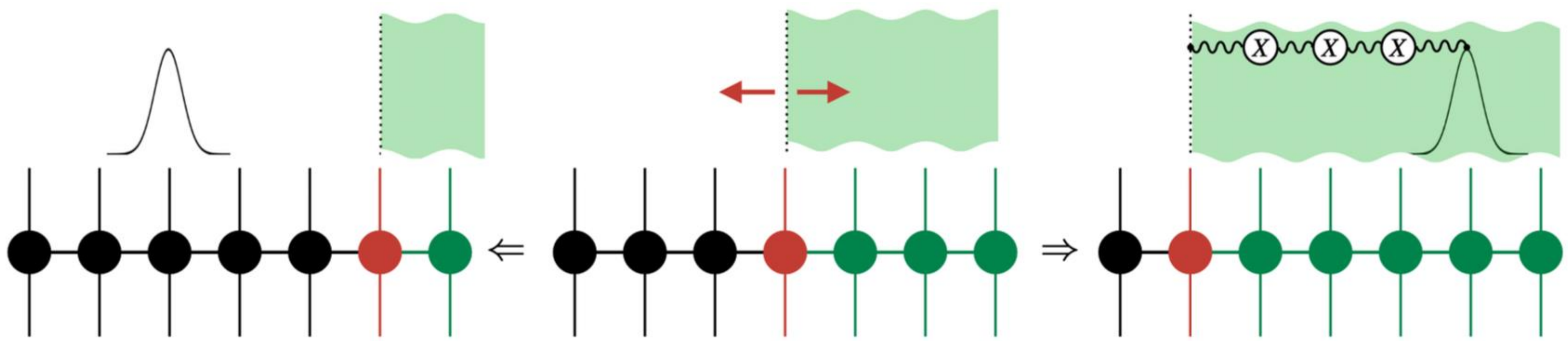


The unitary transformation itself is also unique once relevant symmetry is fixed. Too involved for today's talk ☹

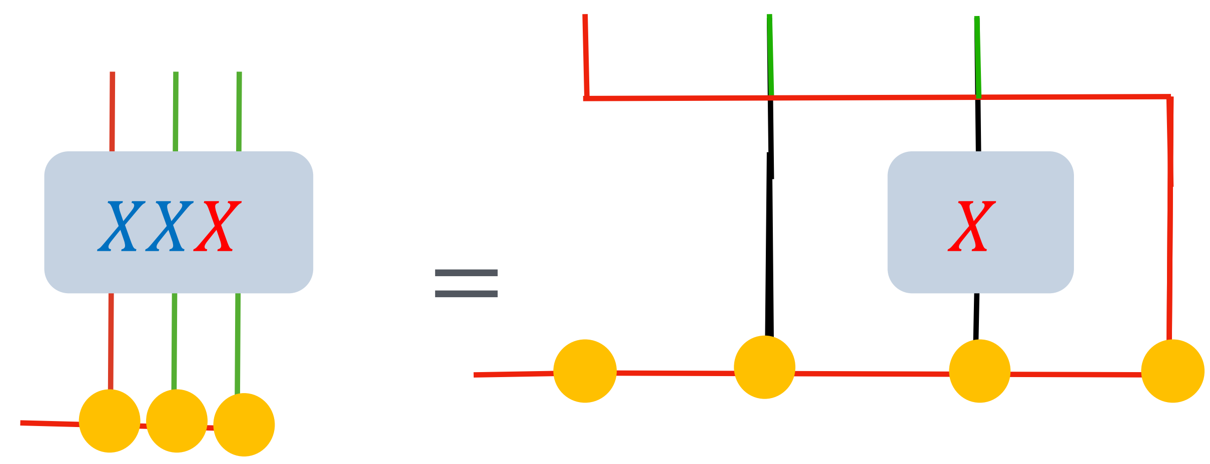


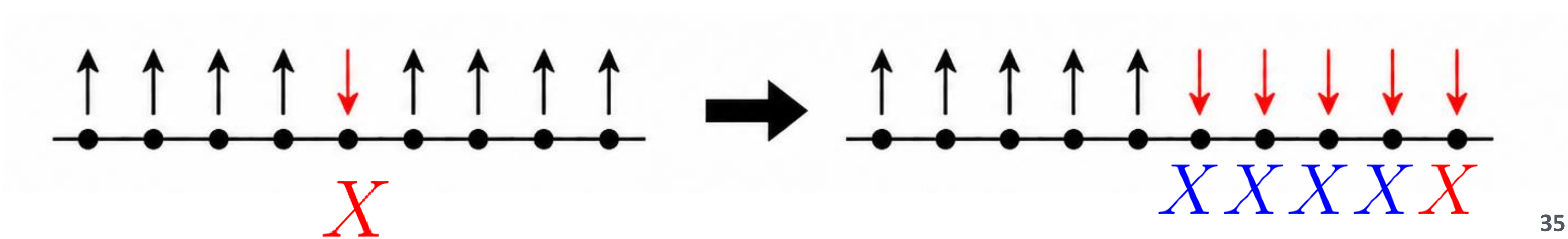
String attachment

MPO creates entanglement between particle and impurity.
The unitary circuit maps the local excitation to non-local one.



$$|\psi_{im}\rangle = U_{KW}^{\dagger}|\psi_{uniform}\rangle$$





Equivalence: Perfect transmission = Duality defect

Duality

H_L ⟷ (Duality) H_R

$\|$

Perfect transmission

Any excitation in H_L can fit in some excitations in H_R

Duality admits perfect transmission with topological string

Beyond Ising

Haldane chain <=> Ferromagnet

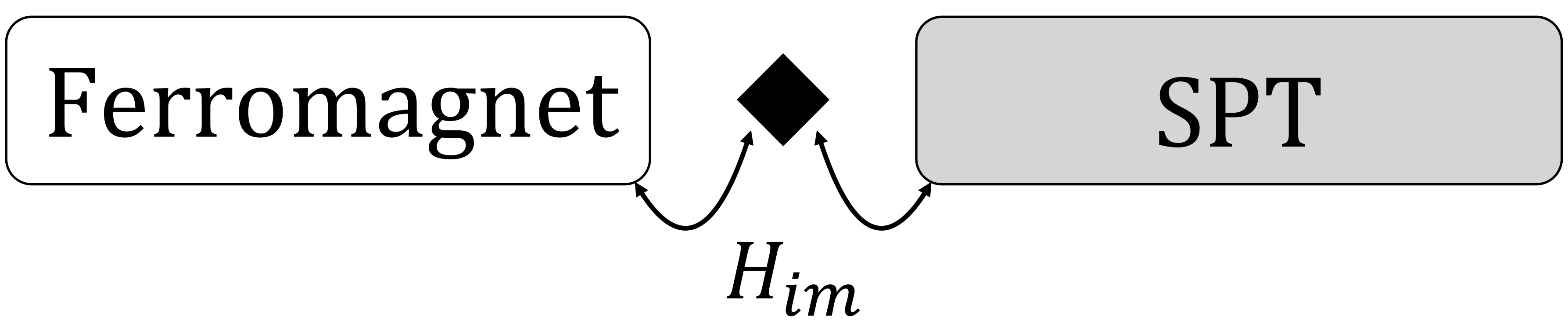
[Kennedy (1992)]

Kennedy-Tasaki transformation is a duality between a SPT phase and ferromagnet phase.

$$S_j'^x = S_j^x e^{i\pi \sum_{k=1}^{j-1} S_k^x} \qquad S_j'^z = e^{i\pi \sum_{k=j+1}^{L} S_k^z} S_j^z$$

$$S_j'^y = e^{i\pi \sum_{k=j+1}^{L} S_k^z} S_j^y e^{i\pi \sum_{k=1}^{j-1} S_k^x}$$

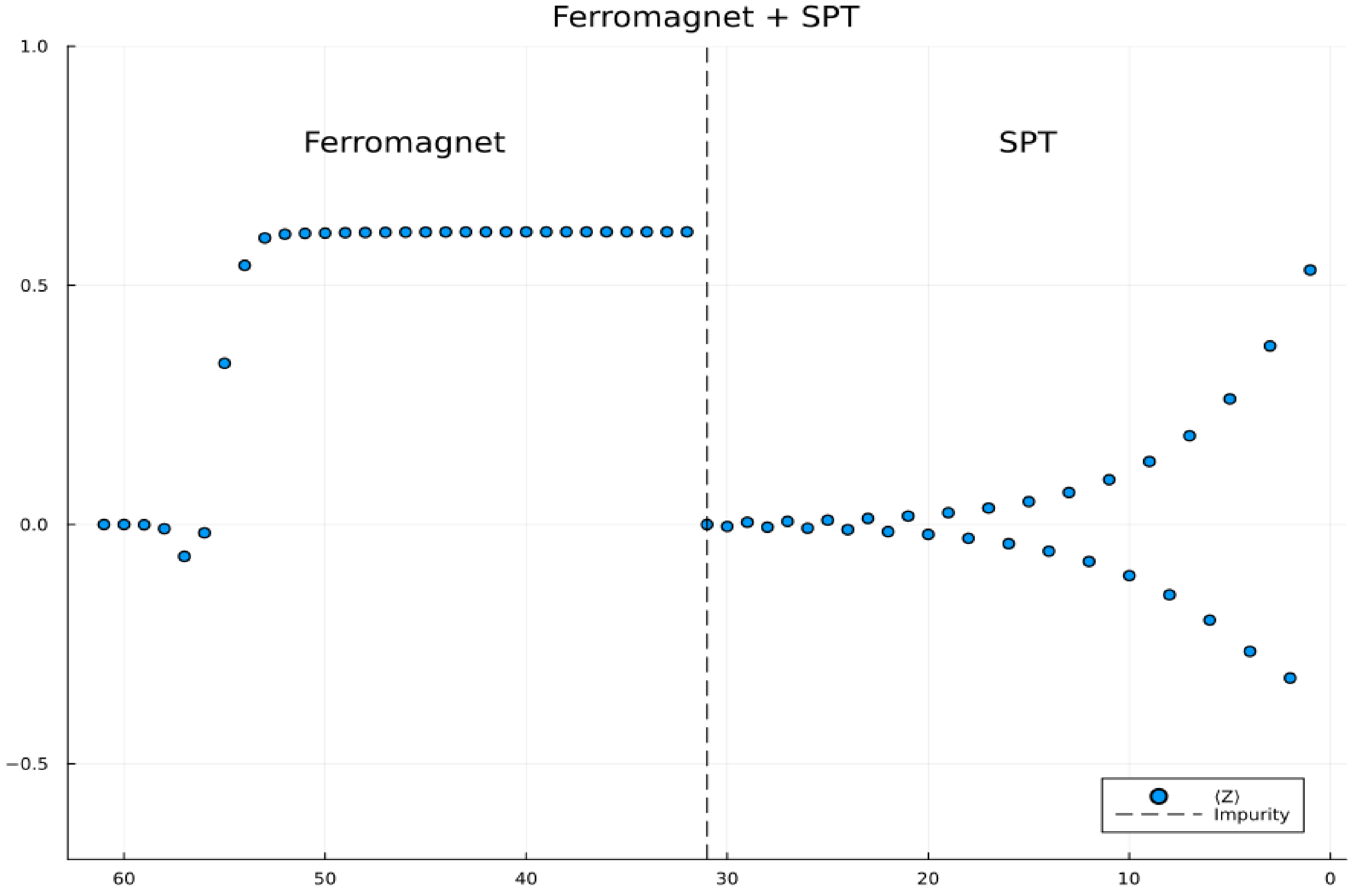
$$H_R = \sum_{i=I+1}^{L-1} (-S_i^x S_{i+1}^x + e^{i\pi S_i^x} S_i^y S_{i+1}^y e^{i\pi S_{i+1}^z} - S_i^z S_{i+1}^z) \qquad H_L = \sum_{i=1}^{I-1} (S_i \cdot S_{i+1})$$



$$H_{im} = S_{I-1}^x \sigma_I^z S_{I+1}^x + i S_{I-1}^y \sigma_I^y S_{I+1}^y e^{i\pi S_{I+1}^z} - S_{I-1}^z \sigma_I^x Z_{I+1}$$

Invisible wavepacket

The wavepacket becomes invisible in SPT phase because it becomes non-local string operator .



$$\langle S_i^z S_j^z \rangle = 0$$

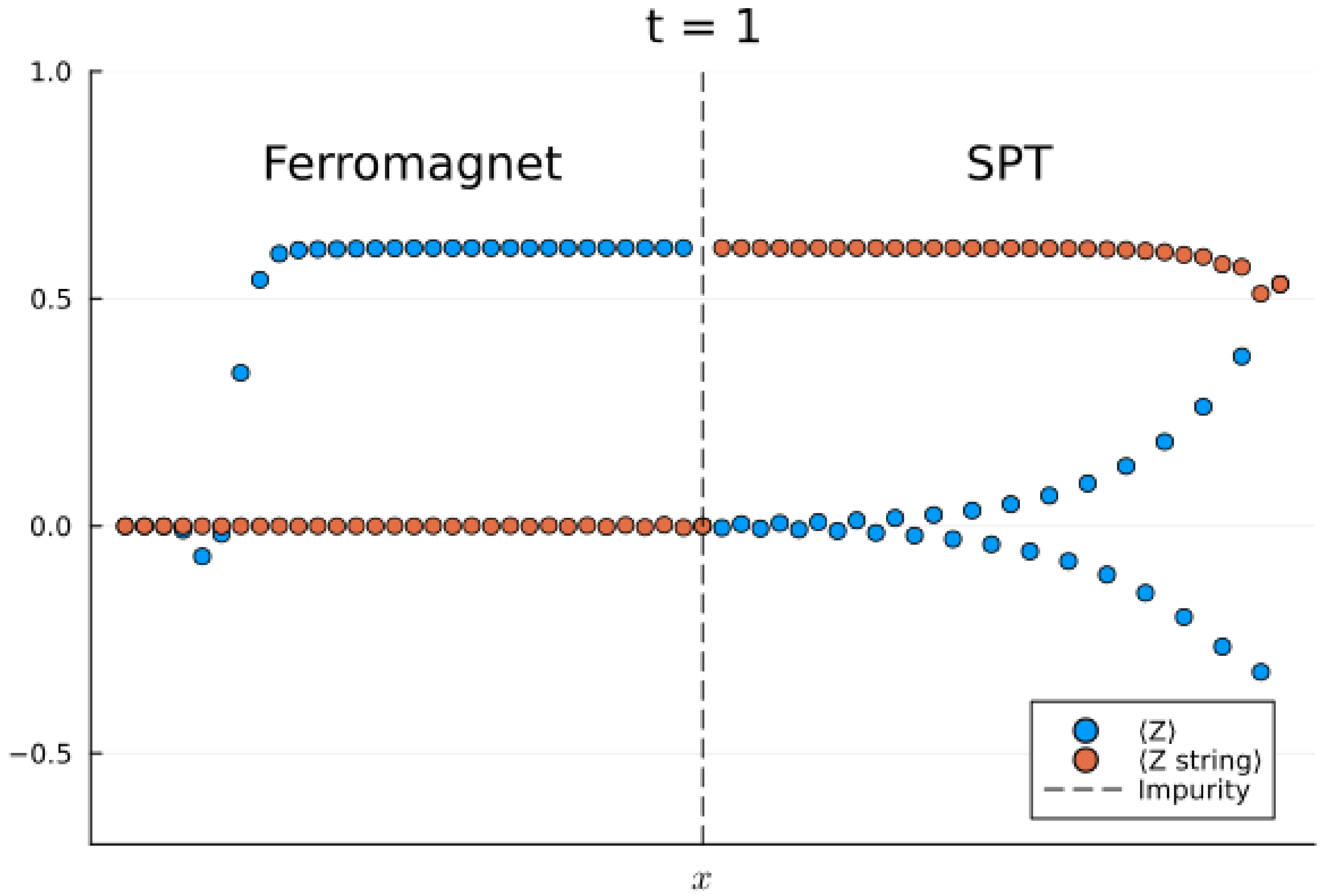
This is a lattice analogue of Callan-Rubakov scattering.

Domain wall => Non-local excitation

The wavepacket becomes visible when one measures a nonlocal string order parameter.

String order parameter

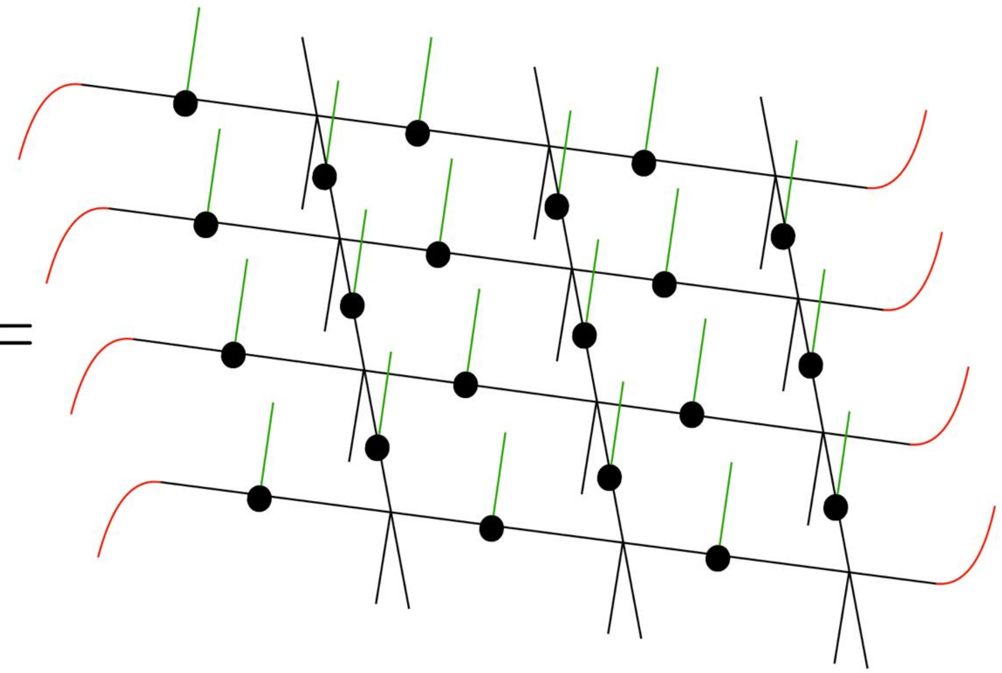
$$\tilde{S}^z_n = e^{i\pi \sum_{j=1}^{n} S^z_j} S^z_n \qquad \langle \tilde{S}^z_i S^z_j \rangle \neq 0$$



Higher dimension: (2+1)d PEPO

The unitary circuit becomes PEPO in two dimensions

$$U_{KW} =$$



TF Ising (paramagnet)

$$H_L = -\sum_{i,j} Z_i Z_j - g \sum_i X_i$$

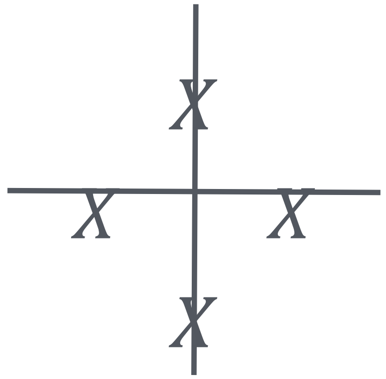
→

Toric code + magnetic field(SPT)

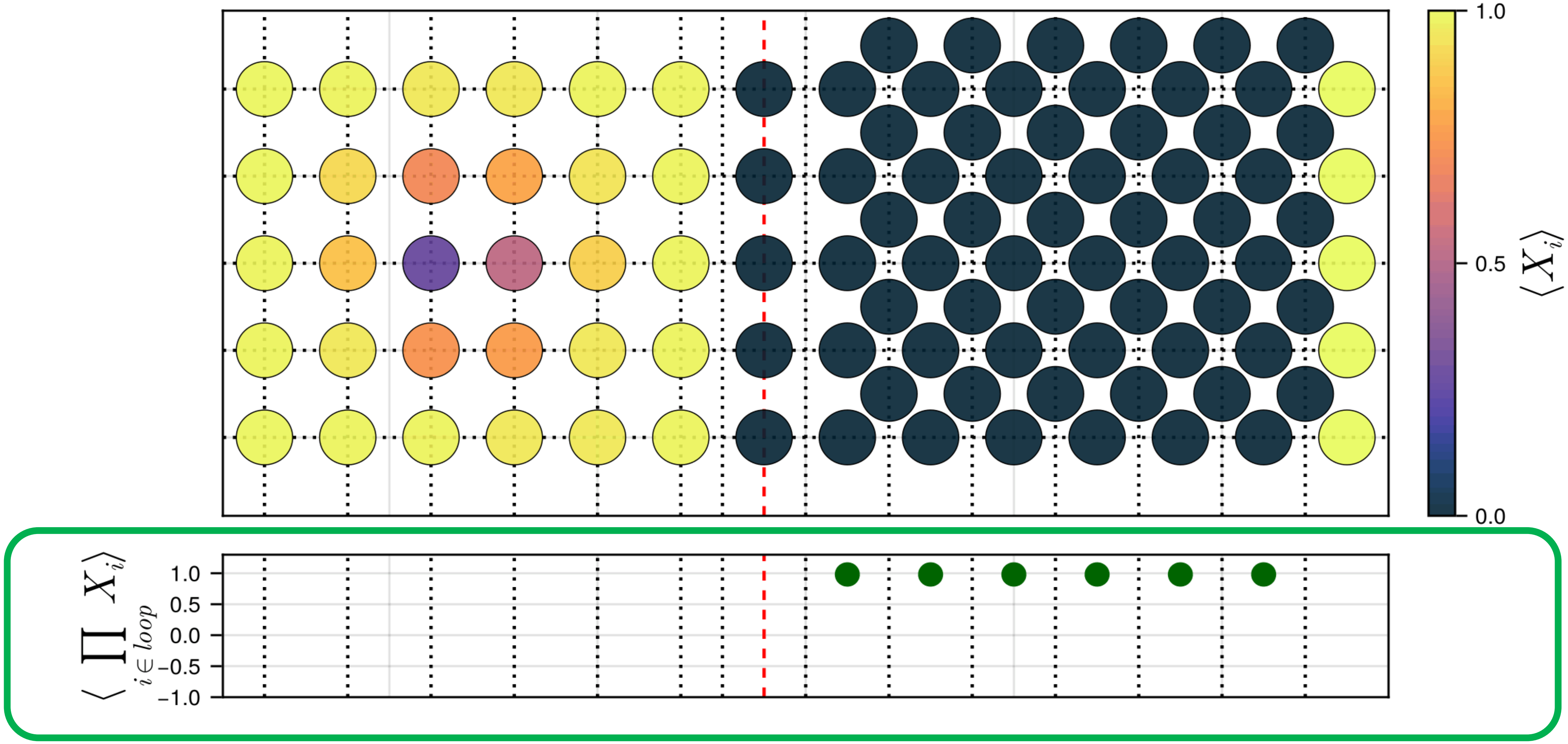
$$H_R = -\sum_i Z_i - g \sum_v A_v$$

$$H_{im} = -\sum_y Z_{(I-1,y)} Z_{(I,y)} Z_{(I+1,y)}$$

$$A_v =$$



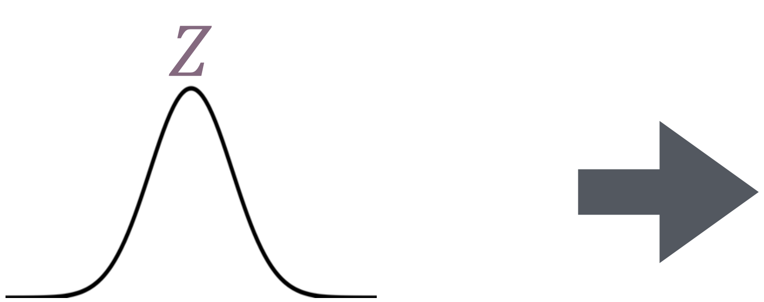
(2+1)d transverse-field Ising model

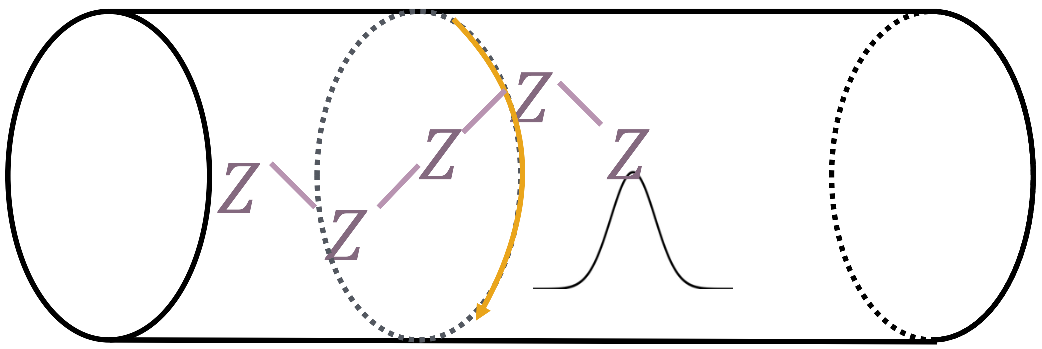


The scattered state is detected by a Wilson loop operator

Spin-flip to string excitation

$$\mathcal{L} = \prod_{i \in \text{loop}} X_i$$



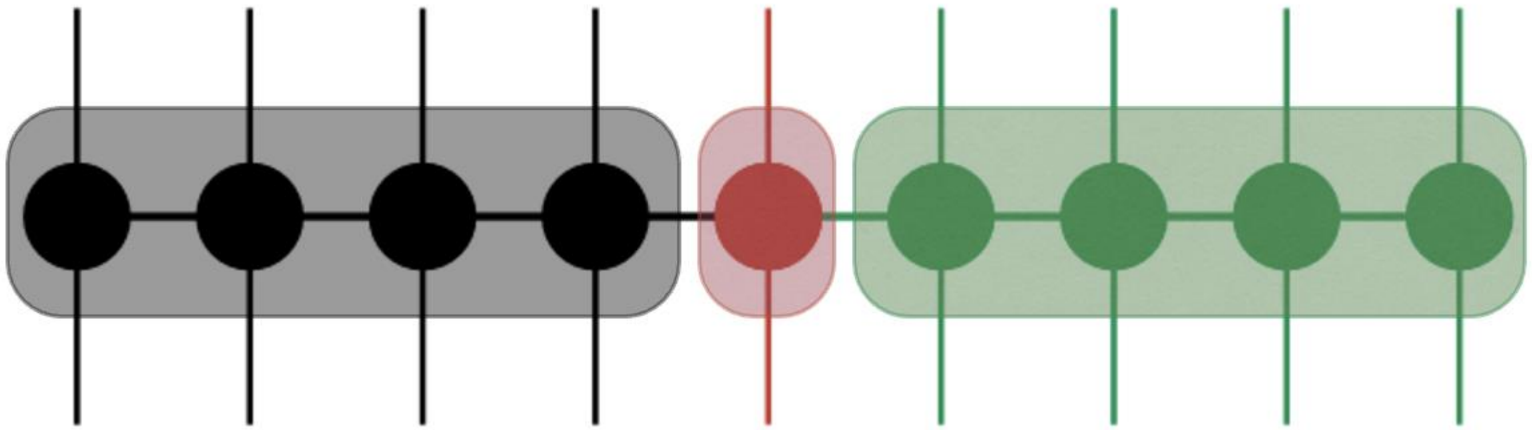


Back to chiral fermions:

Andreev reflection as
a low-energy realization of
a monopole scattering on the lattice

Ising impurity model revisited

The Ising impurity model becomes simple in Majorana representation



This model is constructed by sliding a Majorana zero mode throughout the system.

Majorana representation

$$H_{\text{Ising}} = -\sum_{j=1}^{N-1} X_j X_{j+1} - g\sum_{j=1}^{N-1} Z_j$$

Using the JW transformation, the Ising model becomes a Majorana chain as

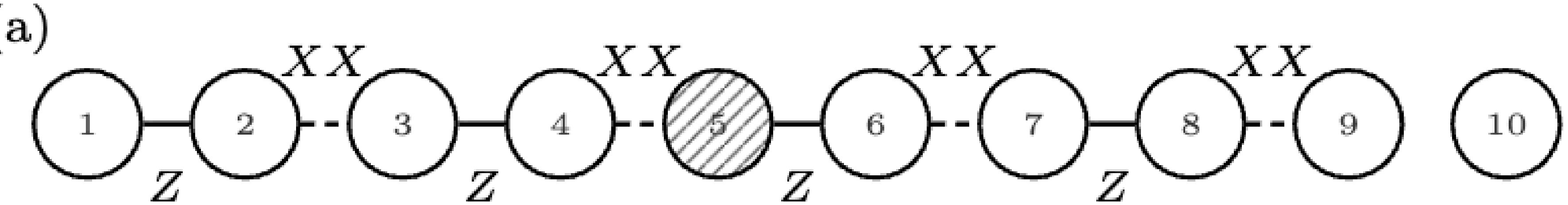
$$\chi_{2j-1} = \left(\prod_{k=1}^{j-1} Z_k\right) X_j, \; \chi_{2j} = \left(\prod_{k=1}^{j-1} Z_k\right) Y_j$$

$$H_{\text{Majorana}} = i\sum_{j=1}^{N-1} [g\chi_{2j-1}\chi_{2j} + \chi_{2j}\chi_{2j+1}]$$

Z_j $\qquad$ $X_j X_{j+1}$

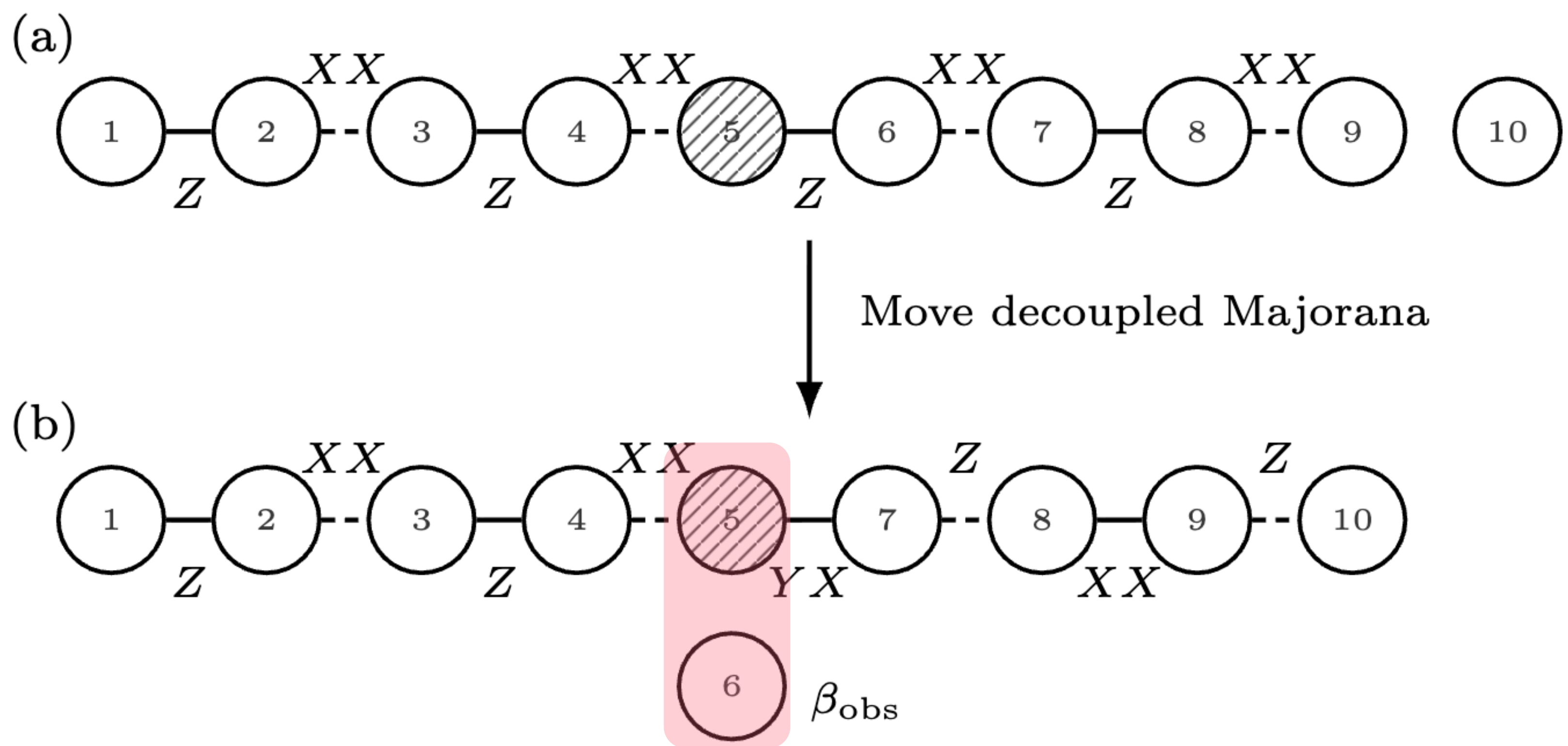
The last Majorana operator is decoupled.

(a)



Majorana zero mode = topological impurity

Moving the decouple does not change the energy spectrum

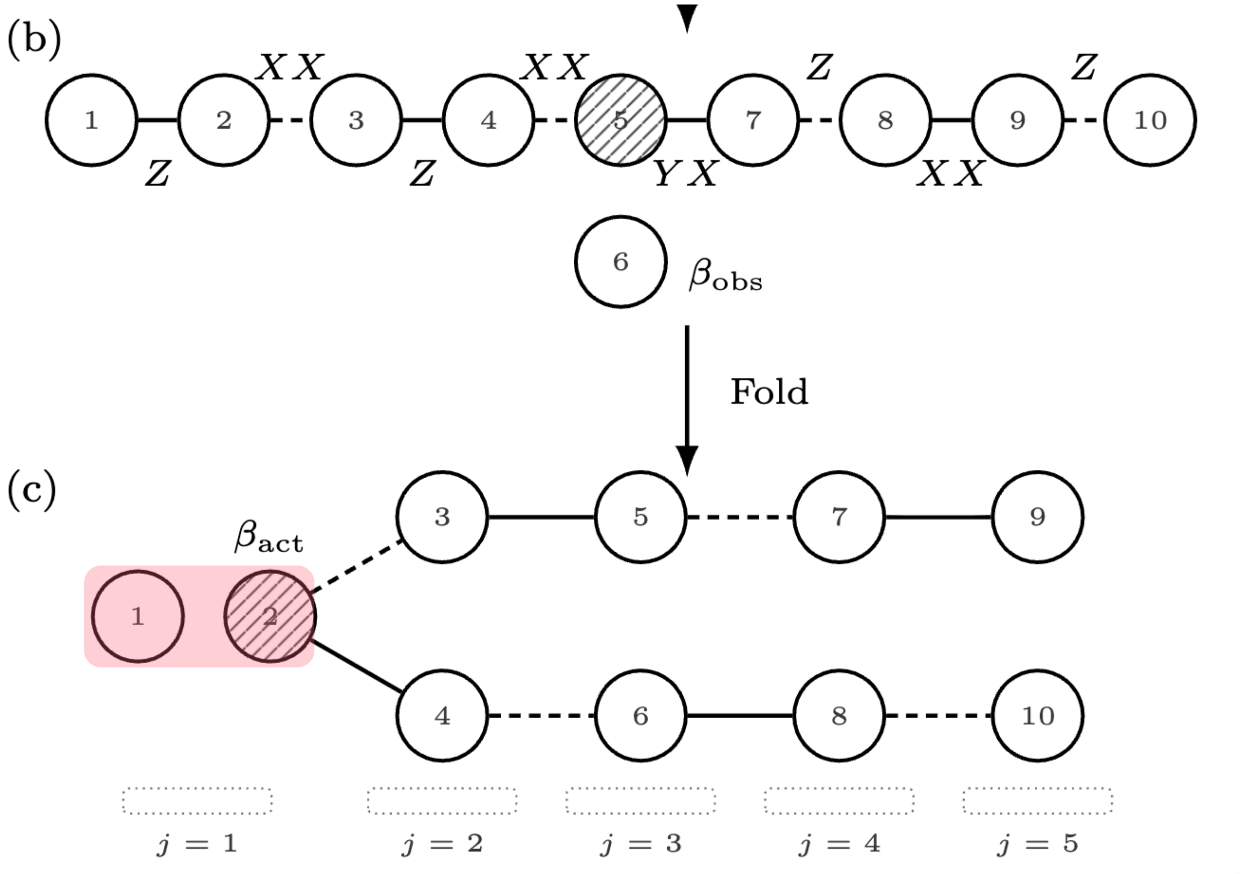


However, the Hamiltonian changes: The Ising model with a duality defect.

$$H_{\text{impurity}} = -\sum_{j=1}^{I-2} X_j X_{j+1} - g\sum_{j=1}^{I-1} Z_j - X_{I-1}X_I - gY_I X_{I+1} - \sum_{j=I+1}^{N} Z_j - g\sum_{j=I+1}^{N-1} X_j X_{j+1}$$

Folding trick

By folding the topological impurity becomes a boundary.



$$H_{\text{EK}} = i\beta_{\text{act}}\left[c_2 + c_2^\dagger + i\left(c_2 - c_2^\dagger\right)\right]$$
$$+2i\sum_{j=2}^{N-1}\left(c_j^\dagger c_{j+1} - c_{j+1}^\dagger c_j\right)$$

The spectator Majorana mode gives the nontrivial quantum dimension $g = \sqrt{2}$.

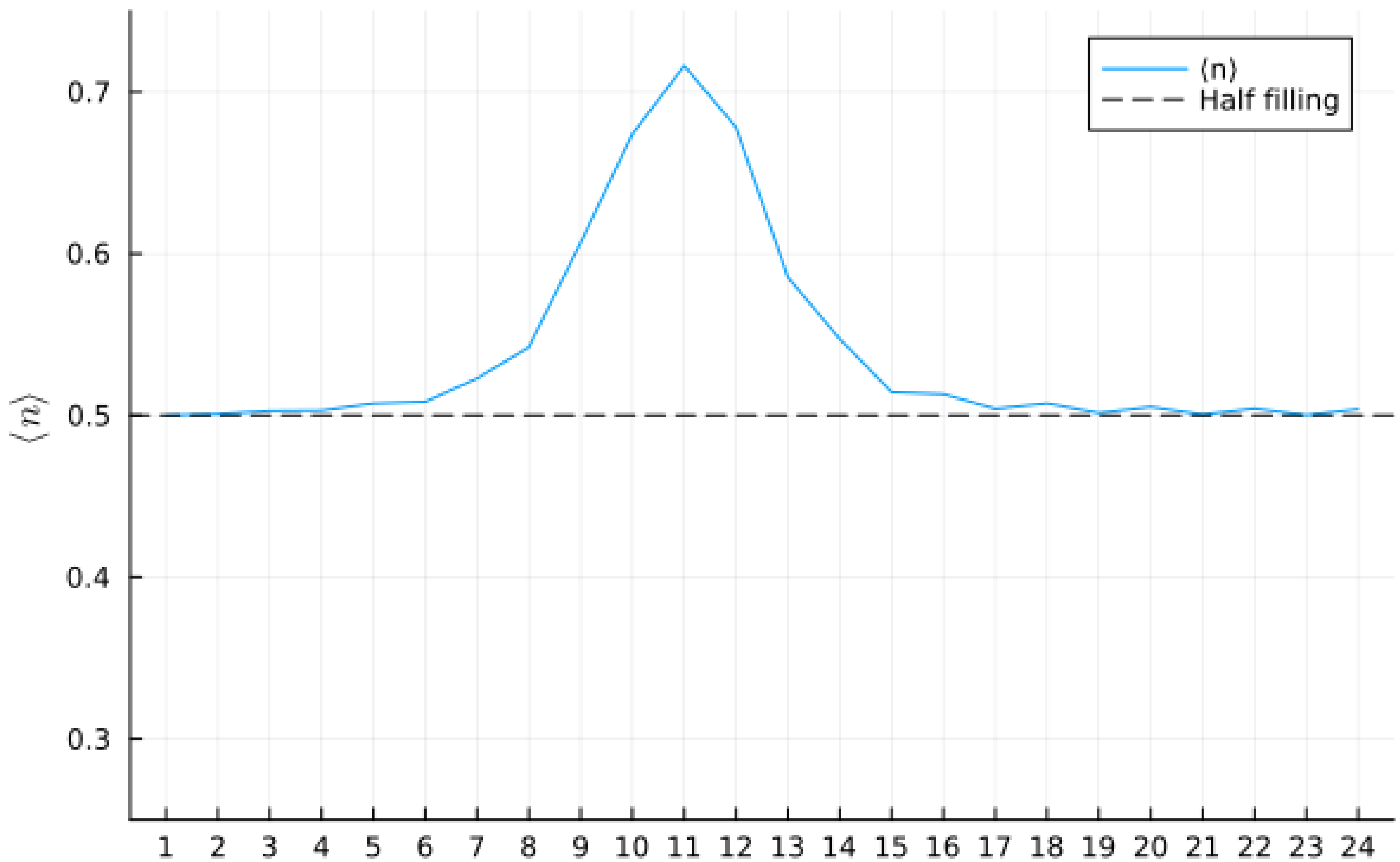
This is **the same** as Maldacena-Ludwig monopole.

Effective low-energy phenomena

The effective field theory of Maldacena-Ludwig is a charge-flip boundary $\psi_{\text{out}} \sim \psi_{\text{in}}^{\dagger}$

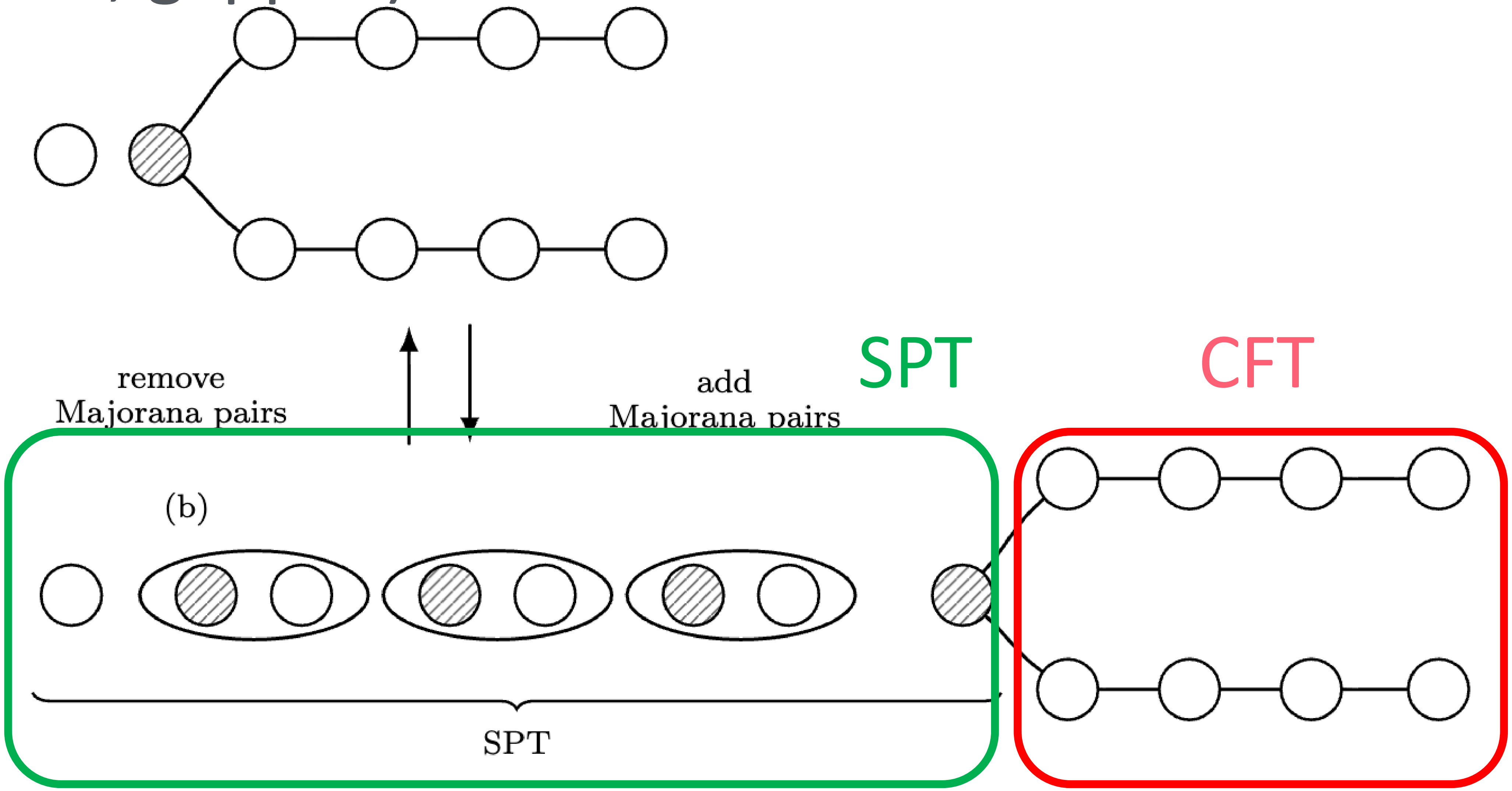
This feature emerges in the low energy of the folded model.

EK t/δt = 1, g = 1.0



Universal phenomena?

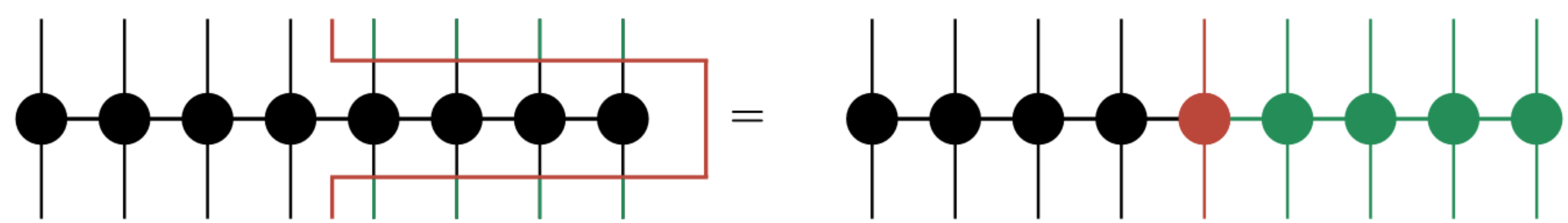
The topological scattering seems to emerge as an RG fixed point of the interface between CFT(self-dual) and SPT(dual, gapped)



The gapped SPT is related to SMG????

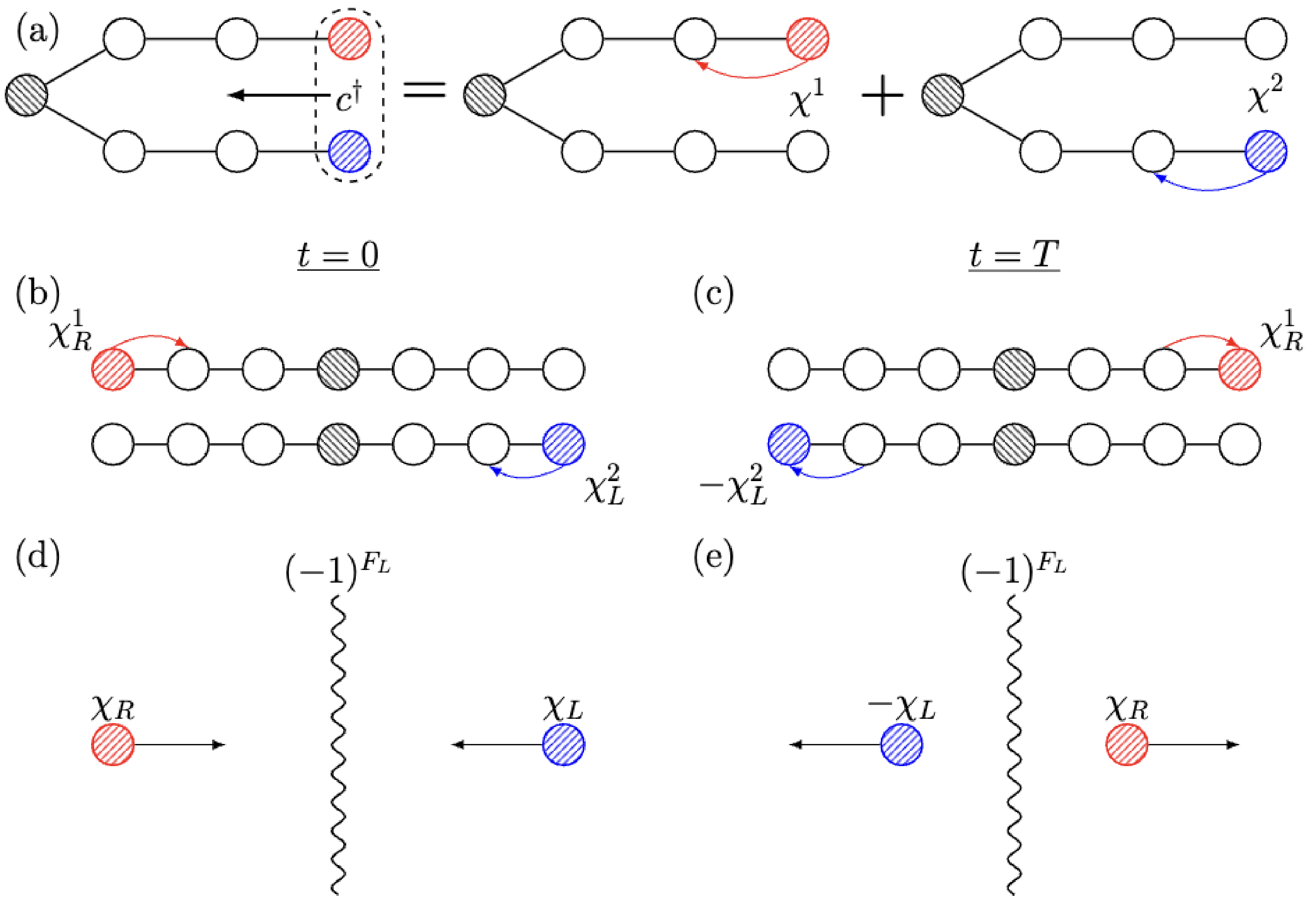
Summary

- Unitarity puzzle = "**Particle disappears**" in local basis
- S-matrix restores unitarity when the Hilbert space is expanded with a new **impurity degree of freedom**
- **Impurity site = virtual leg of duality MPOs**
- Scattering process is **perfect** and new particle is **non-local excitation**
- These scattering might appear in a universal manner between **CFT** and **SPTs (SMGs)**



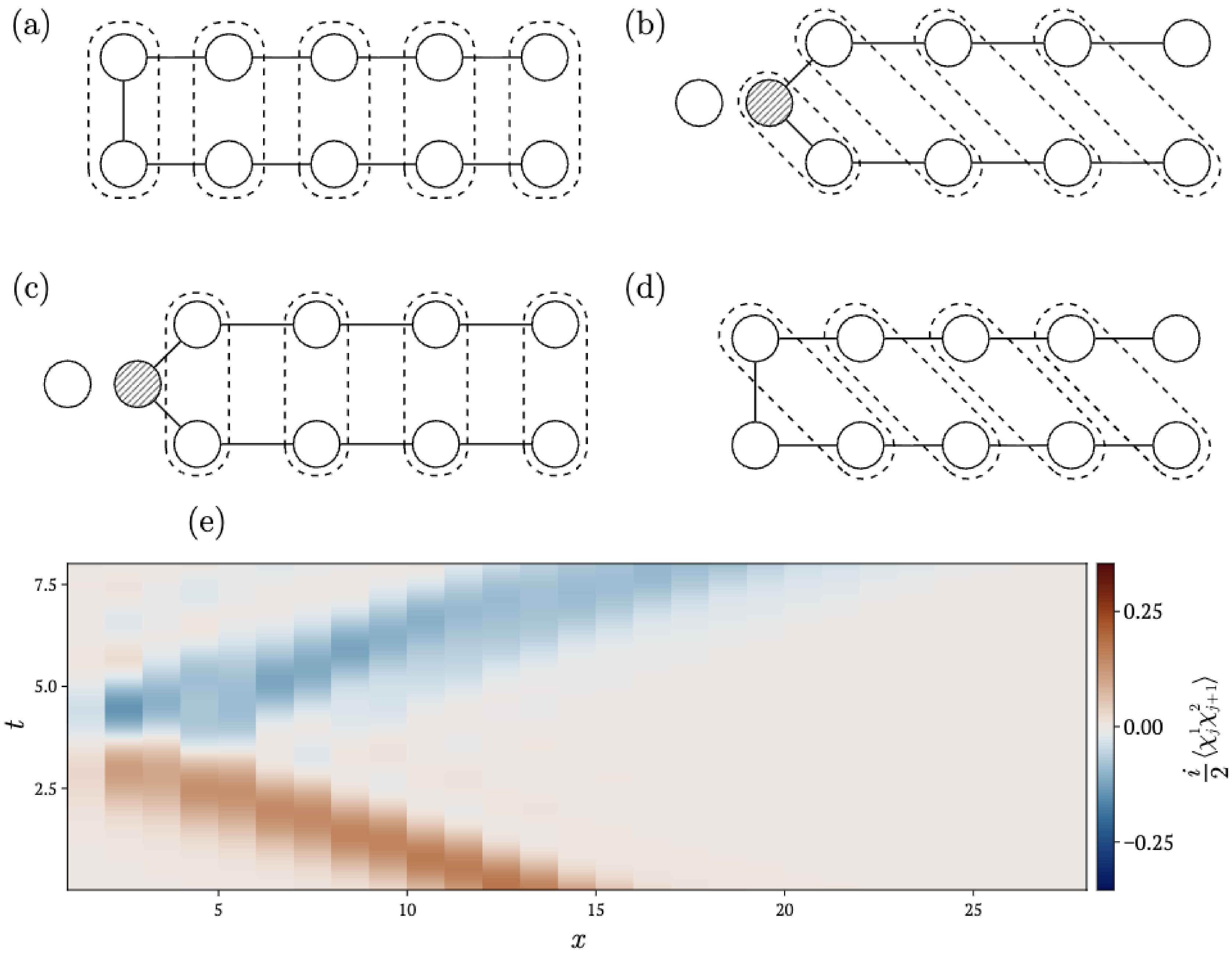
Chiral fermion parity defect

$$c_n \sim \psi_R(x) + (-1)^n \psi_L(x)$$



[Seiberg, Shao (2024), Numasawa's talk]

Andreev reflection for axial charge



[Thacker(1995),Chatterjee (2025), Numasawa's talk]

Building block

$$h'_{j+1} = U^\dagger_{KW} h_j U_{KW} \quad \Longleftrightarrow$$