Fermions and Zeta Function on the Graph

Kazutoshi Ohta

Institute for Mathematical Informatics,

Meiji Gakuin University

Collaboration with So Matsuura, Keio Univ.

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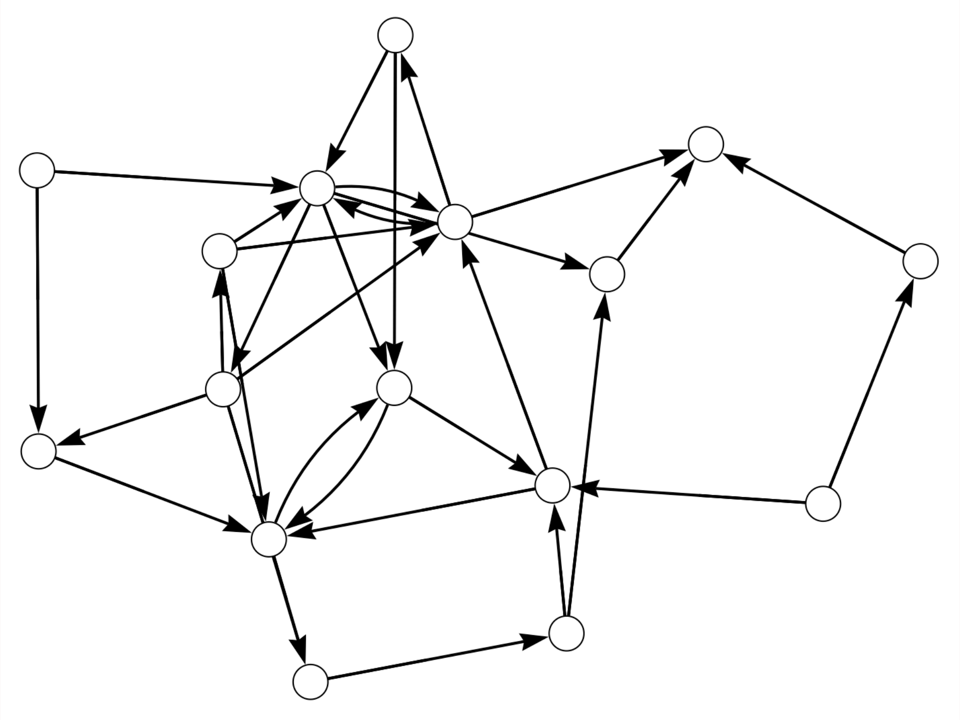
& some additional results in progress

Introduction



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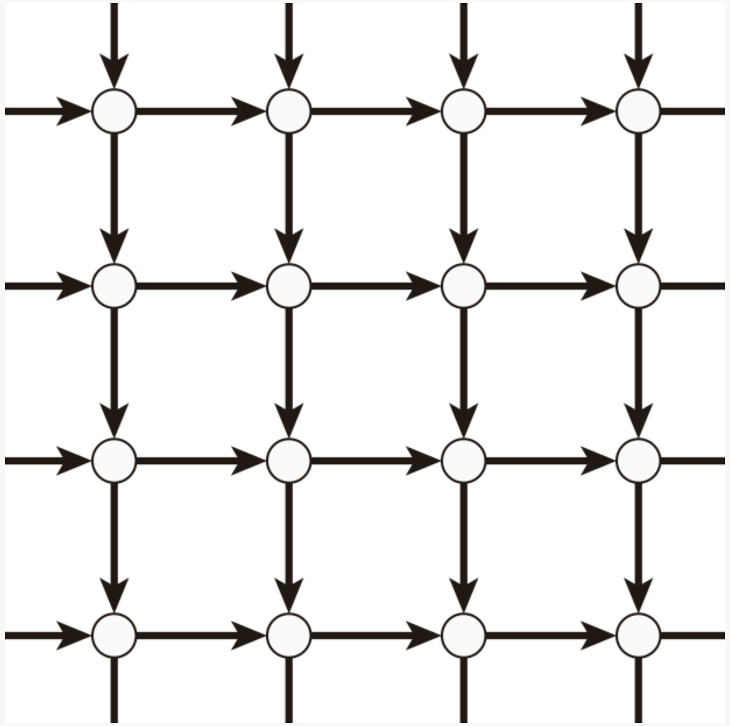
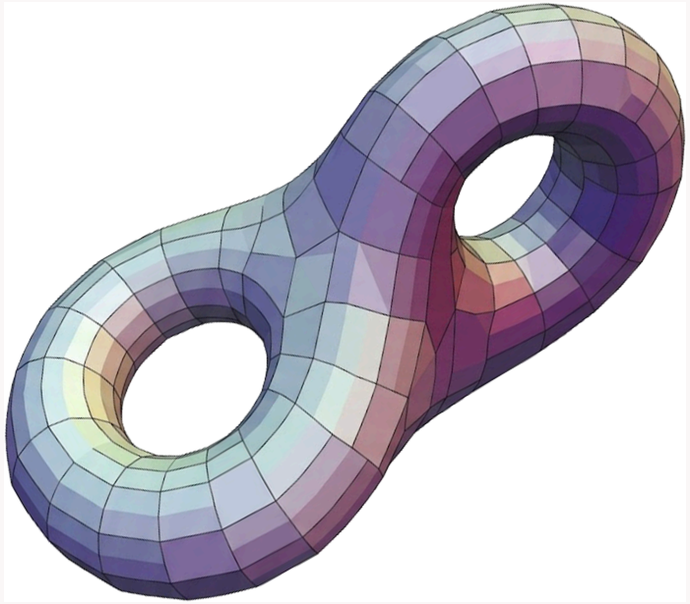


Random Graphs
$\Rightarrow$ graph theory, information theory, network models … **In mathematics**

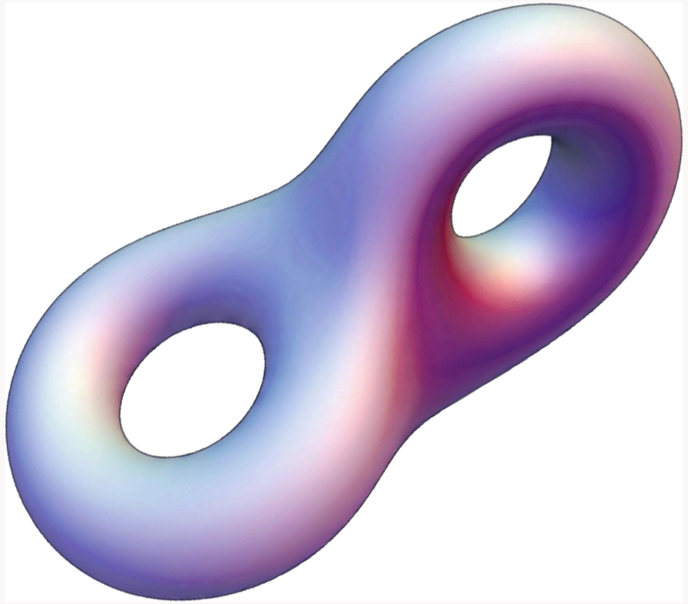
Square Lattice (Grid Graph)
$\Rightarrow$ statistical models, lattice gauge theory … **In physics**

Introduction



$\Rightarrow$

cont. limit



topology of graph/network
(index theorem?)

topology of manifolds
(index theorem)

Fermions on the graph would be important to understand the topology of the graph.

Various approach to fermions on the graph

- Various approaches have been used to construct fermion theories on graphs, such as:
    - Statistical model approach, like Ising model or dimer model on the graph [Kenyon (2002)] [Cimasoni (2009)]
    - Application of the spectral graph theory [Yumoto-Misumi (2021-2023)]
- Later, I will discuss a relation to the statistical model.

We construct the fermions on the generic graphs by utilizing the graph zeta function.

Part I: Construction of the fermoins associated with the graph zeta function

Graph Theory

- A graph Γ consists of sets of the vertices V and edges E: $\Gamma = (V, E)$
- Edges are connected by arrows from "source" vertices $s(e)$ to "target" vertices $t(e)$ for each edge $e \in E$ (directed graph)
- The structure of the graph is given by the adjacency matrix A or incidence matrix L.

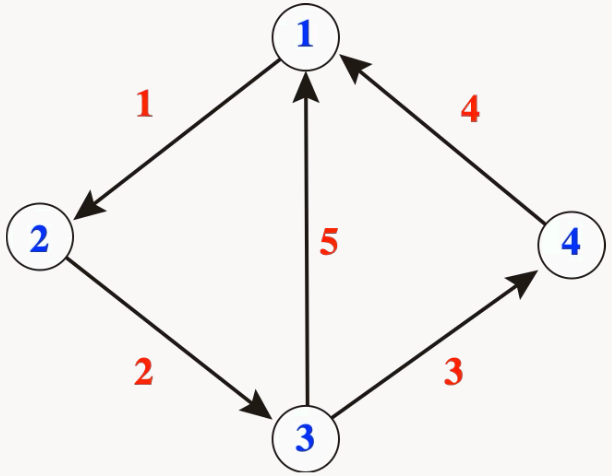
e.g.) double triangle graph ($K_4 - e$)

Adjacency matrix:

$$\begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ A = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Incidence matrix:

$$\begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ L^T = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} -1 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix} \end{array}$$



Basic idea: From graph Laplacian to Dirac operator

Laplacian on the graph (graph Laplacian) Δ is 2nd order difference operator between vertices;

$$\boldsymbol{x}^T \Delta \boldsymbol{x} = \sum_{e \in E} (\boldsymbol{x}^{t(e)} - \boldsymbol{x}^{s(e)})^2 ,$$

which can be written a square of the incidence matrix

$$\Delta = L^T L$$

So we can regard the incidence matrix as the Dirac operator on the graph by analogy of $\Delta = \not{\partial}^2$ in the continuum theory.

This idea was used for the construction of the supersymmetric gauge theory on the graph [Matsuura-KO 2021], which is a generalization of the Sugino model.

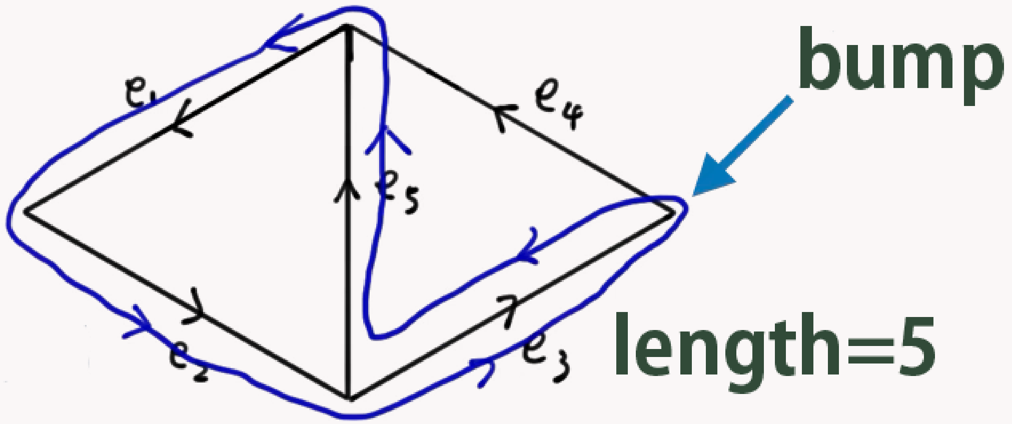
Graph zeta function

Ihara zeta function [Ihara (1966)] and its extension, the Bartholdi zeta function [Bartholdi (2001)] on the graph Γ, express the number of cycles (closed loops) on the graph as coefficients of a polynomial.

$$\zeta_\Gamma(q,u) \equiv \prod_{[C]:\ \text{primitive cycles}} \frac{1}{1 - u^{b(C)} q^{\ell(C)}},$$

where

- primitive cycles: it can not be expressed by $C = (C')^r \quad (r \geq 2)$
- $\ell(C)$: length of the cycle
- $b(C)$: # of the bumps



cf. Riemann zeta function

Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \, ,$$

has an infinite product expression (Euler product)

$$\zeta(s) = \prod_{p\text{: prime numbers}} \frac{1}{1 - p^{-s}} \, .$$

Ihara's theorem

The graph zeta function is given by the determinant of the graph Laplacian deformed by the parameters q and u :

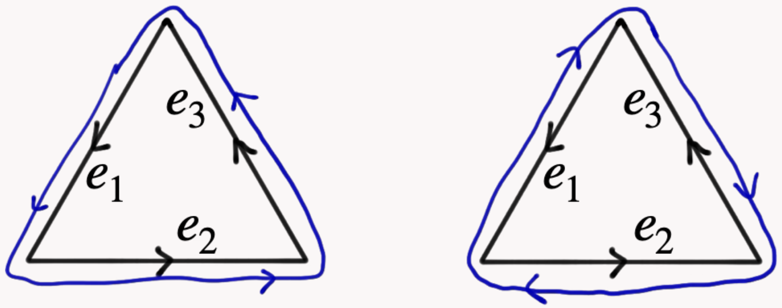
$$\zeta_\Gamma(q,u) = \frac{1}{(1-q^2(1-u)^2)^{n_E-n_V}\det\Delta_{q,u}},$$

where $\Delta_{q,u} \equiv I_{n_V} - qA + q^2(1-u)\left(D-(1-u)I_{n_V}\right)$ 📝

- n_V: # of the vertices, n_E: # of the edges
- A: adjacency matrix, D: degree matrix

📝 $\Delta_{1,0} = D - A = \Delta, \quad \Delta_{q,1} = I - qA$

e.g. Triangle graph (cycle graph) at $u = 0$



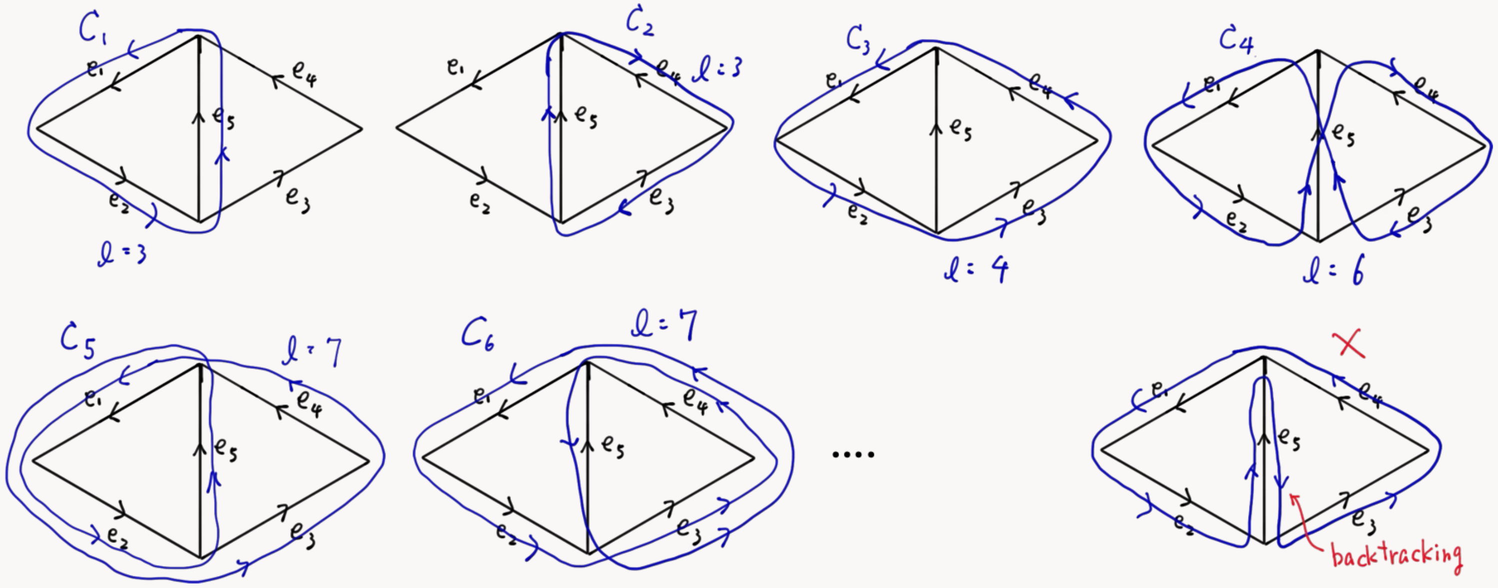
For the triangle graph (C_3-graph), we have only two independent primitive cycles:
$[C] = \{e_1e_2e_3, e_2e_3e_1, e_3e_1e_2\}$, $[\bar{C}] = \{\bar{e}_3\bar{e}_2\bar{e}_1, \bar{e}_1\bar{e}_3\bar{e}_2, \bar{e}_2\bar{e}_1\bar{e}_3\}$

The Ihara zeta function is given by

$$\zeta_{C_3}(q) = \frac{1}{(1-q^3)^2} = 1 + 2q^3 + 3q^6 + 4q^9 + 5q^{12} + \cdots$$

📝 The above series expansion counts all of possible cycles including composites of primitive cycles.

e.g. Double triangle graph ($K_4 - e$) at $u = 0$



$$\zeta_{\mathrm{DT}}(q) = \frac{1}{(1-q^3)^4}\frac{1}{(1-q^4)^2}\frac{1}{(1-q^6)^2}\frac{1}{(1-q^7)^4}\frac{1}{(1-q^9)^4}\frac{1}{(1-q^{10})^{12}}\cdots$$

$$= \frac{1}{1-4q^3-2q^4+4q^6+4q^7+q^8-4q^{10}}$$

$$= 1+4q^3+2q^4+12q^6+12q^7+3q^8+32q^9+52q^{10}+\cdots$$

Our goal

Our goal is to formulate a fermionic theory on a graph such that its partition function exactly equals the inverse of the graph zeta function, i.e.

$$Z_\Gamma = \zeta_\Gamma(q,u)^{-1} \,.$$

This partition function has an infinite product expression

$$Z_\Gamma = \prod_{[C]\text{: primitive cycles}} \left(1 - u^{b(C)} q^{\ell(C)}\right) ,$$

and it reduces to the determinant (Ihara) expression

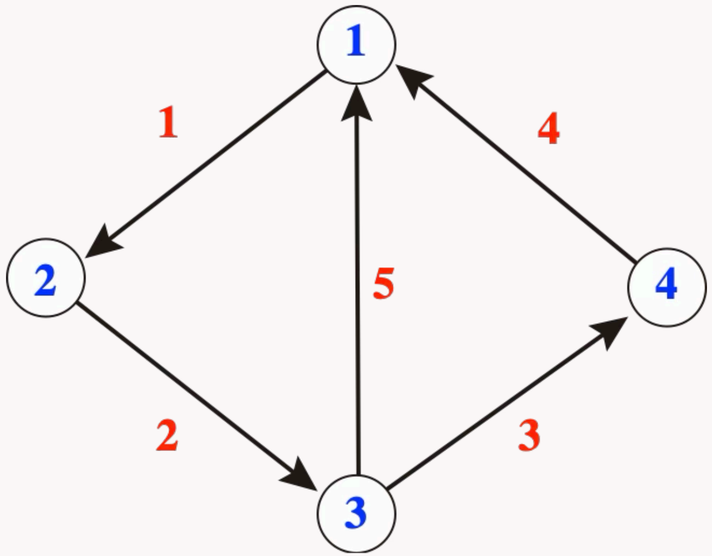
$$Z_\Gamma = (1 - q^2(1-u)^2)^{n_E - n_V} \det \Delta_{q,u} \,,$$

which is a finite ($2n_E$) degree polynomial in q. 😮⁉️

e.g. Double triangle graph ($u = 0$)

$$\prod_{[C]\text{: prime cycles}} \left(1 - q^{\ell(C)}\right) = (1-q^3)^4(1-q^4)^2(1-q^6)^2(1-q^7)^4(1-q^9)^4(1-q^{10})^{12}(1-q^{11})^4(1-q^{12})^6\cdots$$

$$= 1 - 4q^3 - 2q^4 + 4q^6 + 4q^7 + q^8 - 4q^{10} = (1-q^2)\det\Delta_{q,0}$$



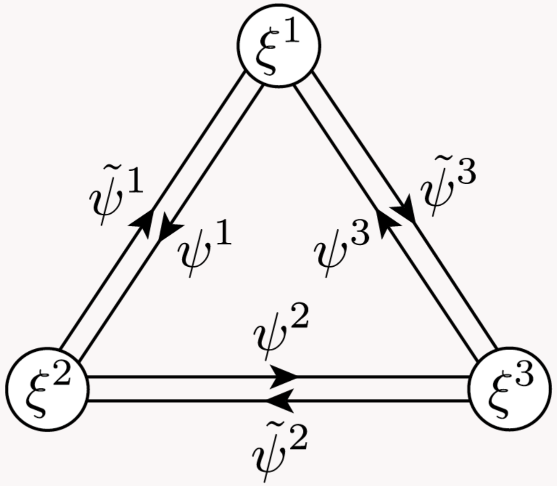
prime cycle = primitive and reduced (without bumps) contributed only at $u = 0$

Construction of the fermions on the graph

1. We first introduce a parameter deformation of the incidence matrix by

$$(L_{q,u})^e{}_v \equiv \begin{cases} 1 & \text{if } v = t(e) \\ -q(1-u) & \text{if } v = s(e), \\ 0 & \text{others} \end{cases} \quad (\tilde{L}_{q,u})^e{}_v \equiv \begin{cases} 1 & \text{if } v = s(e) \\ -q(1-u) & \text{if } v = t(e) \\ 0 & \text{others} \end{cases}$$

2. Put the vertex fermions ξ^v on each $v \in V$ and the edge fermions $(\psi^e, \tilde{\psi}^e)$ on each $e \in E$



3. Construct the Dirac operator and mass matrix by

$$\not{D} + \mathcal{M} = \begin{pmatrix} I_{n_V} & \alpha \tilde{L}_{q,u}^T & \alpha L_{q,u}^T \\ \alpha L_{q,u} & I_{n_E} & -q(1-u) I_{n_E} \\ \alpha \tilde{L}_{q,u} & -q(1-u) I_{n_E} & I_{n_E} \end{pmatrix},$$

where $\alpha = \sqrt{\frac{q}{1-q^2(1-u)^2}}$, which is acting on the Grassmann valued vector

$$\Psi = (\xi, \psi, \tilde{\psi})^T, \quad \bar{\Psi} = (\bar{\xi}, \bar{\psi}, \bar{\tilde{\psi}}).$$

4. Action for the fermion is given by

$$S_F = \bar{\Psi}(\not{D} + \mathcal{M})\Psi .$$

Partition function

The partition function of this model

$$Z_\Gamma(q,u) \equiv \int \mathcal{D}\Psi\mathcal{D}\bar{\Psi}\, e^{-S_F(q,u)} = \det\left(\not{D} + \mathcal{M}\right),$$

coincides with the inverse of the graph zeta function

$$Z_\Gamma(q,u) = (1 - q^2(1-u)^2)^{n_E - n_V} \det \Delta_{q,u} = \zeta_\Gamma(q,u)^{-1}\,.$$

Integrating out the vertex fermions, we obtain another expression (**Hashimoto expression**) of the graph zeta function using **edge adjacency matrix** W and bump matrix J.

$$Z_\Gamma(q,u) = \det\left(q(W+uJ) - I_{2n_E}\right)$$

The equivalence of the Ihara and Hashimoto expression was rigorously proven in [Bass (1992)], where a first-order difference operator, essentially identical to the Dirac operator $\not{D} + \mathcal{M}$, appears (motivating our construction).

Analytically, the fermionic action in the Hashimoto expression

$$S = \bar{\eta}^{\boldsymbol{e}}\left(q(W+uJ) - I_{2n_E}\right)_{\boldsymbol{e}\boldsymbol{e}'}\eta^{\boldsymbol{e}'}$$

is useful. (This model is essentially related to Kac-Ward determinant or Kenyon's construction.)

Generating function of the fermionic cycles

The partition function gives the Witten index counting fermionic states (**fermionic cycles**).

$$\zeta_\Gamma(q,u)^{-1} = 1 + \sum_{[C]} \mu(C) u^{b(C)} q^{\ell(C)} \,,$$

where $\mu(C)$ is a cycle Möbius function defined by

$$\mu(C) = \begin{cases} 0 & \text{if the same directed edge is included somewhere in } C \\ (-1)^F & \text{if } C \text{ contains } F \text{ distinct primitive cycles} \end{cases} .$$

The above state counting can be proved from the Hashimoto expression.

Comparison with the Riemann ζ-function and the Möbius function

The inverse of the Riemann zeta function

$$\zeta(s)^{-1} = \prod_{p\text{: prime numbers}} (1 - p^{-s})$$

is expressed as

$$\zeta(s)^{-1} = \sum_{n=1}^{\infty} \mu(n)\, n^{-s} \,,$$

by using the Möbius function $\mu(n)$.

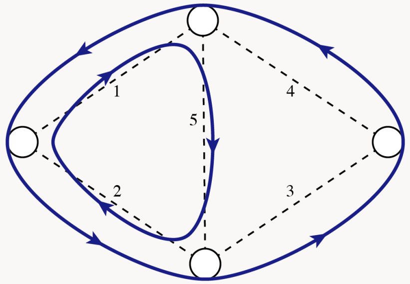
The cycle Möbius function $\mu(C)$ plays the same role as the number-theoretic Möbius function in the Euler product of $\zeta(s)^{-1}$.

e.g) double triangle graph

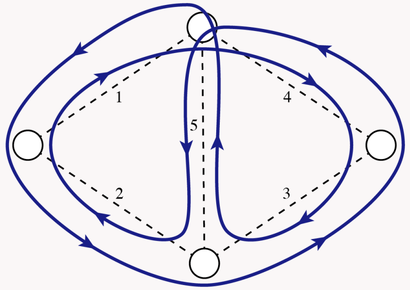
$$\zeta_{\mathrm{DT}}(q)^{-1} = 1 - 4q^3 - 2q^4 + 4q^6 + 4q^7 + q^8 - 4q^{10}$$

$\ell(C)$	fermionic cycles C	F
3	$\Psi_{125}, \Psi_{\bar{5}\bar{2}\bar{1}}, \Psi_{\bar{5}34}, \Psi_{\bar{4}\bar{3}5}$	1
4	$\Psi_{1234}, \Psi_{\bar{4}\bar{3}\bar{2}\bar{1}}$	1
6	$\Psi_{125}\Psi_{\bar{5}\bar{2}\bar{1}}, \Psi_{\bar{5}34}\Psi_{\bar{4}\bar{3}5}, \Psi_{125}\Psi_{\bar{5}34}, \Psi_{\bar{4}\bar{3}5}\Psi_{\bar{5}\bar{2}\bar{1}}$	2
7	$\Psi_{125}\Psi_{\bar{4}\bar{3}\bar{2}\bar{1}}, \Psi_{1234}\Psi_{\bar{5}\bar{2}\bar{1}}, \Psi_{\bar{5}34}\Psi_{\bar{4}\bar{3}\bar{2}\bar{1}}, \Psi_{1234}\Psi_{\bar{4}\bar{3}5}$	2
8	$\Psi_{1234}\Psi_{\bar{4}\bar{3}\bar{2}\bar{1}}$	2
10	$\Psi_{125\bar{4}\bar{3}\bar{2}\bar{1}\bar{5}34}, \Psi_{\bar{4}\bar{3}51234\bar{5}\bar{2}\bar{1}}$	1
10	$\Psi_{125}\Psi_{\bar{5}34}\Psi_{\bar{4}\bar{3}\bar{2}\bar{1}}, \Psi_{1234}\Psi_{\bar{4}\bar{3}5}\Psi_{\bar{5}\bar{2}\bar{1}}$	3

$\Psi_{1234}\Psi_{\bar{5}\bar{2}\bar{1}}, \quad \ell = 7, \quad F = 2$



$\Psi_{\bar{4}\bar{3}51234\bar{5}\bar{2}\bar{1}}, \quad \ell = 10, \quad F = 1$



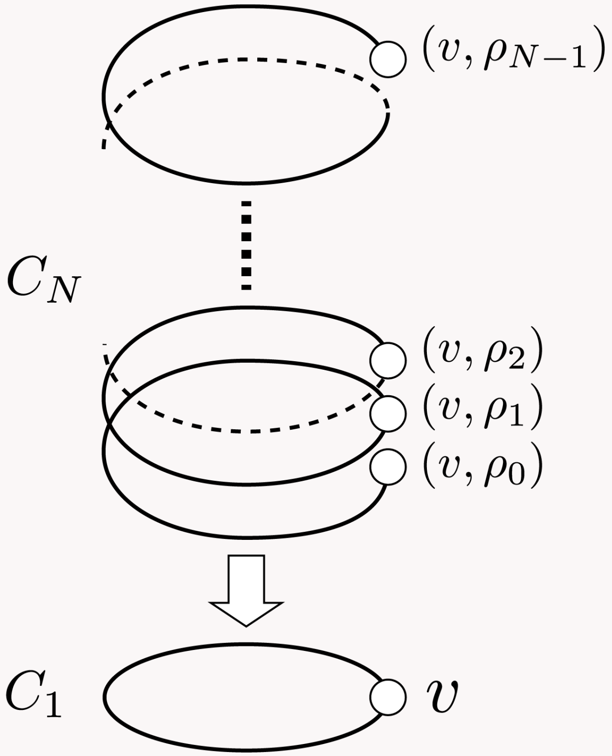
Part II: Grid graph, doublers and index theorem

Covering graph

If the graph has an automorphism structure, we can construct the covering graph by acting the discrete group (generator) to the fundamental graph. (Make copies of the fundamental graph.)

e.g.) cycle graph C_N

1. For C_1, construct the Artin-Ihara L-function for the representation $\rho_n = e^{2\pi i n/N}$ of $\mathbb{Z}_N$:
$$L_{C_1}(q;\rho_n) = \frac{1}{1-(\rho_n+\rho_n^{-1})q+q^2} = \frac{1}{1-2\cos(2\pi n/N)q+q^2}$$
2. The graph zeta function of C_N is given by a product of the L-function (Fourier mode expansion):
$$\zeta_{C_N}(q) = \prod_{n=0}^{N-1} L_{C_1}(q;\rho_n)$$

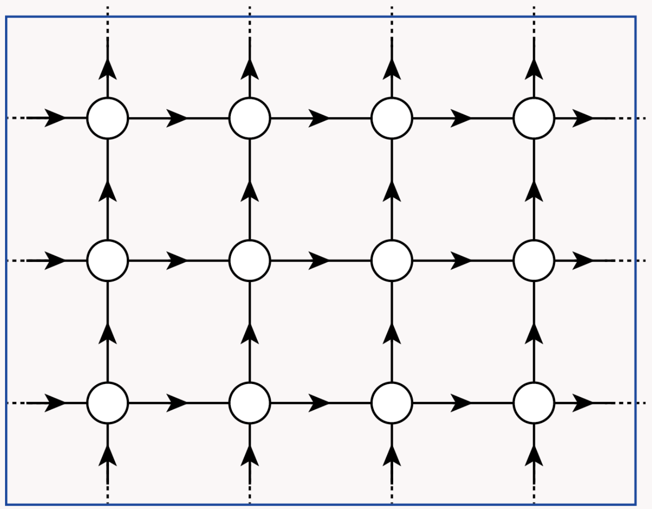


Grid graph

Using the idea of the covering graph and L-function (discrete Fourier transformation), we can construct the graph zeta function for the grid graph, which has translational invariance and periodicity, like the ordinary lattice.

e.g.) 2d square lattice

$$\zeta_{\mathrm{SQ}}(q,u)^{-1} = \prod_{m_1=0}^{N-1}\prod_{m_2=0}^{M-1}(1-q^2(1-u)^2)\left(1+(1-u)(3+u)q^2 - q\hat{A}_{\mathrm{SQ}}(\vec{m})\right),$$



where

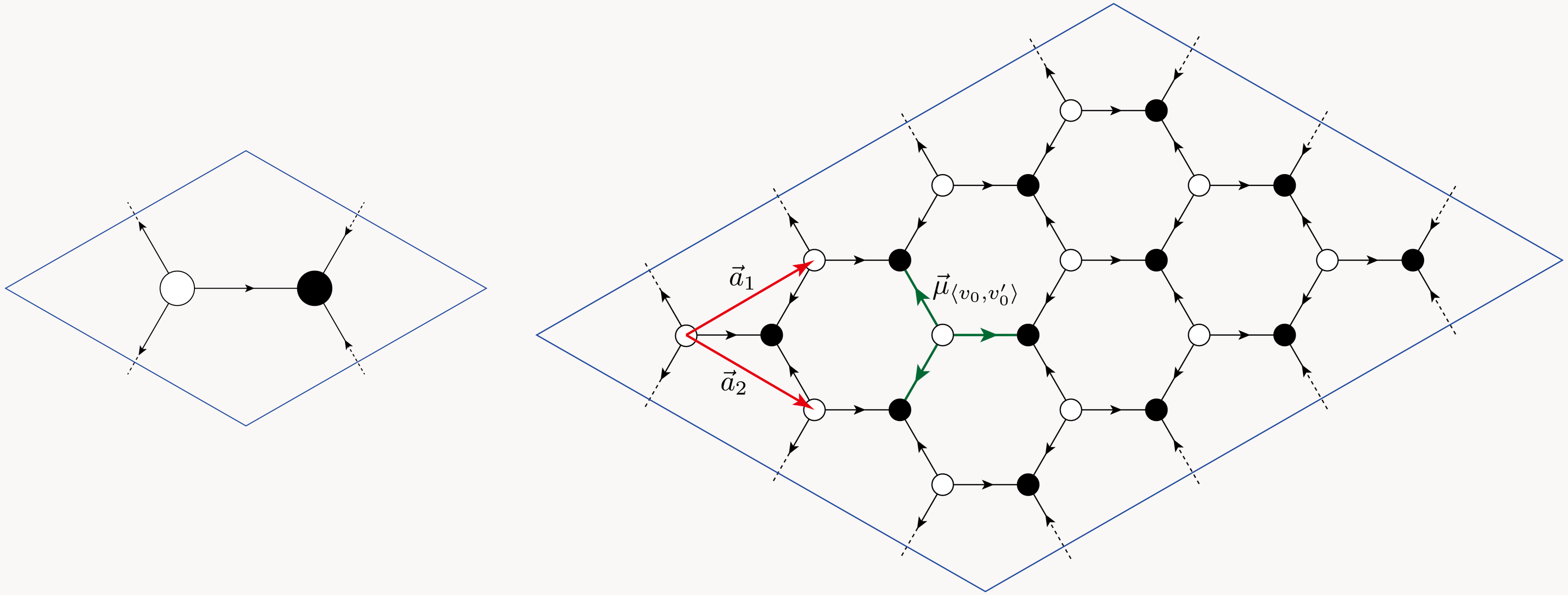
$$\hat{A}_{\mathrm{SQ}}(\vec{m}) = \omega_1^{m_1} + \omega_1^{-m_1} + \omega_2^{m_2} + \omega_2^{-m_2}$$

is the adjacency matrix weighted by the character of $\mathbb{Z}_N \times \mathbb{Z}_M$ and $\omega_1 = e^{2\pi i/N}$, $\omega_2 = e^{2\pi i/M}$.

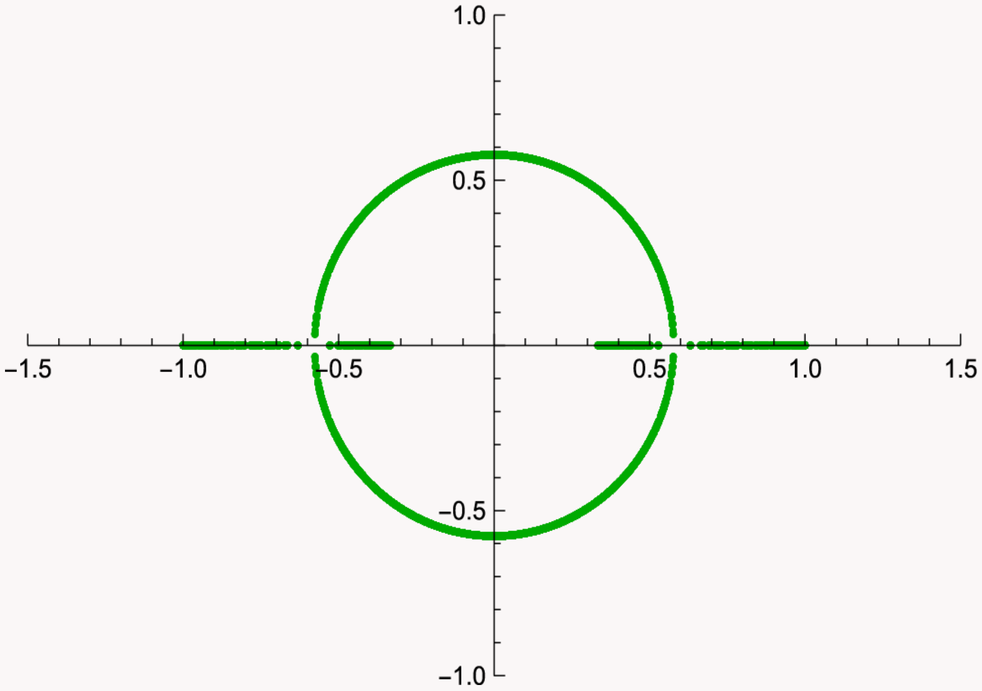
e.g.) 2d honeycomb lattice

$$\zeta_{\mathrm{HC}}(q,u)^{-1} = \prod_{m_1=0}^{N-1}\prod_{m_2=0}^{M-1}(1-q^2(1-u)^2)\det\Big((1+(1-u)(2+u)q^2)I_2 - q\hat{A}_{\mathrm{HC}}(\vec{m})\Big),$$

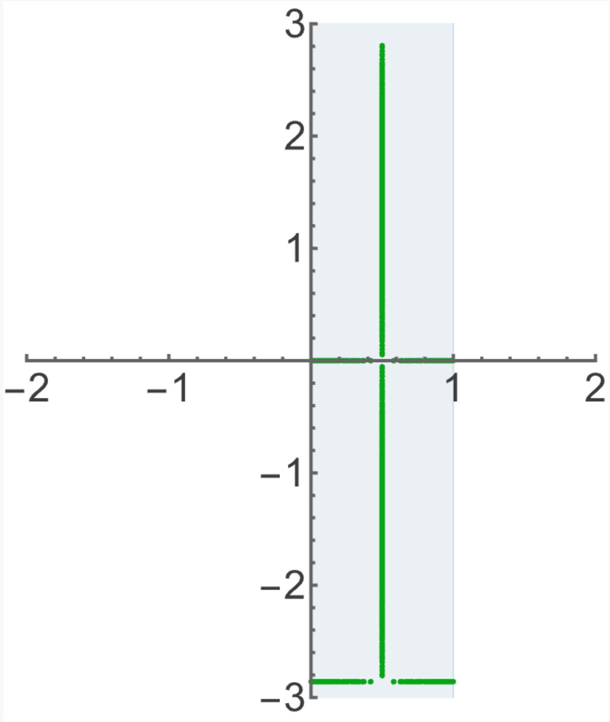
where $\hat{A}_{\mathrm{HC}}(\vec{m}) = \begin{pmatrix} 0 & \omega_1^{\frac{m_1}{3}}\omega_2^{\frac{m_2}{3}} + \omega_1^{\frac{m_1}{3}}\omega_2^{-\frac{2m_2}{3}} + \omega_1^{-\frac{2m_1}{3}}\omega_2^{\frac{m_2}{3}} \\ \omega_1^{-\frac{m_1}{3}}\omega_2^{-\frac{m_2}{3}} + \omega_1^{-\frac{m_1}{3}}\omega_2^{\frac{2m_2}{3}} + \omega_1^{\frac{2m_1}{3}}\omega_2^{-\frac{m_2}{3}} & 0 \end{pmatrix}$



Poles of the graph zeta function on the grid graph



$$\Longrightarrow$$
$$q = 3^{-s}$$



Distribution of poles the graph zeta function (**zeros of the fermion partition function**) on q-plane for 100×100 2d square lattice.

We can see poles along the line of $\mathrm{Re}\, s = \frac{1}{2}$ on s-plane (but there are also extra poles in the critical strip).

Relation to the Riemann hypothesis

- For the regular graph, where each vertex has the same number of edges, if the poles of the graph zeta function appears only on $s = 0, 1$ and line along $\mathrm{Re}\, s = \frac{1}{2}$, this graph is called Ramanujan (satisfies an analogy of the Riemann hypothesis).
- 2d square and honeycomb lattice is regular but not Ramanujan. (We are considering only the nearest neighbor interactions.)
- It is statistically known that 52% of the random regular bipartite graphs are Ramanujan.

Wilson fermion

For 2d square lattice, using the L-function representation, we can see that the denominator of the Dirac propagator (at $u = 0$) behaves

$$\det\left(\not{D} + \mathcal{M}\right) \sim q \sum_{\mu=1}^{2} \left[\sin^2 p_\mu a + (1 - \cos p_\mu a)^2\right] + 1 - 4q + 3q^2 \,.$$

So the fermion associated with the graph zeta function naturally includes the Wilson term.

In the $q \to 1$ limit, this agrees with the supersymmetric one [Misumi (2013)]. In fact, we can embed our fermions into a supersymmetric lattice gauge theory (such as the Sugino model).

Overlap fermion

Setting $\gamma_5 = \begin{pmatrix} 0 & I_{n_E} \\ I_{n_E} & 0 \end{pmatrix}$, we find that the operator $X \equiv qW - I_{2n_E}$ satisfies the so-called γ_5-hermiticity (not only for the grid graph!)

$$\gamma_5 X \gamma_5 = X^\dagger \,.$$

So we can define the overlap operator [Neuberger (1998)] by

$$\not{D}_{\mathrm{ov}} = \frac{1}{a}\left(I_{2n_E} + \frac{X}{\sqrt{X^\dagger X}} \right),$$

which satisfies the Ginsparg-Wilson relation [Ginsparg-Wilson (1982)]

$$\not{D}_{\mathrm{ov}}\gamma_5 + \gamma_5 \not{D}_{\mathrm{ov}} = a \not{D}_{\mathrm{ov}} \gamma_5 \not{D}_{\mathrm{ov}} \,.$$

Poles of the graph zeta function and fermion spectrum

Recalling the inverse of the graph zeta function is a degree $2n_E$ polynomial in q, the Ihara zeta function has poles at $q = q_i$ $(i = 1, 2, \cdots, 2n_E)$. Then, the Ihara zeta function can be factorized into

$$\zeta_\Gamma(q)^{-1} = \det\left(qW - I_{2n_E}\right) = \prod_{i=1}^{2n_E} (q/q_i - 1)\,.$$

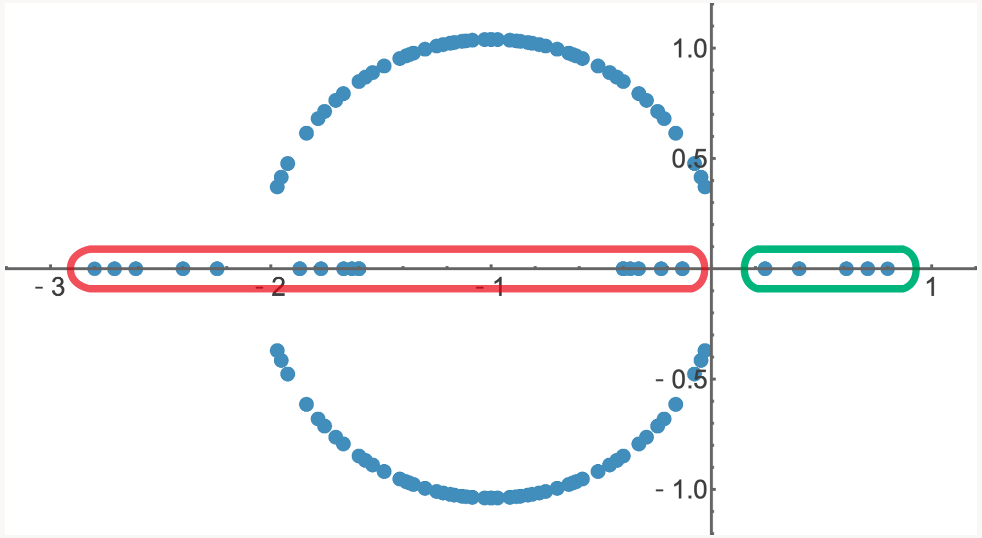
In other words, the operator $X \equiv qW - I_{2n_E}$ has eigenvalues of

$$X \to q/q_i - 1\,.$$

Assuming $0 < q < 1$, we find the following correspondence between the poles of the graph zeta function q_i and eigenvalues λ_i of the overlap operator $\not{D}_{\mathrm{ov}}$:

1. If $\mathrm{Im}\, q_i \neq 0$, the poles appear in pairs of $(q_i, \bar{q}_i)$. This corresponds to the non-zero modes.
2. If $q_i \in \mathbb{R}$ and $q/q_i - 1 > 0$, these poles correspond to the eigenstate of $\lambda_n = \frac{2}{a}$ (doublers).
3. If $q_i \in \mathbb{R}$ and $q/q_i - 1 < 0$, these poles correspond to the eigenstate of $\lambda_n = 0$ (zero modes).

e.g. 2d square lattice (20×20 and $q = 0.6$)



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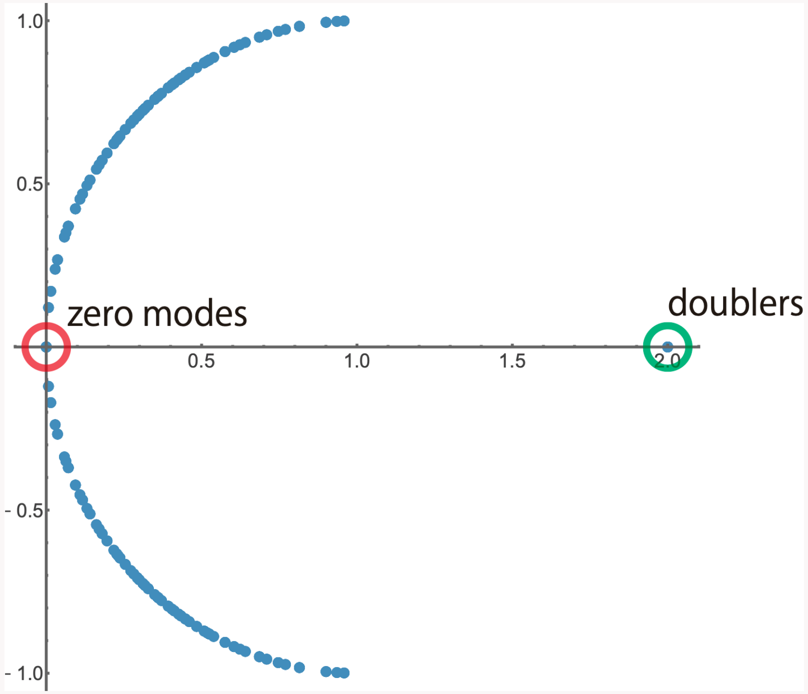


Eigenvalue distribution of

$$X = qW - I_{2n_E}$$

Noting $\det X = \zeta_\Gamma(q)^{-1}$, this is essentially the distribution of poles of the graph zeta function.

Eigenvalue distribution of $\not{D}_{\mathrm{ov}}$

Poles on the real axis contribute to zero modes and doublers.

Index theorem

Introducing a modification of the γ_5 matrix [Lüscher (1998)] by

$$\Gamma_5 \equiv \gamma_5 \left(I_{2n_E} - \frac{a}{2} \not{D}_{\mathrm{ov}} \right) ,$$

the index is defined by

$$\mathrm{Ind}\, \not{D}_{\mathrm{ov}} = \mathrm{Tr}\, \Gamma_5 = n_+ - n_- ,$$

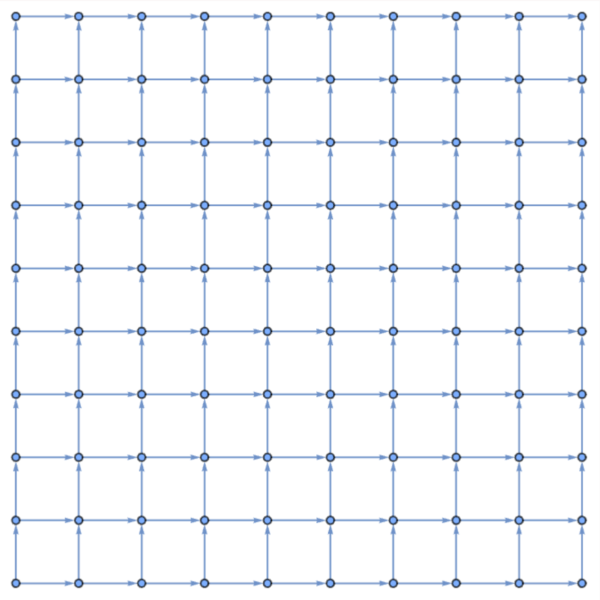
which gives the difference between the numbers of zero modes with positive and negative chirality.

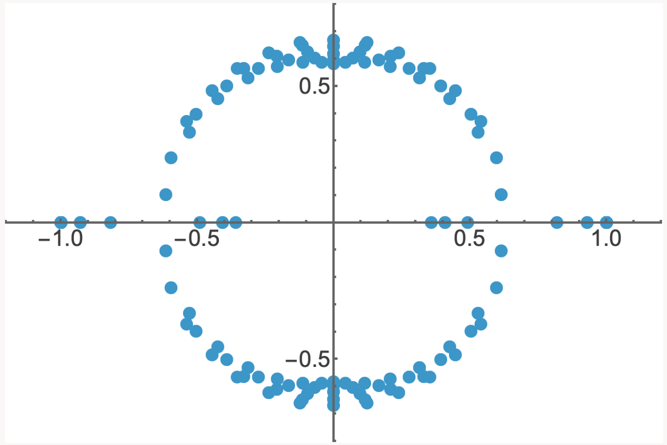
The index is also given by a trace of the matrix sign function of $\gamma_5(qW - I_{2n_E})$

$$\mathrm{Tr}\, \Gamma_5 = -\frac{1}{2} \mathrm{Tr}\, \mathrm{sgn}\, \gamma_5 (qW - I_{2n_E}) ,$$

which is determined by the pole distribution of the graph zeta function.

e.g.) Index on the 2d square lattice (10×10)





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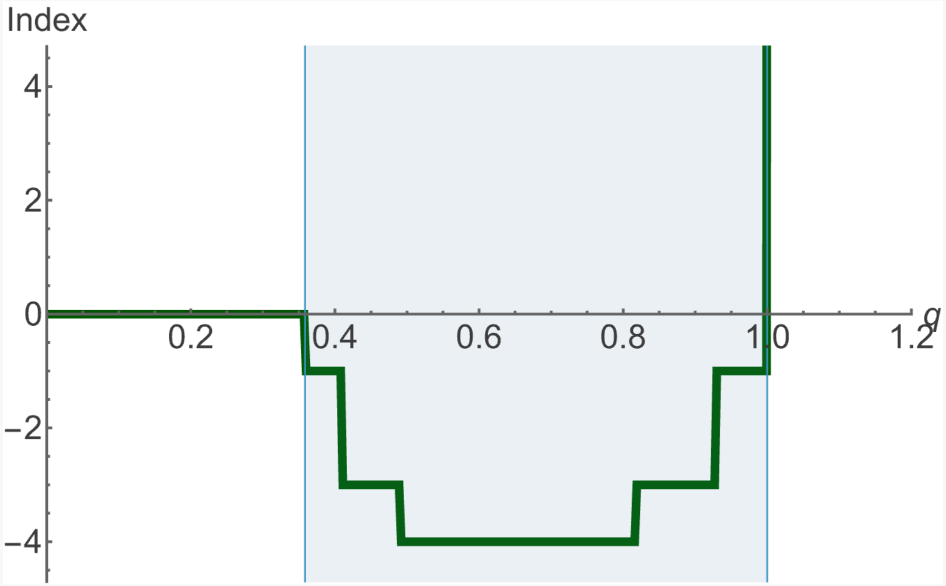
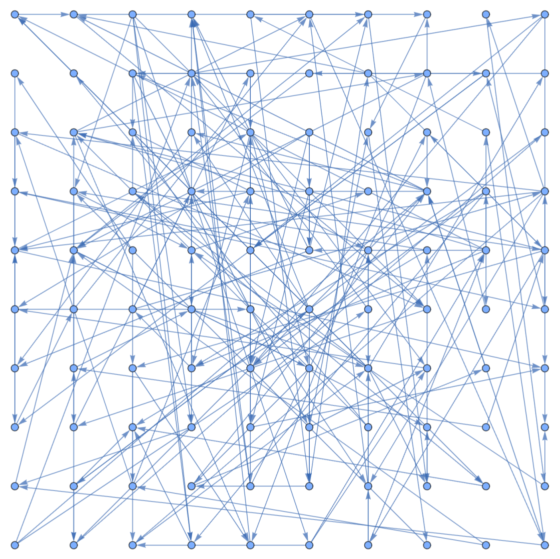
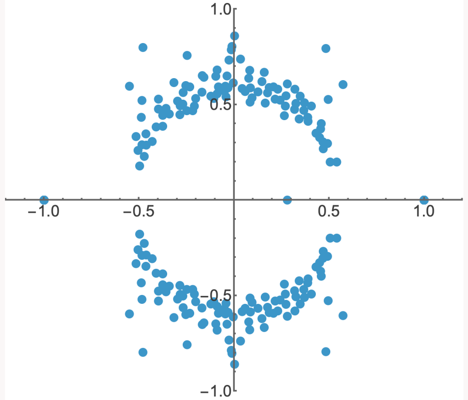


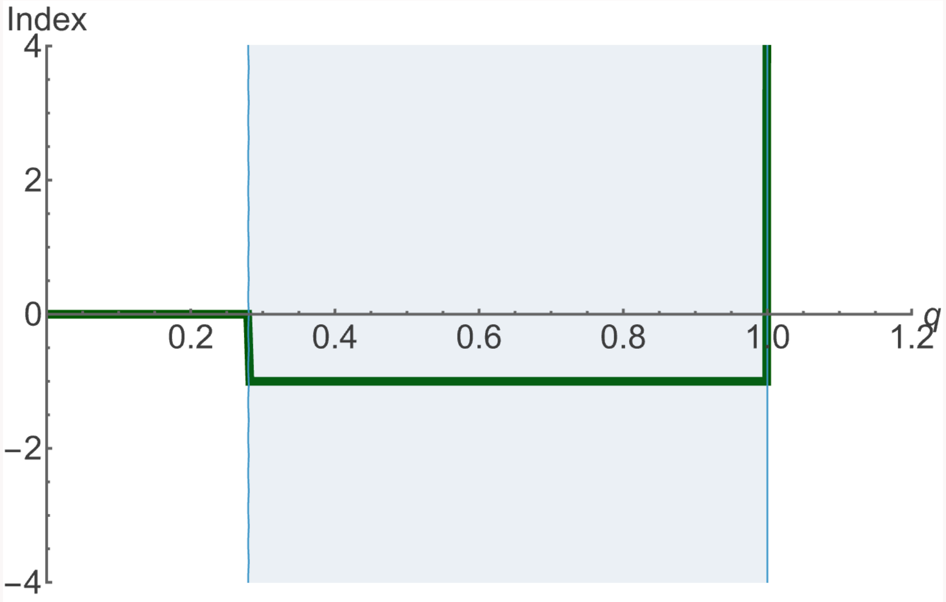
- The index exhibits non-trivial variation within the critical strip.
- $\mathrm{Ind}\, \not{D}_{\mathrm{ov}} = \begin{cases} -1 & \text{if } q \to 1^- \\ n_E - n_V = 80 & \text{if } q \to 1^+ \end{cases}$

e.g.) Index on the random graph generated by the Watts-Strogatz algorithm (rewiring probability $p = 0.8$)





$\Rightarrow$



- The index does not vary so much in the critical strip region.
- $\mathrm{Ind}\, \not{D}_{\mathrm{ov}} = \begin{cases} -1 & \text{if } q \to 1^- \\ n_E - n_V = 100 & \text{if } q \to 1^+ \end{cases}$

Part III: Relation to the statistical model

Winding number

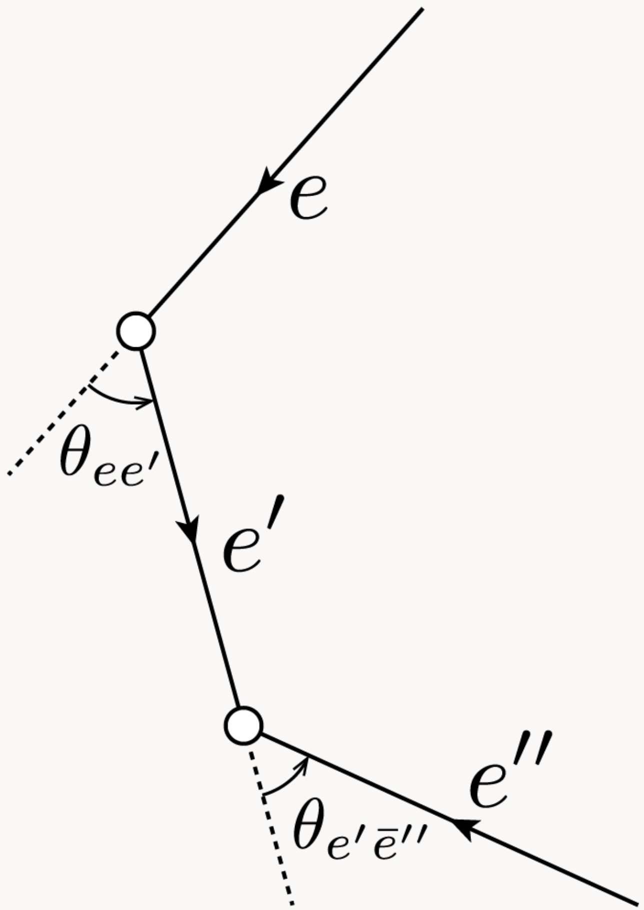
on a planar graph, if we assign a rotation angle in the plane to each edge and consider a weighted adjacency matrix based on these rotation angles, we obtain a graph zeta function that includes the number of rotations of a cycle

$$\tilde{\zeta}_\Gamma(q,u,r) \equiv \prod_{[C]:\text{ primitive cycles}} \frac{1}{1 - r^{w(C)} u^{b(C)} q^{\ell(C)}} ,$$

where

$$w(C) \equiv \frac{1}{2\pi} \sum_{i=1}^{k} \theta_{\boldsymbol{e}_i \boldsymbol{e}_{i+1}} ,$$

for $C = \boldsymbol{e}_1 \boldsymbol{e}_2 \cdots \boldsymbol{e}_k$ with $\boldsymbol{e}_{k+1} = \boldsymbol{e}_1$.



In particular, setting $u = 0$ and $r = -1$, the edge adjacency matrix in the Hashimoto expression coincides with the Kac-Ward matrix [Kac-Ward (1952)] on the general graphs, and the inverse of the graph zeta function gives the partition function of the Ising model on the graph.

$$\tilde{\zeta}_\Gamma(q, u{=}0, r{=}{-1})^{-1} = 2^{-2n_V}(1 - q^2)^{n_E}\left(Z_\Gamma^{\text{Ising}}\right)^2,$$

where $q = \tanh \beta J$

That is, by coupling a $U(1)$ gauge holonomy (winding) to the fermions, the model becomes equivalent to a free-fermion representation of the Ising model on arbitrary graphs.

High temperature expansion of the random-bond Ising model

$$
\begin{aligned}
Z_\Gamma^{\text{Ising}} &= \sum_{\sigma\in\Omega} \exp\left(\beta \sum_{e=\langle u,v\rangle\in E} J_e \sigma_u \sigma_v\right) \\
&= \prod_{e\in E} \cosh \beta J_e \sum_{\sigma\in\Omega} \prod_{e=\langle u,v\rangle\in E} (1+\sigma_u\sigma_v \tanh \beta J_e) \\
&= \prod_{e\in E} \cosh \beta J_e \sum_{\gamma\in\mathcal{E}(\Gamma)} \prod_{e\in E(\gamma)} \tanh \beta J_e \,,
\end{aligned}
$$

where $\mathcal{E}(\Gamma)$ is the Eulerian subgraph, which has even degree vertices and can be drawn by a single stroke-drawn path (including the empty graph).

Setting $z_e = \tan \beta J_e$

C_3 (Triangle Graph)

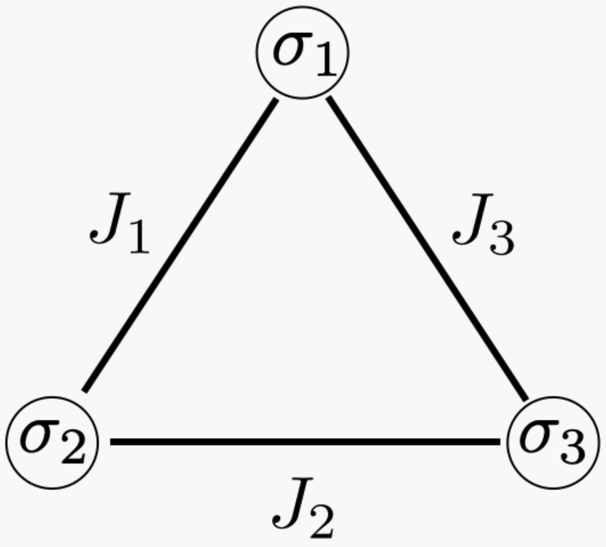
$$\sum_{\sigma_v=\{\pm1\}} (1+\sigma_1\sigma_2 z_1)(1+\sigma_2\sigma_3 z_2)(1+\sigma_3\sigma_1 z_3)$$

$$= \sum_{\sigma_v=\{\pm1\}} \Big[1+\sigma_1\sigma_2 z_1+\sigma_2\sigma_3 z_2+\sigma_1\sigma_3 z_3$$

$$+\sigma_1\sigma_2^2\sigma_3 z_1 z_2+\sigma_1\sigma_2\sigma_3^2 z_2 z_3+\sigma_1^2\sigma_2\sigma_3 z_1 z_3$$

$$+\sigma_1^2\sigma_2^2\sigma_3^2 z_1 z_2 z_3\Big]$$

$$= 2^3\left(1+z_1 z_2 z_3\right)$$



$K_4 - e$ (Double Triangle Graph)

$$\sum_{\sigma_v=\{\pm 1\}} (1+\sigma_1\sigma_2 z_1)(1+\sigma_2\sigma_3 z_2)(1+\sigma_3\sigma_4 z_3)$$

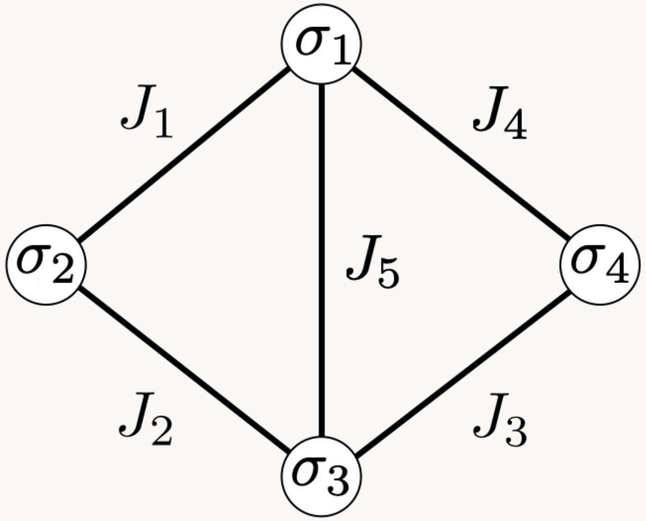
$$\times(1+\sigma_4\sigma_1 z_4)(1+\sigma_1\sigma_3 z_5)$$

$$= 2^4\,(1+z_1z_2z_5+z_3z_4z_5+z_1z_2z_3z_4)$$

If all couplings on the bonds are identical (ordinary Ising model),

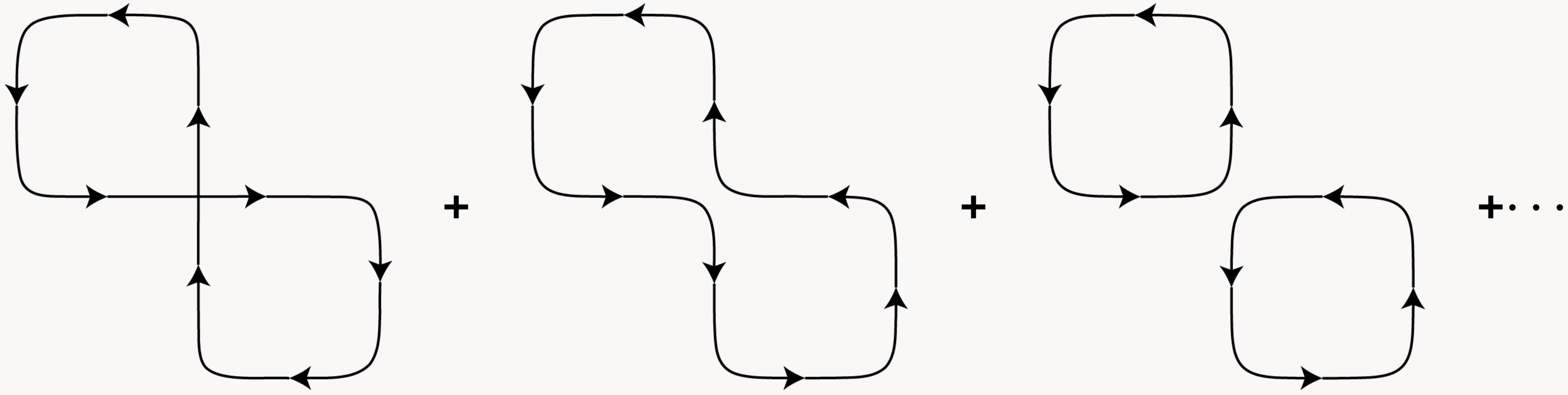
$$z_e = \tanh\beta J = q\,,$$

the partition function of the Ising model on the graph Γ gives the number of the Eulerian subgraphs as a polynomial in q.



Relation to the Ihara zeta function

- The inverse of the Ihara zeta function generates all fermionic cycles, including their directions. (It also contains the signs by the fermion number.)
- The partition function of the Ising model is given by a summation over undirected diagrams (Eularian subgraphs).
- Introducing the winding number and setting $r = -1$, it changes the signs of the fermion number and only the Eularian subgraphs survive by cancellations.
- Since there are two independent directions for each single stroke path, the inverse of the graph zeta function reduces to the square of the partition function of the Ising model as the combination results.



$(-1)^F r^w q^\ell \qquad -r^0 q^8 \qquad -r^1 q^8 \qquad +r^2 q^8$

$$= (r^2 - r + 2 - r^{-1} + r^{-2}) q^8$$

$$\underset{r \to -1}{\longrightarrow} +6q^8$$

$$\tilde{\zeta}(q, r = -1) = (1 + 2q^4 + q^8)^2$$

Grid graph zeta function including the winding number

2d square lattice:

$$\tilde{\zeta}_{\mathrm{SQ}}(q,r)^{-1} = \prod_{m_1=0}^{N-1}\prod_{m_2=0}^{M-1}\left\{(1-q^2)\left(1+3q^2-q\hat{A}_{\mathrm{SQ}}(\vec{m})\right)-(r^{1/2}-r^{-1/2})^2q^4\right\}$$

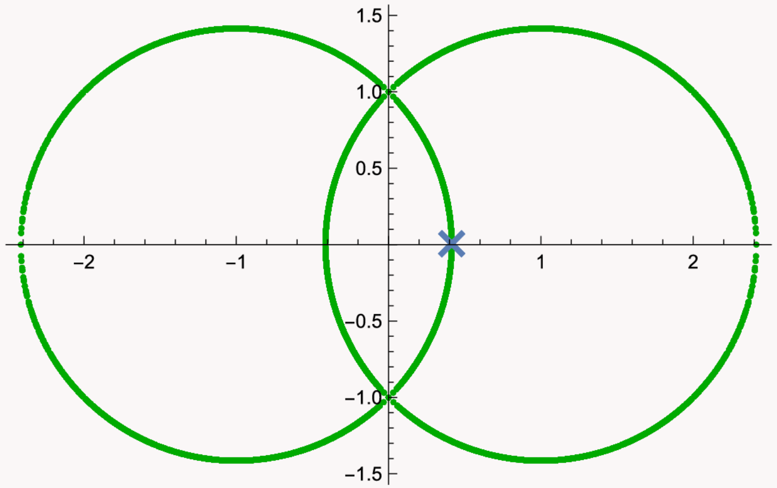
2d honeycomb lattice:

$$\tilde{\zeta}_{\mathrm{HC}}(q,r)^{-1} = \prod_{m_1=0}^{N-1}\prod_{m_2=0}^{M-1}\left\{(1-q^2)\det\left((1+2q^2)I_2-q\hat{A}_{\mathrm{HC}}(\vec{m})\right)-(r^{1/2}-r^{-1/2})^2q^6\right\}$$

📝 The pole distribution of the graph zeta function is modified by the winding parameter r.

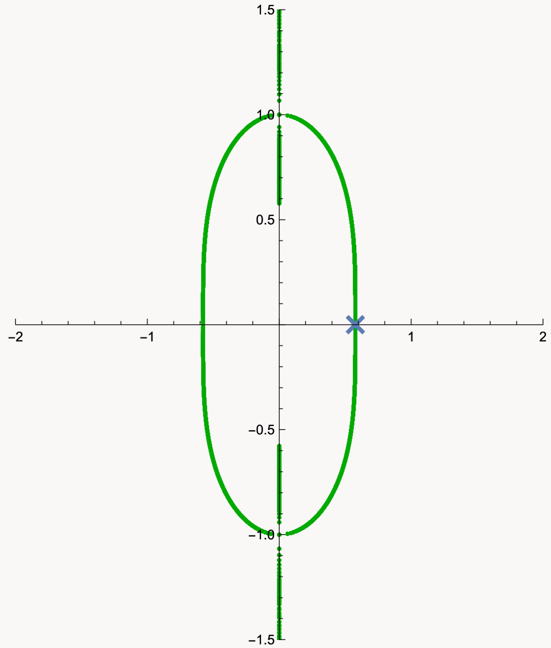
$\Rightarrow$ According to the Lee-Yang theorem, the poles of the graph zeta function (zeros of the partition function) determine the phase transition points of the Ising model on the graph.

Pole distribution (zeros of the partition function) including the winding number ($r = -1$)



$q^* = \sqrt{2} - 1 = 0.414214\cdots$

This agrees with the exact result of the Ising model on 2d square lattice. (We can also see the Kramers-Wannier duality and $J \leftrightarrow -J$ symmetry.)



$q^* = \frac{1}{\sqrt{3}} = 0.57735\cdots$

This agrees with the exact result of the Ising model on 2d honeycomb lattice.

Conclusion and Outlook

- We have constructed a fermion model on arbitrary discrete graphs whose partition function equals the inverse of the graph zeta function.
- Using the covering graph and L-function on the graph, we can generate the partition functions on grid graphs (lattices) with translational symmetry from a single fundamental domain.
- The distribution of zeros of the partition function (poles of the graph zeta function), which are closely related to the Riemann hypothesis in mathematics, would be physically relevant (through the index theorem or the Lee–Yang theorem).
- Combining this fermionic framework with Kazakov-Migdal type bosonic models on graphs opens a route to formulate QCD or supersymmetric gauge theories (gauge fields + matter) on the graphs.