Constraints on the symmetric mass generation approach to lattice chiral gauge theories

Yigal Shamir

Tel Aviv University

with Maarten Golterman

arXiv:2505.20436 PRD 113 (2026) 1, 014503

Why bother?

- EW baryogenesis – non-perturbative dynamics of SM, a chiral gauge theory
- Grand Unified Theories (SU(5), SO(10), ...)
- "Tumbling gauge theory": Raby, Dimopoulos, Susskind 1979

  asymptotically free chiral gauge theory breaks itself spontaneously

  when coupling gets strong: fermion condensate acts as Higgs field
- A fundamental lacuna in our understanding an important class

  of asymptotically free gauge theories!

Outline

- Chiral symmetry, fermion doubling
  The Nielsen-Ninomiya "no-go" theorem
- What works for global chiral symmetry fails for chiral gauge symmetry.
  Why?
- Symmetric Mass Generation (SMG) paradigm:
  gap the doublers with judiciously chosen (strong) interactions.
- The Nielsen-Ninomiya theorem hits back!

$\Rightarrow$ "Check list" for any SMG model

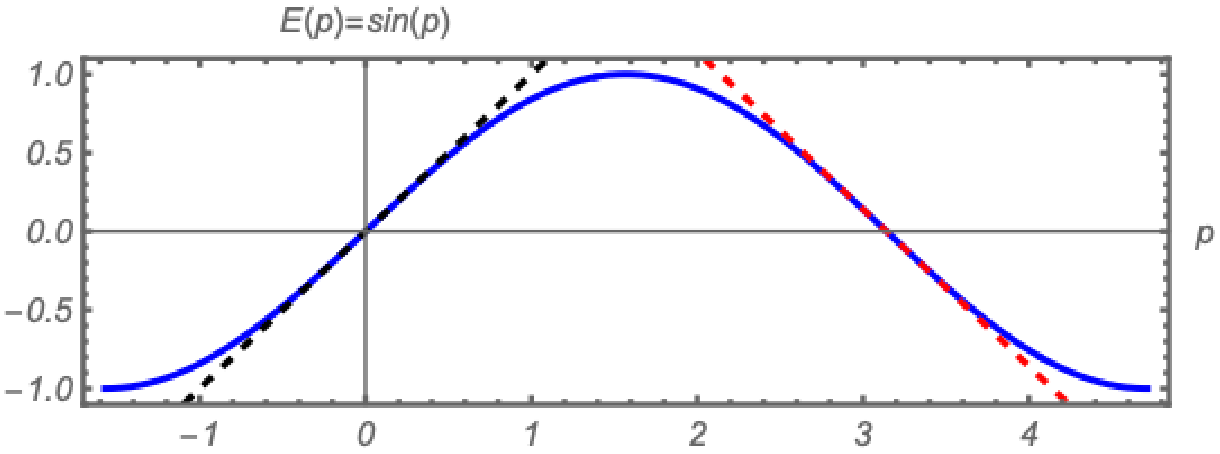
- Other approaches with partial success (time permitting)

Doubling problem: 1-component fermion in $1+1$ dimensions

- Continuum: $E = p$ for RH field
- Lattice: $\partial\psi(x) \Rightarrow (\psi(x+a) - \psi(x-a))/(2a)$

$\Rightarrow E = (1/a)\sin(ap)$

Brillouin zone: "What goes up must come down" Karsten & Smit, 1981



RH fermion

$E = p + \cdots$

LH fermion (doubler)

$E = -(p - \pi/a) + \cdots$

Doubling problem: 1-component fermion in $1+1$ dimensions

- Continuum: $E = p$ for RH field
- Lattice: $\partial\psi(x) \Rightarrow (\psi(x+a) - \psi(x-a))/(2a)$

$\Rightarrow E = (1/a)\sin(ap)$

Brillouin zone: "What goes up must come down" Karsten & Smit, 1981

* Fermion doubling and the anomaly:

- On lattice, H has global U(1) symmetry $\Rightarrow$ exactly conserved current $j_\mu(x)$
- In $d = 1+1$ dims. current conservation on both legs of vac. pol. $\langle j_\mu \; j_\nu \rangle$
  In $d = 3+1$ dims. same for all legs of triangle diagram $\langle j_\mu \; j_\nu \; j_\rho \rangle$
- But if spectrum $=$ RH fields only, these diagrams must be anomalous!
- Doublers solve this problem, since now spectrum $=$ RH $+$ LH $=$ Dirac

Nielsen-Ninomiya theorem (1981)

Consider a lattice hamiltonian for free massless fermions, with conditions:

- Lattice translation invariance ⇒ periodic Brillouin zone
- Relativistic spectrum: massless Weyl fermions ($\vec{p}_c$ = "degeneracy point")

$$
\begin{aligned}
E &= \pm(p - p_c) + \cdots && d = 1+1 \\
H_{2\times 2} &= \pm\,\vec{\sigma}\cdot(\vec{p} - \vec{p}_c) + \cdots && d = 3+1
\end{aligned}
$$

- Dispersion relation has continuous 1st derivative (defines velocity) everywhere in Brillouin zone (follows from locality)
- Exact global symmetry with discrete charges (compact U(1), SU(N), ...)

⇒ In every charge sector, have an equal number of LH and RH fermions.

Hence every exact symmetry becomes vectorlike: no chiral symmetries

Solution for QCD: Wilson fermions

$$\begin{aligned} D_W &= \frac{1}{a}\left(i \sum_\mu \gamma_\mu \sin(ap_\mu) + \sum_\mu (1 - \cos(ap_\mu) \right) + m_0 \\ &\approx i \sum_\mu \gamma_\mu p_\mu + \frac{a}{2} \sum_\mu p_\mu^2 + m_0 \ , \qquad ap \ll 1 \end{aligned}$$

- Naive fermions: doublers at $ap = (\pi, 0, 0, 0), \ldots, (\pi, \pi, 0, 0), \ldots, (\pi, \pi, \pi, \pi)$
- Wilson term $=$ dim-5 operator (formally irrelevant)
- Removes doublers: Masses $= \ m_0 + 2n/a$

  where $n \ = \ \#$(number of momentum components equal to π/a)
- Price: Wilson term breaks chiral symmetry explicitly

$\Longrightarrow$ bare mass m_0 undergoes additive renormalization.

- Works for QCD, requires tuning of bare mass to restore chiral symmetry in continuum limit.

Wilson fermions for chiral gauge theory (Smit-Swift model)?

- Chirality components: $\psi = \psi_L + \psi_R$. Let us couple only ψ_L to gauge field:

$$\begin{aligned} \psi_L(x) &\rightarrow \phi(x)\psi_L(x)\ , & \overline{\psi}_L(x) &\rightarrow \overline{\psi}_L(x)\phi^\dagger(x) \\ \psi_R(x) &\rightarrow \psi_R(x)\ , & \overline{\psi}_R(x) &\rightarrow \overline{\psi}_R(x) \\ U_\mu(x) &\rightarrow \phi(x)U_\mu(x)\phi^\dagger(x+\hat{\mu}) \end{aligned}$$

- ψ_R is free "spectator," decouples in cont. lim. Golterman & Petcher 1989

- Kinetic term is gauge inv [e.g. $\overline{\psi}_L(x)\gamma_\mu U_\mu(x)\psi_L(x+\hat{\mu})$]. Not Wilson term:

$$\overline{\psi}_R(x)\psi_L(x+\hat{\mu}) + \cdots \quad \rightarrow \quad \overline{\psi}_R(x)\phi(x+\hat{\mu})\psi_L(x+\hat{\mu}) + \cdots$$

  Inserting $U_\mu(x)$ won't help, because ψ_L and ψ_R belong to different *irreps*

- What can go wrong? Think about $\phi(x)$ as Higgs field

  $\langle\phi\rangle \neq 0 \Rightarrow$ Higgs phase (not what we want!)

  $\langle\phi\rangle = 0 \Rightarrow \langle$Wilson term$\rangle = 0 \Rightarrow$ doublers are back!

Symmetric Mass Generation paradigm

- Assume: local action, manifest gauge invariance
- Goal: decouple the doublers via "irrelevant" interactions ($e.g.$, multi-fermion interactions) without spontaneously breaking the gauge symmetry
- Anomalies: any symmetry which is anomalous in the target chiral gauge theory must be explicitly broken on the lattice
  (caveat: relax assumption(s) $\Rightarrow$ ways around exist: other approaches)
- Eichten-Preskill model (1986) designed to meet these guidelines, yet fails for similar dynamical reasons as Smit-Swift model Golterman, Petcher, Rivas 1992
- Recently, successful SMG phase claimed for 3-4-5-0 chiral Schwinger model. Anomaly cancellation: $3_L^2 + 4_L^2 = 5_R^2 + 0_R^2$
- How does it evade NN theorem? Or does it?

ZZWY model: SMG for 3-4-5-0 chiral Schwinger model

Zeng, Zhu, Wang and You, PRL 2022

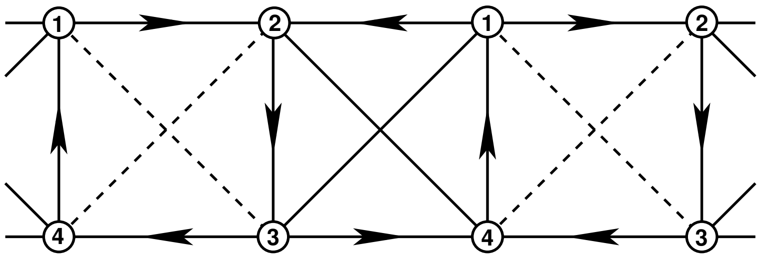
- One-dimensional lattice, continuous time.
- Reduced model: Gauge field turned off

$\Rightarrow$ Need: chiral massless spectrum under to-be-gauged U(1) symmetry

$$H = H_L(\psi_3) + H_L(\psi_4) + H_R(\psi_5) + H_R(\psi_0) + H_{\text{SMG-int}}$$

Edge A: physical fermion

Edge B: mirror fermion (doubler)



- Weak coupling phase: doubled spectrum
- $H_{\text{SMG-int}}$ = two 6-fermion interactions of the Edge-B fields
- Claim: strongly coupled SMG phase, unbroken U(1), gapped "mirrors"

Way out? I. SMG (strong) interactions

- NN theorem is about free hamiltonians.

$\Rightarrow$ Should not apply since SMG interactions are crucial

- However: Generalized no-go theorem — YS 1993, Golterman and YS 2025
- Define 1-particle hamiltonian $H_{\rm eff}^{-1}(\vec{p}) = \left\langle \Psi\, \overline{\Psi} \right\rangle (\vec{p}, \omega = 0)$
- $\{\Psi_a\}$: set of interpolating fields for the massless states (caveat: next)
- Continuum limit of reduced model: want undoubled free massless fermions (because that's what one gets by turning off gauge field in target theory!). Only irrelevant interactions $\Rightarrow$ $H_{\rm eff}(\vec{p})$ has continuous 1st derivative
- Brillouin zone, relativistic massless fermions: same as in free case
- No SSB of to-be-gauged global sym. $\Rightarrow$ discrete charge sectors

$\Rightarrow$ NN theorem will apply to $H_{\rm eff}(\vec{p})$

More about $H_{\text{eff}}(\vec{p})$

- $H_{\text{eff}}^{-1}(\vec{p}) = \mathcal{R}(\vec{p})$ retarded function – convenient analyticity properties
- In a local theory, expect $\mathcal{R}(\vec{p})$ analytic in Brillouin zone, except at degeneracy points: intermediate states with vanishing total energy.
- Relativistic massless spectrum with irrelevant couplings $G \times a^{n-d-1}$ where $n =$ dimension of interaction

$\Rightarrow$ behavior of $H_{\text{eff}}(\vec{p})$ near points $\vec{p}_c$ with zero eigenvalue ($\vec{q} = \vec{p} - \vec{p}_c$):

$$E = \pm q \left(1 + c_1 G^2 (aq)^{2(n-d-1)} \log(q^2)\right) + \cdots \qquad d = 1$$

$$H_{2\times 2} = \pm \vec{\sigma} \cdot \vec{q} \left(1 + c_3 G^2 (aq)^{2(n-d-1)} \log(q^2)\right) + \cdots \qquad d = 3$$

- G irrelevant $\Rightarrow$ $n - d - 1 \geq 1$ $\Rightarrow$ continuous 2nd derivative (at least)
- Caveat: zeros in $\mathcal{R}(\vec{p})$ $\Rightarrow$ poles in $H_{\text{eff}}(\vec{p})$ $\Rightarrow$ NN theorem will not apply

Way out? II. Propagator zeros

- Start: massless Dirac fermion $\frac{\not{p}}{p^2} = P_L\frac{\not{p}}{p^2}P_R + P_R\frac{\not{p}}{p^2}P_L$
- Gap the mirror fermion $\Rightarrow P_L\frac{\not{p}}{p^2}P_R + P_R\frac{\not{p}}{p^2-m^2}P_L$

$\Rightarrow$ This propagator has a zero! Two options:

- ○ "Genuine zero:" from pole in non-local action, acts as a ghost state

  Pelissetto 1988, Golterman & YS 2024

- ○ "Kinematical zero:" gapped RH mirror has a – dynamically generated – LH partner (not the physical LH massless fermion)

- Bound state LH partner generated by same SMG interactions!
- Most likely option in local theory Zeng *et al.* arXiv:2405.05339

Bound state formation (Conjecture)

- Start: massless Dirac fermion $\frac{\not p}{p^2} \;=\; P_L \frac{\not p}{p^2} P_R \;+\; P_R \frac{\not p}{p^2} P_L$
- Gap mirror, create bound state $\quad\Rightarrow\quad P_L \frac{\not p}{p^2} P_R \;+\; \frac{\not p - m}{p^2 - m^2} \;+\; (?)$
- Massive Dirac fermion has elementary RH component (the mirror) and composite LH component: $\mathcal{B}_i = \delta H_{\text{SMG-int}} / \delta \psi_i$
- Similar phenomenon in Eichten-Preskill model Golterman, Petcher, Rivas 1992
- Obtain complete set of interpolating fields by adding suitable composite fields to the elementary fields:

1. Interpolates all massless fermions (in given charge sector)
2. $\mathcal{R}(\vec{p})$ free of propagator zeros $\Rightarrow$ $H_{\text{eff}}(\vec{p}) = \mathcal{R}^{-1}(\vec{p})$ has no poles.

Generalized no-go theorem

Assume:

- Local reduced model, regular lattice
- Continuum limit is free, relativistic, with massless fermions
- Analyticity: lattice hamiltonian has finite range, fermions only (technical)
- Unbroken to-be-gauged global symmetry, discrete charge sectors
- Conjecture: Complete set of interpolating fields (no propagator zeros)

$\Longrightarrow$ $H_{\text{eff}}(\vec{p})$ satisfies all conditions of NN theorem $\Rightarrow$ vector-like spectrum

"Check list" for any SMG model!

ZZWY: limited numerical evidence (Edge A: massless, Edge B: gapped).

No investigation yet of propagator zeros, bound states.

Compute vacuum polarization! (Counts # massless fields)

Two dimensions vs. four dimensions

- 4-fermion interactions (no derivatives) are renormalizable in 2 dimensions!
- In ZZWY model, 4-fermion interactions which respect all symmetries will be induced $\Rightarrow$ 1st derivative of $H_{\rm eff}(\vec{p})$ not continuous!

Two options:

- ○ Cancel induced 4-fermion interactions by tuning counterterms (Difficult! SMG phase might or might not survive)

$\Rightarrow$ $H_{\rm eff}(\vec{p})$ will have continuous 1st derivative; nogo theorem applies!

- ○ "Live with" the 4-fermion interactions (ZZWY model).
  Nogo theorem doesn't apply, but continuum-limit theory not what we want!
  E.g., 4-fermion interactions can gap *all* massless fermions without SSB

$\Longrightarrow$ Limited lessons from 2 dimensional theories!

Other approaches

Lüscher: modified chiral symmetry

Neuberger 1997, Lüscher 1999 – 2000

- QCD: seek modified chiral sym. $\delta\overline{\psi} = \overline{\psi}\gamma_5$, $\delta\psi = \gamma_5(1 - aD)\psi \equiv \hat{\gamma}_5\psi$
- Overlap operator, Ginsparg-Wilson relation: $\gamma_5 D + D\gamma_5 = aD\gamma_5 D$
- D^{-1} anticommutes except coinciding points: $[D^{-1}\gamma_5 + \gamma_5 D^{-1}]_{xy} = a\gamma_5\delta_{xy}$

$\Longrightarrow$ Axial anomaly from variation of fermion measure

- Chiral gauge theory: modified chiral projectors

  $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$ now also $\hat{P}_\pm = \frac{1}{2}(1 \pm \hat{\gamma}_5)$, since $\hat{\gamma}_5^2 = 1$.

- Project chiral field in lattice action: $S = \overline{\psi} P_+ \, D \, \hat{P}_- \psi$
- Integration measure of $\hat{P}_-\psi$ depend on gauge field $\Longrightarrow$ integrability conds.
- Obstructions: perturbative anomaly, Witten anomaly, . . . ?
- General solution for U(1) only (requires one extra condition)

Kaplan: domain wall fermions 1992, with Grabowska and Sen 2015 – to date

- RH and LH chiral 4d modes live on opposite boundaries of 5d space
- QCD: couple to 4d gauge field (independent of 5th coordinate)
  massive 5d bulk decouples, effective 4d theory satisfies GW relation
- Chiral gauge theory: aim to couple 4d gauge field only to physical fermion
  on "near" wall in continuum limit.
- "Anomaly inflow": bulk 5d dof's decouple if anomaly-free spectrum
- Try to decouple doubler on far wall by damping 4d gauge field
  with 5th coordinate (classical propagation)
- Problem: too much global symmetry! Golterman & YS 2024, Kaplan & Sen 2024
  Conserved fermion number current for each 5d DWF field
- Related problem: topological sectors

Gauge-fixing approach: $U(1)$

Bock, Golterman, YS 1995 – 2002

- Chiral Wilson action removes doublers at weak coupling.
  Price: not gauge invariant ⇒ longitudinal gauge dof's couple to fermions.
- Add renormalizable gauge-fixing term to lattice action "Rome approach" 1990

$$S = \frac{1}{g^2}F^2 + \frac{1}{\xi g^2} \boxed{(\partial_\mu A_\mu)^2} + \text{chiral Wilson action} + \text{c.t.}$$

- Continuum limit at novel critical point with required chiral spectrum.
  Restore gauge invariance and unitarity to all orders in pert. theory.
  Many counterterms, but only $m^2 A_\mu^2$ is relevant.
- Reduced model: $A_\mu \to \partial_\mu \phi$ hence $(\partial_\mu A_\mu)^2 \to (\Box\phi)^2$
- Evade generalized no-go theorem! E.g., "charged" two-point function has LH elementary fermion, while RH channel is composite of "neutral" fermion and the higher-derivative scalar ϕ (1st derivative not continuous).

Gauge-fixing approach: $SU(N)$

Golterman & YS 2004 – to date

- Add ghost fields
- Equivariant BRST gauge fixing: $SU(N) \to \frac{SU(N)}{U(1)^{N-1}}$ Schaden 1999

$\Longrightarrow$ Evades lattice Gribov copies Neuberger 1987

- Remaining $U(1)^{N-1}$ is gauge fixed as before.
- Coupling $\tilde{g}^2 = \xi g^2$ that controls longitudinal sector is asymptotically free!

Open questions

- Non-perturbative unitarity (equivariant BRST should help)
- Dynamics of gauge fixing sector
- Even reduced model hard to simulate

In conclusion

- Nielsen-Ninomiya theorem generalizes to interacting models
- Symmetric mass generation might not be the most promising approach

  Key issue 1. Role of propagator zeros—bound states!

  Key issue 2. “SMG anomaly paradigm” does not seem to help:

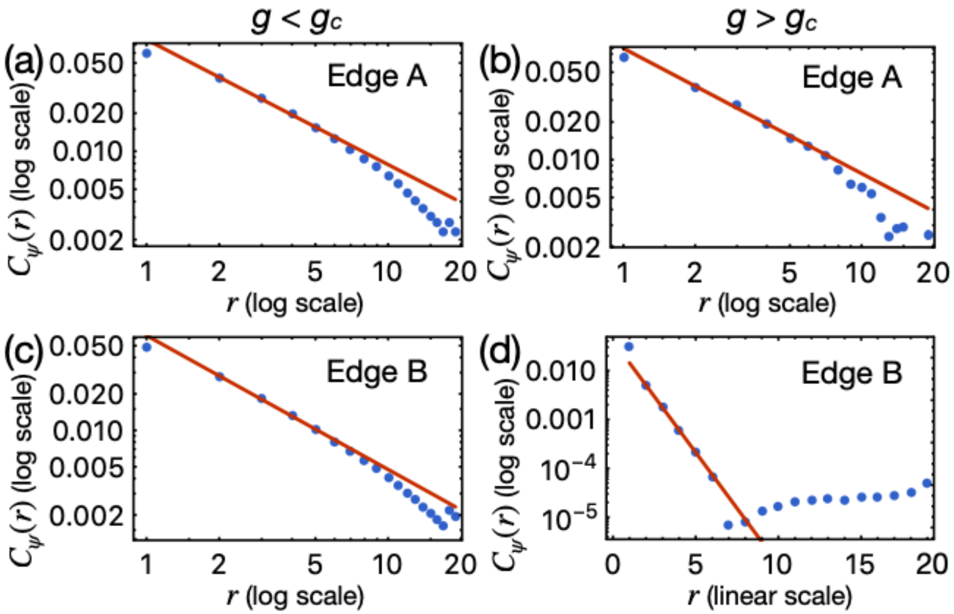
    Reduced model (no gauge field) does not seem to care about anomalies

    while it is constrained by the generalized no-go theorem.
- Other approaches obtained partial success, yet open issues remain

Thank you

ZZWY model: SMG for 3-4-5-0 chiral Schwinger model

figure from Zeng, Zhu, Wang and You:



Edge A: physical (chiral) fermions
Edge B:
"mirror" fermions to be gapped

Left: Weak interaction ($g < g_c$)
both massless (log-log scale!)
Right: Strong interaction ($g > g_c$)
Edge A massless (log-log)
Edge B gapped (linear-log)

Gauge-fixing approach: $U(1)$

Bock, Golterman, YS 1995 – 2002

- Chiral Wilson action $\Rightarrow$ longitudinal gauge dof's couple to fermions
- Reduced model: keep only longitudinal dof's

  continuum: $A_\mu(x) \to \partial_\mu \phi(x)$, lattice: $U_\mu(x) \to \phi(x) I \phi^\dagger(x+\hat{\mu})$
- Field redefinitions: $\psi_L(x) = \phi(x)\psi_L^c(x)$ $\psi_R(x) = \psi_R^s(x)$
- Kinetic term, e.g.: $\bar{\psi}_L(x)\,\gamma_\mu\,[U_\mu(x)\psi_L(x+\hat{\mu}) - U_\mu^\dagger(x-\hat{\mu})\psi_L(x-\hat{\mu})]$

$$\to \bar{\psi}_L(x)\,\gamma_\mu\,\phi(x)[\phi^\dagger(x+\hat{\mu})\psi_L(x+\hat{\mu}) - \phi^\dagger(x-\hat{\mu})\psi_L(x-\hat{\mu})]$$

$$\to \bar{\psi}_L^c(x)\,\gamma_\mu\,[\psi_L^c(x+\hat{\mu}) - \psi_L^c(x-\hat{\mu})]$$

- Wilson term – becomes Yukawa-like coupling

$$\bar{\psi}_R(x)[\psi_L(x+\hat{\mu}) + \psi_L(x-\hat{\mu})]$$

$$\to \bar{\psi}_R^s(x)[\phi(x+\hat{\mu})\psi_L^c(x+\hat{\mu}) + \phi(x-\hat{\mu})\psi_L^c(x-\hat{\mu})]$$

- Reduced model invariant under global $U(1)_{\text{charged}} \times U(1)_{\text{spectator}}$

Gauge-fixing approach, continued

- Add renormalizable gauge-fixing term to lattice action "Rome approach" 1990

$$S = \frac{1}{g^2}F^2 + \frac{1}{\xi g^2}\boxed{(\partial_\mu A_\mu)^2} + \text{chiral Wilson action} + \text{c.t.}$$

- Reduced model via $A_\mu \to \partial_\mu \phi$, controls dynamics of $\phi(x)$

$$S(\text{reduced}) = \frac{1}{\xi g^2}\boxed{(\Box\phi)^2} + \text{chiral Wilson action}(\psi_L^c, \psi_R^s, \phi) + \text{c.t.}$$

- Evade generalized no-go theorem!

  E.g., charged two-point function ($Q_{\text{charged}} = 1, \quad Q_{\text{spec.}} = 0$):

  LH channel: elementary massless fermion ψ_L^c

  RH channel: composite of ψ_R^s and ϕ^* (1st derivative not continuous)