Exact $\mathrm{SL}(2,\mathbb{Z})$-Structure of Lattice Maxwell Theory with θ-term in Modified Villain Formulation

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Theoretical and Mathematical Sciences

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Motivation

We'd like to extend a mass shift and a discrete chiral symmetry on the lattice Schwinger model [Dempsey et al., 2022] to $3+1D$.

$$H(\theta) = \frac{a}{2\beta}\left(\Pi_n + \frac{\theta}{2\pi}\right)^2 + \underbrace{\frac{1}{i2a}\left(\chi^\dagger_{n+1} e^{iA_n}\chi_n - h.c.\right) + m(-1)^n \chi^\dagger_n \chi_n}_{\text{Staggered fermion}}$$

$$[A_n, \Pi_m] = i\delta_{nm},\ \left\{\chi^\dagger_n, \chi_m\right\} = \delta_{nm}$$

The chiral transformation is given by $\chi_x \to \chi_{x+1}$ and H changes to

$$H(\theta) \to H(\theta+\pi) - 2\left(m + \frac{a}{8\beta}\right)(-1)^n \chi^\dagger_n \chi_n.$$

$\longrightarrow$ H has the chiral symmetry at $m = -\frac{a}{8\beta}$ (mass shift).

$\theta \to \theta + \pi$: Chiral anomaly

[Shao et al., 2025]: Onsager algebra.

[Numasawa]: Lieb-Schultz-Mattis anomaly (Jun 30, 3:30 PM)

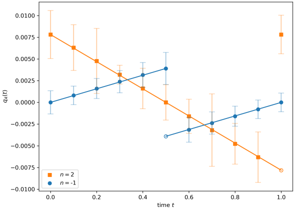
Chiral Transformation on $(3+1)$D Lattice

In $(3+1)$D, a fermion Hamiltonian [Kogut and Susskind, 1975] is given by

$$H = \frac{1}{2a} \sum_{j=1}^{3} \left(\eta_j(x) \chi^\dagger_{x+\hat{\mu}} e^{iA_{x,\mu}} \chi_x - h.c. \right) + m\epsilon(x) \chi^\dagger_x \chi_x$$

$$\eta_1(x) = 1,\ \eta_2(x) = (-1)^{x_1},\ \eta_3(x) = (-1)^{x_1+x_2},\ \epsilon(x) = (-1)^{x_1+x_2+x_3}.$$

- Chiral transformation:
  $\chi_x \to \Gamma_5 \chi_x = i(-1)^{x_2} \chi_{x+\hat{1}+\hat{2}+\hat{3}}$
- Chiral Anomaly:
  $\frac{\mathrm{d}}{\mathrm{d}t} \left\langle \chi^\dagger \Gamma \chi \right\rangle \to -2 \frac{1}{2\pi^2} F \wedge F$
  [SA, Kikukawa, and Takemoto, 2025]



$\longrightarrow$ This result impiles the 2π shift of a θ-term.

[Gioia and Thorngren, 2025]: Weyl fermions (Jul 2, 11:00 AM).

[Misumi, Onogi, and Yamaoka 2026]: boundary theory (Jul 3, 11:00 AM).

Towards the Mass Shift and Discrete Chiral Symmetry

To derive the mass shift, we need to construct a θ-term with

$$\theta \to \theta + 2\pi$$

under the chiral transformation.

$\longrightarrow$ It can be interpreted as $\underset{\text{One duality in } (3+1)\text{D Maxwell}}{\underbrace{\mathcal{T}\text{-transformation}}}$.

In this talk, we consider a $(3+1)$D Maxwell theory.

- Analyze the duality structure
- Determine the θ-term.

[Shao et al., 2026]: $(3+1)$D compact scalar theory (Jun 30, 9:30 AM)

[Furukawa, 2026]: $(2+1)$D fractonic field theory (July 2, 15:50)

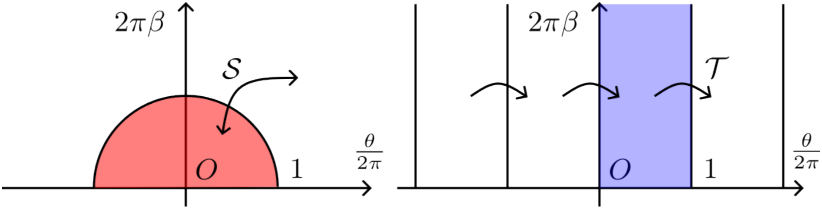
$\mathrm{SL}(2,\mathbb{Z})$-duality in the $(3+1)$D continuum space

The Maxwell theory in $(3+1)$D with θ term

$$Z[\beta,\theta] = \int DA \exp\left(-\frac{\beta}{2}F^2 - i\frac{\theta}{8\pi^2}F\wedge F\right)$$

is invariant under

$$\mathcal{S}: \tau \to -\frac{1}{\tau},\ \mathcal{T}: \tau \to \tau + 1 \quad (\tau = \frac{\theta}{2\pi} + i2\pi\beta)$$



$\mathcal{S}$ and $\mathcal{T}$ generate $\mathrm{SL}(2,\mathbb{Z}) = \left\langle S^2 = 1, (ST)^3 = 1 \right\rangle$ group.

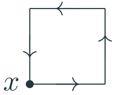
Lattice Gauge Action

We consider an N^4 lattice.

- Covariant difference: $U_{x,\mu} = \exp(iaA_{x,\mu})$

$$\frac{1}{a}(U_{x,\mu}\chi(x + a\hat{\mu}) - \chi(x)) \to (\partial_\mu + iA_\mu)\chi(x) \ (a \to 0)$$

- Field strength:



$$= U_{x,1}U_{x+\hat{1},2}U^{-1}_{x+\hat{2},1}U^{-1}_{x,2} = \exp(iadA) \to 1 + iadA$$

- Kinetic term

$$S = \beta\left(1 - \frac{1}{2}(\exp(iadA) + \exp(-iadA))\right) \to \frac{\beta}{2}(adA)^2$$

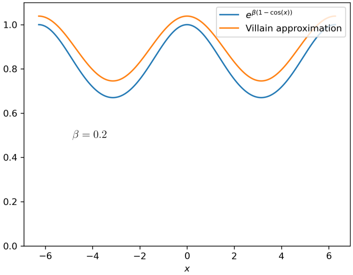
In this talk, we fix $a = 1$.

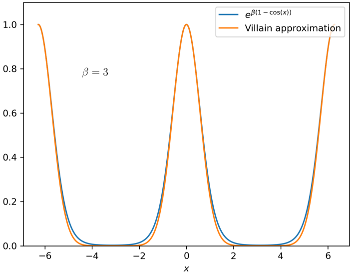
Villain Approximation

To establish the $\mathcal{T}$-duality, we need the integer topological charge.

⟶ Villain Approximation [Villain, 1975]

$$\exp(-\beta(1-\cos(x))) \simeq \sum_{n\in\mathbb{Z}} \exp\left(-\frac{\beta}{2}(x+2\pi n)^2\right)$$





If β is large enough, two functions coincide.

Villain Formulation

The Wilson partition function can be approximated by

$$Z_{\mathsf{Wilson}} = \int DA^e \exp(-\beta(1-\cos(dA^e))) \quad \left(\int DA^e = \int_{-\pi}^{\pi} \prod_{x,\mu} \frac{dA^e_{x,\mu}}{2\pi}\right)$$

$$\simeq \int DA^e \sum_{\{n\}} \exp\left(-\frac{\beta}{2}(dA^e + 2\pi n)^2\right). \quad \left(\sum_{\{n\}} = \prod_{\substack{x\\ \mu<\nu}} \sum_{n_{x,\mu\nu}}\right)$$

- $U_{x,\mu} = \exp\left(iA^e_{x,\mu}\right)$: Electric gauge field
- $dA^e + 2\pi n$: Gauge strength
- $n_{x,\mu\nu} \in \mathbb{Z}$: Manetic Flux $\sim B$
- $dn_{x,\mu\nu\rho}$: Monopole $\sim \nabla B$

$\mathcal{T}$-duality and θ term in Villain Formulation

The topological charge is defined by [Sulejmanpasic and Gattringer, 2019]

$$Q_0[dA^e + 2\pi n] = \frac{1}{8\pi^2} \sum_x (dA^e + 2\pi n) \cup (dA^e + 2\pi n)_{x,1234}$$

$$= \frac{1}{8\pi^2} \sum_x \sum_{\substack{\mu<\nu \\ \rho<\sigma}} \epsilon_{\mu\nu\rho\sigma} (dA^e + 2\pi n)_{x,\mu\nu} (dA^e + 2\pi n)_{x+\hat{\mu}+\hat{\nu},\rho\sigma}.$$

If monopoles are absent ($dn = 0$), Q_0 takes an integer value.

⟶ The partition function with θ term is $\mathcal{T}$-dual!

$$Z[\beta, \theta] = \int DA^e \sum_{\{n\}} \exp\left(-\frac{\beta}{2}(dA^e + 2\pi n)^2 - i\theta Q_0\right) \underbrace{\delta(dn = 0)}_{\text{Admissibility condition}}$$

in Wilson action

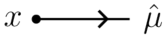
⟶ Modified Villain Formulation

Modified Villain Formulation

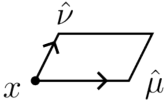
Introducing 3-form A^m, the partition function can be rewritten by

$$Z[\beta,\theta] = \int DA^e \sum_{\{n\}} \exp\left(-\frac{\beta}{2}(dA^e + 2\pi n)^2 - i\theta Q_0\right)\delta(dn = 0)$$

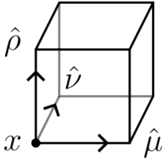
$$= \int DA^e DA^m \sum_{\{n\}} \exp\left(-\frac{\beta}{2}(dA^e + 2\pi n)^2 - i\theta Q_0 + iA^m dn\right).$$



$A^e_{x,\mu} \in [-\pi,\pi)$



$n_{x,\mu\nu} \in \mathbb{Z}$



$A^m_{x,\mu\nu\rho} \in [-\pi,\pi)$

Loop Operators

Let $\dot{\gamma}$ be a unit tangent vector of a contour γ.

- Electric Wilson loop:

$$W_e^{q_e}(\gamma) = \exp\left(i \oint_\gamma q_e A^e\right) = \exp(iq_e\dot{\gamma}A^e)$$

Electrically charged particle with $q_e \in \mathbb{Z}$ emerges on γ

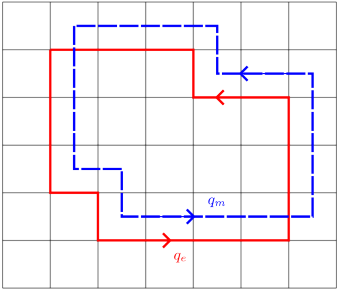
- Magnetic Wilson loop:

$$W_m^{q_m}(\gamma) = \exp(-iq_m\dot{\gamma} \cup A^m)$$

Monopole $q_m \in \mathbb{Z}$ appears on γ

- Dyonic Wilson loop:

$$\begin{aligned} W_d^{(q_e,q_m)}(\gamma) &= W_e^{q_e}(\gamma)W_m^{q_m}(\gamma) \\ &= \exp(iq_e\dot{\gamma}A^e - iq_m\dot{\gamma} \cup A^m) \end{aligned}$$

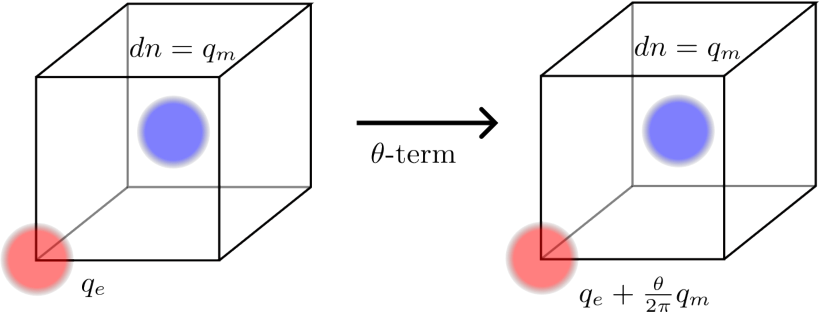


$W_d^{(q_e,q_m)}(\gamma)$

[Thorngren, 2015; Jacobson and Sulejmanpasic, 2023]: Framed Wilson loop

Witten Effect

Monopoles obtain the electric charge in the presence of the θ-term [Witten, 1979].



Shifting θ by 2π,

$$(q_e, q_m) \to (q_e + q_m, q_m)$$

Poisson Summation

$\mathcal{S}$-duality is described by Poisson summation for 2-form n.

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} F(m) \quad \left(F(y) = \int_{-\infty}^{\infty} dx\, f(x) \exp(i2\pi xy) \right)$$

If $f(x) = \exp\left(-\frac{1}{4\pi} 2\pi\beta (a + 2\pi x)^2 + ibx\right)$, $F(y)$ is given by

$$F(y) = \frac{1}{\sqrt{2\pi\beta}} \exp\left(-\frac{1}{4\pi} \frac{1}{2\pi\beta} (b + 2\pi y)^2 - iay \right) \exp\left(-i\frac{ab}{2\pi} \right)$$

The Poisson summation converts

$$2\pi\beta \to \frac{1}{2\pi\beta}, \;\; a \to b, \;\; b \to -a$$

[Anosova et al., 2022; Choi et al., 2022; Sulejmanpasic and Gattringer, 2019; Gorantla et al., 2021]

$\mathcal{S}$-duality at $\theta = 0$

$\mathcal{S}$-duality is described by Poisson summation for 2-form n,

$$\begin{aligned} Z[\beta, 0] &= \int DA^e DA^m \sum_{\{n\}} \exp\left(-\frac{1}{4\pi} 2\pi\beta (dA^e + 2\pi n)^2 + i A^m dn \right) \\ &\propto \int DA^e DA^m \sum_{\{m\}} \exp\left(-\frac{1}{4\pi} \frac{1}{2\pi\beta} (\partial A^m + 2\pi m)^2 - i A^e \partial m \right) \\ &= Z\left[\frac{1}{4\pi^2\beta}, 0 \right], \end{aligned}$$

where $a(db) = (-1)^{|a|}(\partial a) b$ and

$$dA^e(\partial A^m + 2\pi m) = -A^e \partial(\partial A^m + 2\pi m) = -2\pi A^e \partial m$$

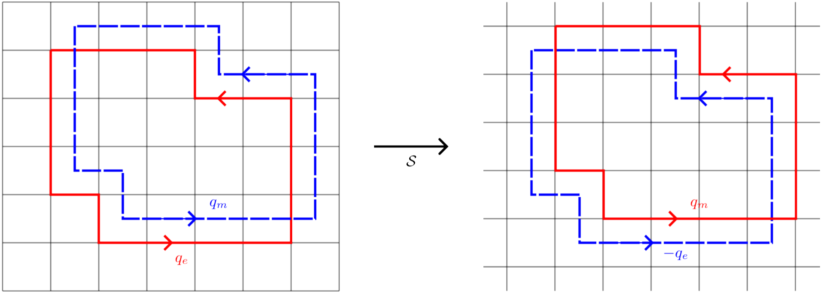
That is, $\mathcal{S}$ converts

$$2\pi\beta \to \frac{1}{2\pi\beta}, \quad A^e \to A^m, \quad A^m \to -A^e$$

$\mathcal{S}$-transformation for Loop Operators

$\mathcal{S}$ exchanges A^e and A^m,

$$W_e \leftrightarrow W_m, \;\; (q_e, q_m) \to (q_m, -q_e)$$



Note that the framing is flipped.

$\mathcal{S}$-duality with θ term

In the presence of the θ-term, the action is described by

$$M = M_{x,\mu\nu|y,\rho\sigma} = \underbrace{2\pi\beta\delta_{x,y}\delta_{\mu\rho}\delta_{\nu\sigma}}_{\text{Kinetic Term}} + \underbrace{i\frac{\theta}{4\pi}(\epsilon + {}^t\epsilon)}_{\theta\text{-term}} \quad (\epsilon = \epsilon_{\mu\nu\rho\sigma}\delta_{x,y-\hat{\rho}-\hat{\sigma}}).$$

However, the inverse M^{-1} is non-local due to θ-term.

$\longrightarrow$ This does not seem to have the $\mathcal{S}$-duality.

[Anosova et al., 2022] suggest an overlap-type action

$$M_{AGS} = \sqrt{(2\pi\beta)^2 + \left(\frac{\theta}{2\pi}\right)^2}\frac{M}{\sqrt{M^*M}}$$

$\longrightarrow$ The $\mathcal{S}$-duality is exact, whereas the $\mathcal{T}$-duality is not.

$\mathcal{S}$-duality with θ term

In the presence of the θ-term, the action is described by

$$M = M_{x,\mu\nu|y,\rho\sigma} = \underbrace{2\pi\beta\delta_{x,y}\delta_{\mu\rho}\delta_{\nu\sigma}}_{\text{Kinetic Term}} + \underbrace{i\frac{\theta}{4\pi}(\epsilon + {}^{t}\epsilon)}_{\theta\text{-term}} \quad (\epsilon = \epsilon_{\mu\nu\rho\sigma}\delta_{x,y-\hat{\rho}-\hat{\sigma}}).$$

However, the inverse M^{-1} is non-local due to θ-term.

$\longrightarrow$ We prove the $\mathcal{S}$-duality.

[Anosova et al., 2022] suggest an overlap-type action

$$M_{AGS} = \sqrt{(2\pi\beta)^2 + \left(\frac{\theta}{2\pi}\right)^2}\frac{M}{\sqrt{M^*M}}$$

$\longrightarrow$ The $\mathcal{S}$-duality is exact, whereas the $\mathcal{T}$-duality is not.

Our Work

In this talk, we revisit the ultra-local partition function

$$Z[\beta,\theta] = \int DA^e DA^m \sum_{\{n\}} \exp\left(-\frac{\beta}{2}(dA^e + 2\pi n)^2 - i\theta Q_0 + iA^m dn\right)$$

$$Q_0 = \frac{1}{8\pi^2}(dA + 2\pi n) \cup (dA + 2\pi n)$$

and show that the non-local effect can be removed by an appropriate $\mathcal{S}$-transformation.

$\longrightarrow$ The theory has full $\mathrm{SL}(2,\mathbb{Z})$ duality

We also derive the $\mathrm{SL}(2,\mathbb{Z})$-transformation laws of W_d,

$$\mathcal{S} : \left\langle W_d^{(q_m,-q_e)}(\gamma) \right\rangle_{\tilde{\beta},\tilde{\theta}} = e^{iu\pi\left[\frac{\theta}{2\pi}q_m^2 + 2q_e q_m - \frac{\tilde{\theta}}{2\pi}q_e^2\right]} \left\langle W_d^{(q_e,q_m)}(\gamma) \right\rangle_{\beta,\theta}$$

$$\mathcal{T} : \left\langle W_d^{(q_e-q_m,q_m)}(\gamma) \right\rangle_{\beta,\theta+2\pi} = (-1)^{vq_m^2} \left\langle W_d^{(q_e,q_m)}(\gamma) \right\rangle_{\beta,\theta}.$$

where u and v are real constants determined by γ.

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Exact Calculation of the Path Integral

We first check the $\mathcal{S}$-duality by calculating the partition function. By using the gauge fixing techniques [Peng et al., 2025], we get

$$\begin{aligned} Z[\beta,\theta] =& C\frac{1}{\sqrt{\beta}^{3(N^4-1)}}\left(\sum_{n_1}\sum_{n_2}\exp\left(-\frac{\beta}{2}4\pi^2(n_1^2+n_2^2)-i\theta n_1 n_2\right)\right)^3 \\ =& \sqrt{\frac{\tilde{\beta}}{\beta}}^{3N^4} Z[\tilde{\beta},\tilde{\theta}], \quad \left(\frac{\tilde{\theta}}{2\pi}+i2\pi\tilde{\beta}=-\frac{1}{\frac{\theta}{2\pi}+i2\pi\beta}\right), \end{aligned}$$

where $C>0$ only depends on the volume of the lattice space. (Cf. This result agrees with continuum predictions [Witten, 1995].)

The ultra-local partition function has the $\mathcal{S}$-duality!

[Murayama]: 2D case

Projectors

Why does the ultra-local theory appear not to possess $\mathcal{S}$-duality?

$\longrightarrow$ $\mathcal{S}$ is not described by the simple Poisson summation when $\theta \neq 0$.

To construct the appropriate $\mathcal{S}$-transformation, we consider the Hodge decomposition for 2-form n,

$$n = dn^{(1)} + \partial n^{(3)} + \bar{n},$$

where $n^{(r)}$ is a $\mathbb{C}$ r-form, and $\bar{n}$ is a 2-form.

Especially, $\bar{n}$ satisfies $d\bar{n} = \partial\bar{n} = 0$, and

$$\bar{n}_{\mu\nu} = \frac{1}{\sqrt{N^4}} \sum_{\substack{x \\ \mu<\nu}} n_{x,\mu\nu}.$$

We define projectors,

$$P_d n = dn^{(1)},\ P_\partial n = \partial n^{(3)},\ P_0 n = \bar{n}$$

Non-local θ-term

Setting $F^e = dA^e + 2\pi n$ and

$$\epsilon_{x,\mu\nu|y,\rho\sigma} = \epsilon_{\mu\nu\rho\sigma}\delta_{x+\hat{\mu}+\hat{\nu},y},$$

we define a non-local θ-term

$$\begin{aligned} Q_{NL}[F^e] =& \frac{1}{8\pi^2} F^e \cdot \left[P_d \epsilon P_\partial + P_\partial {}^t\epsilon P_d + P_0 \frac{\epsilon + {}^t\epsilon}{2} P_0 \right] F^e \\ =& Q_0[F^e] + \underbrace{\frac{1}{8\pi^2} F^e \cdot \left[P_d \frac{\epsilon - {}^t\epsilon}{2} P_\partial - P_\partial \frac{\epsilon - {}^t\epsilon}{2} P_d \right] F^e}_{\text{Non-local}}. \end{aligned}$$

However, $P_\partial n = 0$ without monopoles $(dn = 0)$!

$\longrightarrow Q_{NL}[F^e] = Q_0[F^e]$

$\mathcal{S}$-duality with θ term

We construct the $\mathcal{S}$-trasformation:

1. Replace Q_0 with Q_{NL}

$$\frac{\beta}{2}(F^e)^2 + i\theta Q_0[F^e] - iA^m dn$$
$$\to \frac{1}{4\pi} F^e \underbrace{\left(2\pi\beta + i\frac{\theta}{2\pi}\left[P_d \epsilon P_\partial + P_\partial {}^t\epsilon P_d + P_0 \frac{\epsilon + {}^t\epsilon}{2} P_0\right]\right)}_{= M_{NL}(\beta,\theta)} F^e - iA^m dn.$$

2. Apply Poisson summation formula

$$M_{NL}(\beta,\theta)^{-1} = M_{NL}(\tilde{\beta},\tilde{\theta})$$

3. Replace Q_{NL} with Q_0

By incorporating this non-local procedure into the $\mathcal{S}$,
the action remains ultra-local!

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θ-term with Dyonic Matters

Since $dn \neq 0$ in the presence of Dyonic matters,

$$Q_{NL} \neq Q_0.$$

We seek an extension of Q satisfying:

1. Ultra-locality.
2. $Q[F^e] = Q_{NL}[F^e] + \mathcal{O}\big((dn)^2\big)$.
3. $Q = Q_0$ when $dn = 0$.

We find

$$\begin{aligned} Q[F^e] &= \frac{1}{8\pi^2} F^e \cup F^e + \frac{1}{4\pi} F^e \cup_1 dn \\ &= \frac{1}{2\pi} dA^e \cup n + \frac{1}{2}(n \cup n + n \cup_1 dn). \end{aligned}$$

A similar topological charge is defined in a lattice Chern-Simons theory.

[Jacobson and Sulejmanpasic, 2023]

Higher Cup Product

- Definition:

$$
\begin{aligned}
(\alpha^{(2)} \cup_1 \beta^{(3)})_{x,1234} = {} & (\alpha_{x,34} + \alpha_{x+\hat{3},24} + \alpha_{x+\hat{2}+\hat{3},14})\,\beta_{x,123} \\
& + (\alpha_{x,14} + \alpha_{x+\hat{4},13} + \alpha_{x+\hat{3}+\hat{4},12})\,\beta_{x+\hat{1},234} \\
& - (\alpha_{x+\hat{4},23} + \alpha_{x+\hat{2}+\hat{4},13})\,\beta_{x,124} + \alpha_{x,34}\,\beta_{x+\hat{3},124} \\
& - (\alpha_{x,24} + \alpha_{x+\hat{4},23})\,\beta_{x+\hat{2},134} + \alpha_{x+\hat{3}+\hat{4},12}\,\beta_{x,134}.
\end{aligned}
$$

- Leibniz Rule:

$$
\begin{aligned}
d(\alpha^{(2)} \cup_1 \beta^{(2)}) = {} & (d\alpha) \cup_1 \beta + (-1)^{2} \alpha \cup_1 d\beta \\
& + (-1)^{2+2+1} (\alpha \cup \beta - (-1)^{2\times 2} \beta \cup \alpha)
\end{aligned}
$$

Expectation Value of Dyonic Loop

$$\left\langle W_d^{(q_e,q_m)}(\gamma)\right\rangle_{\beta,\theta} = \frac{1}{Z[\beta,\theta]}\int DA^e DA^m \sum_{\{n\}} \exp\left(-\frac{\beta}{2}(F^e)^2 - i\theta Q[F^e] + iA^m dn\right) \times \exp(i(q_e\dot{\gamma}A^e - q_m\dot{\gamma}\cup A^m))$$

By integrating A^m field, we get a monopole constraint

$$\epsilon_{\mu\nu\rho\sigma}(dn)_{x+\hat{\mu},\nu\rho\sigma} = -q_m\dot{\gamma}_{x,\mu}.$$

$\longrightarrow$ Monopole appears on γ

$\mathcal{T}$-transformation and Witten Effect

$\theta \to \theta + 2\pi$ induces

$$\left\langle W_d^{(q_e,q_m)}(\gamma) \right\rangle_{\beta,\theta+2\pi} = \left\langle W_d^{(q_e,q_m)}(\gamma) \exp(-i2\pi Q[F^e]) \right\rangle_{\beta,\theta}$$

Using $dn = \epsilon q_m \dot{\gamma}$ and

$$2\pi Q[F^e] = A^e \cup dn + \pi\underbrace{(n \cup n + n \cup_1 dn)}_{\text{Pontryagin Square } \mathfrak{P}(n)} = \underbrace{-q_m \dot{\gamma} \cdot A^e}_{\text{Witten Effect}} + \pi \mathfrak{P}(n),$$

we observe the Witten effect

$$\left\langle W_d^{(q_e,q_m)}(\gamma) \right\rangle_{\beta,\theta+2\pi} = \left\langle W_d^{(q_e+q_m,q_m)}(\gamma) \exp(i\pi\mathfrak{P}(n)) \right\rangle_{\beta,\theta},$$

$$(q_e, q_m) \to (q_e + q_m, q_m).$$

[Kobayashi et al., 2025]: $\mathfrak{P}(n)$ is related to a higher form anomaly (July 1, 2:00 PM)

Pontryagin Square $\mathfrak{P}(n)$

Definition: $\mathfrak{P}(n) = n \cup n + n \cup_1 dn$

Properties

- If $dn' = 0$, $\mathfrak{P}(n') \in 2\mathbb{Z}$
- $\mathfrak{P}(n + n') = \mathfrak{P}(n) + 2\mathbb{Z}$

$\mathfrak{P}(n)$ only depends on $dn = \epsilon q_m \dot{\gamma}$ (up to an even number).
$\longrightarrow$ $\mathfrak{P}(n)$ becomes a constant in the path integral

$$\mathfrak{P}(n) = q_m^2 v + 2\mathbb{Z} \quad (v \in \mathbb{Z}).$$

Then we get

$$\left\langle W_d^{(q_e,q_m)}(\gamma) \right\rangle_{\beta,\theta+2\pi} = (-1)^{v q_m^2} \left\langle W_d^{(q_e+q_m,q_m)}(\gamma) \right\rangle_{\beta,\theta} .$$

(v is a quantity analogous to the self-linking number of γ.)

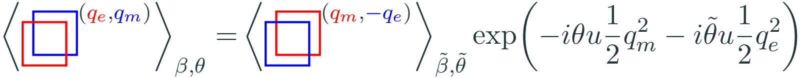
Poisson Summation

To apply the Poisson summation, we need to extract Q_{NL} from Q,

$$Q[F^e] = Q_{NL}[F^e] + \frac{1}{2}\underbrace{(P_\partial n \cup_1 dP_\partial n)}_{= \mathcal{O}\left((dn)^2\right)}.$$

By using $dn = \epsilon q_m dn$, the second term becomes a constant,

$$P_\partial n \cup_1 dP_\partial n = q_m^2 u.$$



Electrically and magetically objects are exchanged,

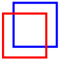
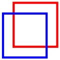
$$(q_e, q_m) \to (q_m, -q_e).$$

However, its framing is flipped.

Framing of Dyonic Loop

$\mathcal{P}$: Applying the Poisson summation

We can flip the framing by alternating $\mathcal{P}$ and $\mathcal{T}$ in three times,

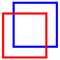
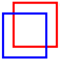
$\Bigg\langle$  $^{(q_e,q_m)}\Bigg\rangle_{\beta,\theta} = \Bigg\langle$  $^{(q_e,q_m)}\Bigg\rangle_{\beta,\theta} \exp\left(i2\pi u(q_e + \frac{\theta}{2\pi}q_m)q_m\right).$

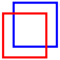
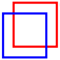
The phase factor can be interpreted as the AB phase between the electric charge $q_e + \frac{\theta}{2\pi}q_m$ and the magnetic charge q_m.

[Jackiw and Rebbi, 1976; Goldhaber, 1976; Hasenfratz and 't Hooft, 1976]

$\mathcal{S}$-transformation

Using

$\Big\langle$  $^{(q_e,q_m)}\Big\rangle_{\beta,\theta} = \Big\langle$  $^{(q_m,-q_e)}\Big\rangle_{\tilde{\beta},\tilde{\theta}} \exp\left(-i\theta u\frac{1}{2}q_m^2 - i\tilde{\theta}u\frac{1}{2}q_e^2\right),$

$\Big\langle$  $^{(q_e,q_m)}\Big\rangle_{\beta,\theta} = \Big\langle$  $^{(q_e,q_m)}\Big\rangle_{\beta,\theta} \exp\left(i2\pi u(q_e + \frac{\theta}{2\pi}q_m)q_m\right),$

we have

$$\left\langle W_d^{(q_e,q_m)}(\gamma)\right\rangle_{\beta,\theta} = \exp\left(-iu\pi\left[\frac{\theta}{2\pi}q_m^2 + 2q_eq_m - \frac{\tilde{\theta}}{2\pi}q_e^2\right]\right) \times \left\langle W_d^{(q_m,-q_e)}(\gamma)\right\rangle_{\tilde{\beta},\tilde{\theta}}.$$

$\left\langle W_d^{(q_e,q_m)}(\gamma)\right\rangle_{\beta,\theta}$ is $\mathcal{S}$ covariance!

$\mathrm{SL}(2,\mathbb{Z})$-structure

$$\mathcal{S} : \left\langle W_d^{(q_m,-q_e)}(\gamma) \right\rangle_{\tilde{\beta},\tilde{\theta}} = e^{iu\pi\left[\frac{\theta}{2\pi}q_m^2+2q_eq_m-\frac{\tilde{\theta}}{2\pi}q_e^2\right]} \left\langle W_d^{(q_e,q_m)}(\gamma) \right\rangle_{\beta,\theta}$$

$$\mathcal{T} : \left\langle W_d^{(q_e-q_m,q_m)}(\gamma) \right\rangle_{\beta,\theta+2\pi} = \underbrace{(-1)^{vq_m^2} \left\langle W_d^{(q_e,q_m)}(\gamma) \right\rangle_{\beta,\theta}}_{\text{A new class of dyonic operator}}.$$

We find four kinds of the dyonic operator,

$$\begin{tikzcd}
W_d^{(q_e,q_m)} \arrow[loop left, "\mathcal{S}"] \arrow[r, leftrightarrow, "\mathcal{T}"] & (-1)^{vq_m}W_d^{(q_e,q_m)} \arrow[r, leftrightarrow, "\mathcal{S}"] & (-1)^{vq_e}W_d^{(q_e,q_m)} \arrow[loop right, "\mathcal{T}"] \\
& (-1)^{v(q_e+q_m)}W_d^{(q_e,q_m)} \arrow[loop right, "{\mathcal{S},\mathcal{T}}"] &
\end{tikzcd}$$

The signs are related to their statistics of Dyonic operators.

(Cf. [Ang et al., 2020; Kan et al., 2024])

Non-spin Maxwell Theory

Similar $\mathrm{SL}(2,\mathbb{Z})$-structure is found in a non-spin Maxwell theory .

Maxwell theory on non-spin manifold ($w_2 \neq 0$)

$$\begin{tikzcd}
W_{\mathrm{b}}^{q_e}T_{\mathrm{b}}^{q_m} \arrow[loop left, "\mathcal{S}"] \arrow[r, leftrightarrow, "\mathcal{T}"] & W_{\mathrm{b}}^{q_e}T_{\mathrm{f}}^{q_m} \arrow[r, leftrightarrow, "\mathcal{S}"] & W_{\mathrm{f}}^{q_e}T_{\mathrm{b}}^{q_m} \arrow[loop right, "\mathcal{T}"] \\
 & W_{\mathrm{f}}^{q_e}T_{\mathrm{f}}^{q_m} \arrow[loop right, "\mathcal{S},\mathcal{T}"] &
\end{tikzcd}$$

The subscripts b/f express the statistics of Wilson/'t Hooft loops.

$$W_{\mathsf{f}}^{q_e}(\gamma = \partial\Sigma) = (-1)^{q_e \int_\Sigma w_2} \exp\left(i q_e \oint_\gamma A^e \right)$$

$$\leftrightarrow (-1)^{q_e v} \exp\left(i q_e \oint_\gamma A^e \right)$$

The parameter v plays the same role of 2nd Stiefel Whitney class w_2.

Contents

Introduction

$\mathrm{SL}(2,\mathbb{Z})$-duality in Pure Maxwell

$\mathrm{SL}(2,\mathbb{Z})$-structure of Dyonic Operators

Summary

Summary

We employed the ultra-local lattice Maxwell theory in the modified Villain formulation.

- Established the full $\mathrm{SL}(2,\mathbb{Z})$ duality.
- Determined the modular transformation laws of W_d with

$$Q[F^e] = \frac{1}{8\pi^2} F^e \cup F^e + \frac{1}{4\pi} F^e \cup_1 dn$$

[Outlook]

- Canonical Formalism (work in progress)
- Mass Shift
- Lattice-exact Cardy–Rabinovici model

[Cardy, 1982; Cardy and Rabinovici, 1982; Katayama and Tanizaki, 2025]

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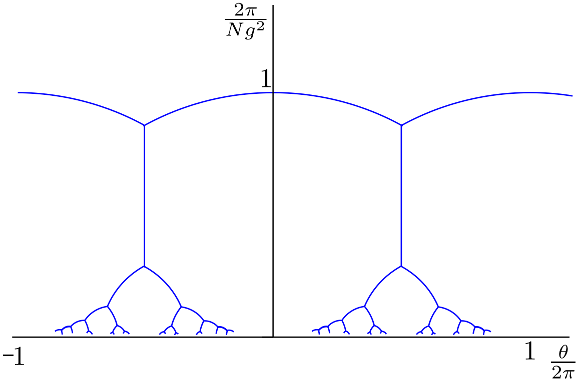
Cardy-Rabinovici model

- A toy model of lattice Maxwell with a charge-N Higgs
- Assumed full $\mathrm{SL}(2,\mathbb{Z})$-duality
- Rich phase structure [Cardy, 1982; Cardy and Rabinovici, 1982]
- Non-invertible symmetries and Gravitational anomalies

[Honda and Tanizaki, 2020; Hayashi and Tanizaki, 2022]



Definitions of Projectors

We denote the Fourier transformation of r-form α by

$$\alpha'_{\mu_1\mu_2\cdots\mu_r}(p) = \sum_x \frac{e^{-ipx}}{\sqrt{N^d}} \alpha_{x,\mu_1\mu_2\cdots\mu_r}$$

Projectors are determined by

$$(P_d\alpha)_{x,\mu_1\mu_2\cdots\mu_r} = \sum_{p\neq 0} \frac{e^{ipx}}{\sqrt{N^d}} \sum_{j=1}^{r} \sum_{\mu_{r+1}=1}^{d} \frac{(-1)^{r+j} f_{\mu_j} f^*_{\mu_{r+1}} \alpha'_{\mu_1\cdots\mathring{\mu}_j\cdots\mu_{r+1}}(p)}{|f(p)|^2},$$

$$(P_\partial\alpha)_{x,\mu_1\mu_2\cdots\mu_r} = \sum_{p\neq 0} \frac{e^{ipx}}{\sqrt{N^d}} \sum_{j=1}^{r+1} \sum_{\mu_{r+1}=1}^{d} \frac{(-1)^{r+j+1} f_{\mu_j} f^*_{\mu_{r+1}} \alpha'_{\mu_1\cdots\mathring{\mu}_j\cdots\mu_{r+1}}(p)}{|f(p)|^2},$$

$$(P_0\alpha)_{x,\mu_1\mu_2\cdots\mu_r} = \frac{1}{\sqrt{N^d}} \alpha'_{\mu_1\mu_2\cdots\mu_r}(0),$$

with $f_\mu(p) = e^{ip_\mu} - 1$ and $|f(p)|^2 = \sum_\sigma f^*_\sigma f_\sigma(p)$.

Integerness of Topological Charge

$$\begin{aligned} Q_0[F^e] =& \frac{1}{8\pi^2} F^e \cup F^e \\ =& \frac{1}{8\pi^2} F^e \left[P_d \frac{\epsilon + {}^t\epsilon}{2} P_\partial + P_\partial \frac{\epsilon + {}^t\epsilon}{2} P_d + P_0 \frac{\epsilon + {}^t\epsilon}{2} P_0 \right] F^e \end{aligned}$$

Assuming $dn = 0$, we get $P_\partial F^e = 0$ and

$$\begin{aligned} Q_0[F^e] =& \frac{1}{8\pi^2} (dA + 2\pi n) P_0 \frac{\epsilon + {}^t\epsilon}{2} P_0 (dA + 2\pi n) \\ =& \bar{n}_{12}\bar{n}_{34} - \bar{n}_{13}\bar{n}_{24} + \bar{n}_{14}\bar{n}_{23}. \end{aligned}$$

$(P_0 n)_{12} = \bar{n}_{12}$ becomes an integer because

Symmetry

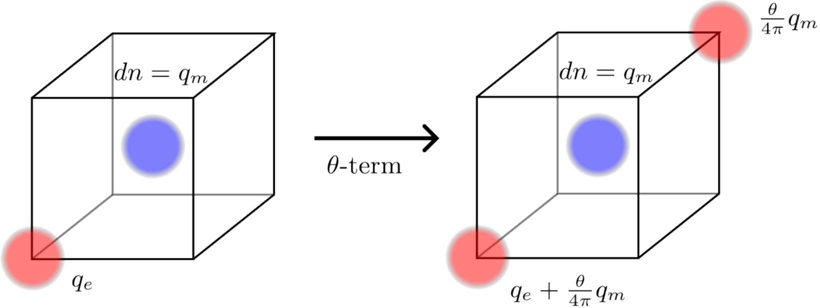
$$Z[\beta,\theta] = \int DA^e DA^m \sum_{\{n\}} \exp\left(-\frac{\beta}{2}(dA^e + 2\pi n)^2 - i\theta Q_0 + iA^m dn\right)$$

$$Q_0 = \frac{1}{8\pi^2}(dA + 2\pi n) \cup (dA + 2\pi n)$$

- $\mathbb{R}$ 0-form gauge symmetry: $A^e \to A^e + d\lambda^{(0)}$
- $\mathbb{Z}$ 1-form gauge symmetry: $A^e \to A^e + 2\pi k^{(1)},\ n \to n - dk^{(1)}$
- $\mathbb{R}$ 1-form global symmetry: $A^e \to A^e + \lambda^{(1)}\ (d\lambda^{(1)} = 0)$

Witten Effect under Q_0

$$Q_0 = \frac{1}{8\pi^2}(dA^e + 2\pi n) \cup (dA^e + 2\pi n)$$
$$= \frac{1}{4\pi}(A^e \cup dn - dn \cup A^e) + \frac{1}{2} n \cup n$$



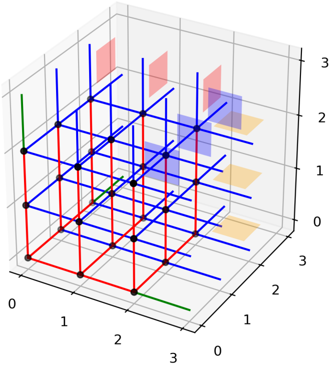
The induced charge is spitted.

Gauge Fixing [Peng et al., 2025]

Assuming $dn = 0$, n can be decomposed into

$$n = dg + \check{h} \ (g, \ \check{h} \in \mathbb{Z})$$

- $A^e \to 0$ via 0-form symm.
- $A^e \in \mathbb{R}$ after absorbing g by 1-form gauge symm.
- $n \to \check{h}$ is assigned on colored plaquettes.
- $A^e \to 0$ via 1-form global symm.



Degrees of freedom:

$$\#\{A^e \in \mathbb{R}\} = 3(N^4 - 1), \ \#\check{h} = 6$$

Implementing the Path Integral

$$
\begin{aligned}
Z[\beta,\theta] &= \int DA^e \sum_{\{n\}} \exp\left(-\frac{\beta}{2}(dA^e+2\pi n)^2 - i\theta Q_0\right)\delta(dn=0) \\
&= \left(\prod_{l:\text{red}} \int_{-\pi}^{\pi} \frac{dA_l^e}{2\pi}\right)\left(\prod_{l:\text{green}} \int_{-\pi}^{\pi} \frac{dA_l^e}{2\pi}\right)\left(\prod_{l:\text{blue}} \int_{-\infty}^{\infty} \frac{dA_l^e}{2\pi}\right)\sum_{\{\check{h}\}} \\
&\quad \times \exp\left(-\sum_x \left[\frac{\beta}{2}(dA^e+2\pi\check{h})^2 + i\theta\frac{1}{2}P_0\check{h}\cup P_0\check{h}\right]\right) \\
&= C\frac{1}{\sqrt{\beta}^{3(N^4-1)}}\left(\sum_{n_1}\sum_{n_2}\exp\left(-\frac{\beta}{2}4\pi^2(n_1^2+n_2^2) - i\theta n_1 n_2\right)\right)^3 \\
&= \frac{C}{\sqrt{\beta}^{3(N^4-1)}}\left(\left|\vartheta\begin{bmatrix}0\\0\end{bmatrix}(0,2\tau)\right|^2 + \left|\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}(0,2\tau)\right|^2\right)^3,
\end{aligned}
$$

Applying $\mathcal{S}$-transformation

$$
\begin{aligned}
Z[\beta,\theta] &= \int DA^e DA^m \sum_{\{n\}} \exp\left(-\frac{1}{4\pi} F^e M(\beta,\theta) F^e + i A^m dn\right) \\
&= \int DA^e DA^m \sum_{\{n\}} \exp\left(-\frac{1}{4\pi} F^e M_{NL}(\beta,\theta) F^e + i A^m dn\right) \\
&= \int DA^e DA^m \sum_{\{n\}} \exp\left(-\frac{1}{4\pi} F^m M_{NL}(\tilde{\beta},\tilde{\theta}) F^m + i A^e dm\right) \\
&= \int DA^e DA^m \sum_{\{n\}} \exp\left(-\frac{1}{4\pi} F^m M(\tilde{\beta},\tilde{\theta}) F^m + i A^e dm\right)
\end{aligned}
$$

Comparison with previous studies

- Naive ultra-local action:

$$M = 2\pi\beta + i\frac{\theta}{2\pi}\left[P_d\frac{\epsilon + {}^t\epsilon}{2}P_\partial + P_\partial\frac{\epsilon + {}^t\epsilon}{2}P_d + P_0\frac{\epsilon + {}^t\epsilon}{2}P_0\right]$$

- [Anosova et al., 2022]:

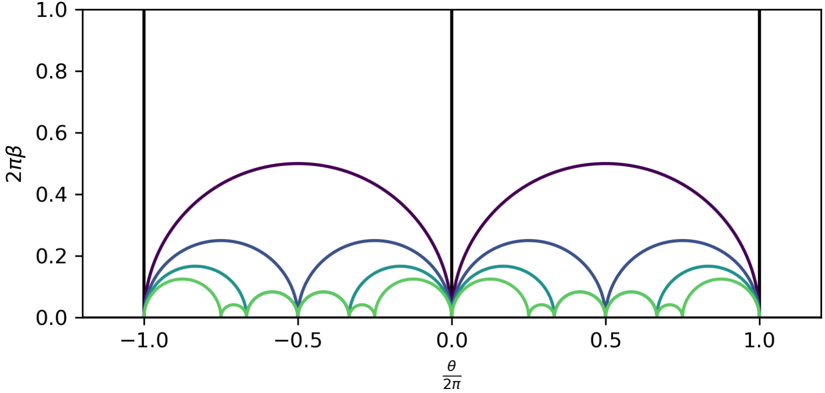
$$M_{AGS} = \sqrt{(2\pi\beta)^2 + \left(\frac{\theta}{2\pi}\right)^2}\frac{M}{\sqrt{M^*M}}$$

- Our work:

$$M_{NL} = 2\pi\beta + i\frac{\theta}{2\pi}\left[P_d\epsilon P_\partial + P_\partial{}^t\epsilon P_d + P_0\frac{\epsilon + {}^t\epsilon}{2}P_0\right]$$

Sign Problem

We can avoid the sign problem on the solid lines.



Canonical Formalism

- $\mathcal{T}$-transformation:

$$\mathcal{T} = \exp\left(i\frac{1}{4\pi}(A^e \cup (dA^e + 2\pi n) + 2\pi n \cup A^e + A^e \cup_1 2\pi dn) \right)$$

- $\mathbb{Z}$ 1-form transformation: $A^e \to A^e + 2\pi k,\ n \to n - dk$

$$U(k) = \exp\left(i2\pi k(\Pi^e + \frac{1}{2\pi}\partial A^m) \right)$$

$\mathcal{T}$ is not invariant under $U(k)$,

$$\mathcal{T}U(k) = \underbrace{U(k)\exp(i\pi(k \cup dk + dk \cup_1 n))}_{= V(k)}\mathcal{T}.$$

Putting a spin $|s = (-1)^n\rangle$, $U(k) \sim X$ and $(-1)^n \sim Z$. Then,

$$V(k) \simeq XZ$$

$V(k) = 1 \longrightarrow$ Fermion system

[Chen and Kapustin, 2019; Feng et al., 2026]