Lattice 2D $U(1)$ chiral gauge theory and magnetically charged vertex operators via bosonization

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- OM, S. Onoda and H. Suzuki, PTEP **2024**, no.6, 063B01 (2024), arXiv:2403.03420 [hep-lat].
- cf. S. Onoda, 2505.05050, 2606.12358.

Workshop landscape

- Chiral gauge theories force us to keep three notions separated:
  - gauge invariance,
  - locality of the regularized theory,
  - anomaly matching and anomalous Ward identities.
- Non-perturbative approaches.
  1. Hamiltonian and bosonic constructions: exact microscopic symmetries and anomaly realization.
  2. (Modified) Villain constructions: duality, integer topological charge, and magnetic sectors are manifest.
  3. Wilson/overlap constructions: standard Euclidean gauge variables, locality, index, and measure integrability.
- For chiral gauge theory, the decisive question is not only the continuum limit.
- What can be made exact at finite cutoff?

Strong consistency criterion

- Let $Z[A]$ be the regulated functional for a would-be chiral gauge theory.
- Background gauge field: It is acceptable, and necessary, that

$$Z[A^\Lambda] = \mathrm{e}^{\mathrm{i}\mathcal{A}_{\mathrm{lat}}[\Lambda, A]} Z[A]$$

  for an anomalous theory.
- Dynamical gauge field: The path integral over A should be defined as a gauge theory only when

$$\mathcal{A}_{\mathrm{lat}}[\Lambda, A] = 0 \mod 2\pi.$$

- How does a formulation fail when the gauge representation is anomalous?
  - No-go for lat anomalous gauge theory [Kikukawa–Suzuki '07] $\to$ low-energy effective theory with gauge-boson mass

Manifest gauge invariance at finite cutoff

- Use bosonization as a 2D laboratory, but organize the story around the Wilson/overlap.
    - ★ Modified Villain and Hamiltonian approaches are extremely sharp for anomalies and dualities.
    - ★ Still, Wilson-type link variables remain the common language of Euclidean lattice gauge theory.
- A natural Wilson-type criterion: Can gauge invariance be made exact at finite cutoff only in anomaly-free cases?
- Obstacle is the gauge-field dependent measure or its bosonized counterpart.

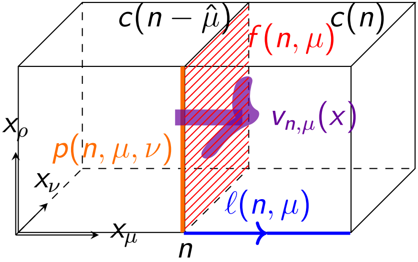
- This talk: *A chiral gauge theory should be a gauge theory on the lattice when and only when its gauge anomaly cancels. In 2D, bosonization lets us test this principle with magnetically charged vertex operators on a finite Wilson-type lattice.*

Wilson-type topology without monopoles

- A Wilson-type lattice is not topologically blind.
- With *admissibility*, reconstruct bundle data from compact links.

U_ℓ compact links → admissibility → topological sectors

  - Topology from compact links: admissibility and interpolation give $Q \in \mathbb{Z}$ on a finite lattice [Lüscher '84].
  - Coupling to higher-form backgrounds: with a $\mathbb{Z}_N^{[1]}$ background, Q can become fractional, giving a lattice mixed anomaly. [Abe–OM–Suzuki, Abe–OM–Onoda–Suzuki–Tanizaki].



edge ↓

4D cell $c(n)$
3D face $f(n,\mu)$
2D plaquette $p(n,\mu,\nu)$
1D link $\ell(n,\mu)$

Excision method and bosonization

- Admissibility forbids monopoles in the bulk. (Witten effect?)
- We introduce "excision method" in 2D compact scalar theory.
  - Magnetic objects are not given in terms of link variables; they are changes of geometry.

(removing sites and links)

- Villain-type lattice formulation of 2D $U(1)$ chiral gauge theory were devised [Berkowitz–Cherman–Jacobson '23].
  - See also [DeMarco–Lake–Wen '23, Thorngren–Preskill–Fidkowski '26, Seifnashri '26].
  - 4D: $U(1)$ [Lüscher '99], $SU(2) \times U(1)$ [Kikukawa–Nakayama '00, Kadoh–Kikukawa '07]
- We propose yet another Wilson-type lattice formulation of 2D $U(1)$ chiral gauge theory.

Admissibility: topology at finite cutoff

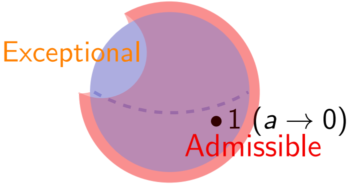
- For a compact $U(1)$ link field (lattice constant $a = 1$),

$$U_p = \prod_{\ell \in \partial p} U_\ell, \qquad F_p := \frac{1}{\mathrm{i}} \ln U_p, \qquad -\pi < F_p \leq \pi.$$

- Admissibility restricts plaquettes to a smooth region,

$$\sup_p |F_p| < \epsilon.$$

  - $v_{n,\mu}(x)$ is written in terms of $e^{\mathrm{i}yF_p}$ and more complicated factors ($y = x - n$). Note that $(-1)^y$ is ill-defined.
- Then branch jumps are excluded, and Bianchi identities become exact lattice statements.
- Other example: $SU(2)$ ($\sim$ sphere)



  - Admissible lattice gauge fields: well-defined conf space $\sim$ disk
  - Exceptional region
    - ★ Topological freezing
    - ★ Monopole as lattice artifact

Exercise: Bianchi identities in 4D $U(1)$

- For the gauge potential,

$$A_\ell := \frac{1}{\mathrm{i}} \ln U_\ell, \qquad -\pi < A_\ell \leq \pi,$$

there exists $N_p \in \mathbb{Z}$ such that

$$F_p = (\mathrm{d}A)_p + 2\pi N_p, \quad F_{n,\mu\nu} = \Delta_\mu A_{n,\nu} - \Delta_\nu A_{n,\mu} + 2\pi N_{n,\mu\nu}.$$

- Bianchi identities are given by

$$\mathrm{d}F_p = 0.$$

1. $\epsilon_{\mu\nu\rho\sigma}\Delta_\nu F_{\rho\sigma}$ has 6 F_p-terms. So $|\epsilon_{\mu\nu\rho\sigma}\Delta_\nu F_{\rho\sigma}| < 6\epsilon$.
2. On the other hand, $\mathrm{d}F_p = 2\pi \mathrm{d}N_p$.
3. Since $\epsilon_{\mu\nu\rho\sigma}\Delta_\nu N_{\rho\sigma} \in \mathbb{Z}$, the bound $|\epsilon_{\mu\nu\rho\sigma}\Delta_\nu N_{\rho\sigma}| < 1$ implies that it vanishes.

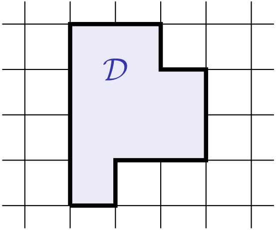
We assume $\epsilon \leq \frac{\pi}{3}$.

Magnetic objects on a finite Wilson-type lattice

- Magnetic monopole:
  - Magnetic defect operators provide nontrivial topology.
  - Quite heavy but significant in nonperturbative dynamics.
  - Maxwell equation w/ monopole current j_{m}: $\mathrm{d}\star F = j_{\mathrm{e}}$, $\mathrm{d}F = j_{\mathrm{m}}$.
- Admissibility implies the absence of bulk magnetic sources:

$$\mathrm{d}F = 0 \quad \text{on the admissible lattice.}$$

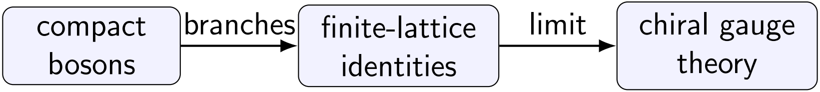
  Magnetic operator cannot be ordinary local plaquette excitation.
- Excision method [Abe–OM–Onoda–Suzuki–Tanizaki]



  - remove a small region $\mathcal{D}$;
  - impose magnetic data on $\partial\mathcal{D}$;
  - take the continuum limit with $\mathrm{size}(\mathcal{D}) = O(a/\epsilon)$.
  - represent magnetic singularity as a boundary of the lattice, $M(\partial\mathcal{D})$.

Present problem

- We now apply the same finite-lattice logic to a bosonized 2D chiral gauge theory.
- Bosonization makes the anomaly a classical variation of compact scalar fields.
- Magnetic vertex operators are unavoidable because fermion operators contain dual vertices.
- The "hole" construction gives a finite-lattice definition of these dual vertices.



- Anomaly cancellation is checked before the limit.

Continuum target theory

- For each flavor $\alpha = 1, \ldots, N_f$, introduce vector and axial charges $q_{V,\alpha}$ and $q_{A,\alpha}$:

$$\begin{aligned} D_\alpha &= \gamma_\mu\big(\partial_\mu + \mathrm{i}A_\mu(q_{V,\alpha} + q_{A,\alpha}\gamma_5)\big) \\ &= \gamma_\mu\big(\partial_\mu + \mathrm{i}A_\mu(q_{R,\alpha}P_R + q_{L,\alpha}P_L)\big), \\ q_{R,\alpha} &= q_{V,\alpha} + q_{A,\alpha}, \qquad q_{L,\alpha} = q_{V,\alpha} - q_{A,\alpha}, \\ P_{R,L} &= \frac{1 \pm \gamma_5}{2} \end{aligned}$$

- The gauge anomaly coefficient is

$$\sum_\alpha \left(q_{R,\alpha}^2 - q_{L,\alpha}^2\right) = 4 \sum_\alpha q_{V,\alpha} q_{A,\alpha}.$$

  Thus the condition is

$$\sum_\alpha q_{V,\alpha} q_{A,\alpha} = 0.$$

Bosonization dictionary used here

- A chiral fermion sector is represented by compact scalars $\phi_\alpha \sim \phi_\alpha + 2\pi$:

$$S_B = \sum_\alpha \left[\frac{R^2}{4\pi} \int_{M_2} |\mathrm{d}\phi_\alpha + 2q_{A,\alpha} A|^2 + \frac{\mathrm{i}}{2\pi} q_{V,\alpha} \int_{M_2} A \wedge (\mathrm{d}\phi_\alpha + 2q_{A,\alpha} A) \right].$$

- We assume $q_{V,\alpha} \in \mathbb{Z}$ and $2q_{A,\alpha} \in \mathbb{Z}$.
- The scalar radius R belongs to continuum matching.
- As an equivalence of quantum theories:
  - ⋆ Hilbert space / state spectrum
  - ⋆ Partition functions with background fields
  - ⋆ Correlation functions of corresponding operators
  - ⋆ Symmetry realization and anomalies
  - ⋆ Locality and operator algebra
  - ⋆ Topological sectors

What is exact, and what is continuum matching?

Finite lattice statements:

- compact variables and integer branch fields are exact;
- admissibility gives exact lattice Bianchi identities;
- anomaly cancellation gives exact gauge invariance;
- magnetic charge is an integer boundary observable.

Continuum statements:

- the bosonic theory flows to the desired chiral fermion theory;
- operator dimensions and spin structures are matched appropriately;
  - Spin-structure dependence may require additional topological factors, such as an Arf invariant in particular cases.
- the zero-mode selection rule agrees with the index theorem.

Example: the 3450 model

- A standard anomaly-free example is

$$(q_{R,1}, q_{L,1}) = (5, 3), \qquad (q_{R,2}, q_{L,2}) = (0, 4).$$

Equivalently,

$$(q_{V,1}, q_{A,1}) = (4, 1), \qquad (q_{V,2}, q_{A,2}) = (2, -2).$$

- Then

$$\sum_\alpha q_{V,\alpha} q_{A,\alpha} = 4 \cdot 1 + 2 \cdot (-2) = 0.$$

- This is the type of charge assignment for which the gauge-anomaly obstruction to exact gauge invariance vanishes.

Compact variables and branch fields

- On a square lattice Γ, use

$$\mathrm{e}^{\mathrm{i}\phi_\alpha(n)}, \qquad U(n,\mu) = \mathrm{e}^{\mathrm{i}A_\mu(n)}.$$

- The gauge-invariant covariant difference is

$$\begin{aligned} D_\mu\phi_\alpha(n) &= \frac{1}{\mathrm{i}} \ln \left[\mathrm{e}^{-\mathrm{i}\phi_\alpha(n)} U(n,\mu)^{2q_{A,\alpha}} \mathrm{e}^{\mathrm{i}\phi_\alpha(n+\hat{\mu})}\right] \\ &= \Delta_\mu\phi_\alpha(n) + 2q_{A,\alpha}A_\mu(n) + 2\pi\ell_{\alpha,\mu}(n). \end{aligned}$$

Here $\ell_{\alpha,\mu}(n) \in \mathbb{Z}$ records the branch of the logarithm.

Gauge-field strength and integer data

- The compact gauge field is written as

$$F_{\mu\nu}(n) = \frac{1}{\mathrm{i}} \ln\big[U(n,\mu)U(n+\hat{\mu},\nu) \\ \times U(n+\hat{\nu},\mu)^{-1}U(n,\nu)^{-1}\big] \\ = \Delta_\mu A_\nu(n) - \Delta_\nu A_\mu(n) + 2\pi N_{\mu\nu}(n),$$

  where $N_{\mu\nu}(n) \in \mathbb{Z}$ is branch data.
- The anomaly and magnetic charge are controlled by these integer branch fields, not only by smooth "continuum-looking" fields.

Admissibility gives exact lattice identities

- For all flavors, impose smoothness bounds such as

$$\sup_{n,\mu} |D_\mu \phi_\alpha(n)| < \varepsilon, \qquad \sup_{n,\mu,\nu} |2q_{A,\alpha} F_{\mu\nu}(n)| < \delta.$$

- The axial-vector current is given by the "magnetic" current

$$\epsilon_{\mu\nu} D_\nu \phi_\alpha(n).$$

Then, we require the gauge invariant conditions

$$\begin{aligned} &|\Delta_\mu \ell_{\alpha,\nu}(n) - \Delta_\nu \ell_{\alpha,\mu}(n) - 2q_{A,\alpha} N_{\mu\nu}(n)| \\ &= |\Delta_\mu D_\nu \phi_\alpha(n) - \Delta_\nu D_\mu \phi_\alpha(n) - 2q_{A,\alpha} F_{\mu\nu}(n)| \\ &\leq \frac{2}{\pi}\varepsilon + \frac{1}{2\pi}\delta < 1, \end{aligned}$$

where

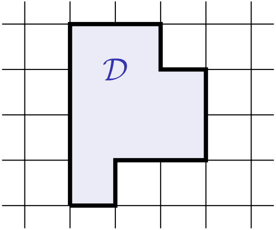
$$0 < \varepsilon < \frac{\pi}{2}, \qquad 0 < \delta < \min(\pi, 2\pi - 4\varepsilon).$$

- They also forbid a magnetic source in the smooth bulk.

Magnetic vertex operators: the obstruction

- In continuum bosonization, a dual vertex operator carries magnetic charge:
$$e^{\mathrm{i} m_\alpha \widetilde{\phi}_\alpha(x)}.$$
- The magnetic current is schematically
$$j_\alpha^{(m)} = \frac{1}{2\pi} \mathrm{d}\phi_\alpha.$$
- For a smooth field, $\mathrm{d}j_\alpha^{(m)} = 0$ identically.
- Admissibility gives a Bianchi identity, so magnetic charge cannot appear as an ordinary local excitation on a uniform lattice.

Excision method: the operator is a hole



- Remove a small region $\mathcal{D}$ from the lattice.
- The operator insertion is encoded by boundary data on $\partial\mathcal{D}$.
- The shape of $\mathcal{D}$ is irrelevant in the continuum limit if its size is $O(a)$.
- The Bianchi identity holds away from the hole.

$$M_{\{m_\alpha\}}(\mathcal{D}) \quad \leftrightarrow \quad \prod_\alpha \mathrm{e}^{\mathrm{i} m_\alpha \widetilde{\phi}_\alpha}.$$

Magnetic charge around the hole

- For a loop C on Γ, define

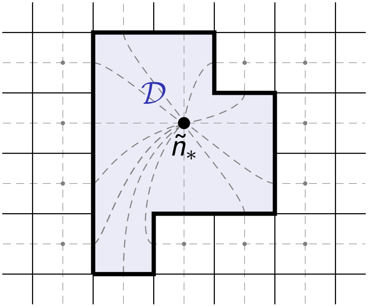
$$m_\alpha(C) = \frac{1}{2\pi} \sum_{(n,\mu)\in C} D_\mu \phi_\alpha(n) - \frac{2q_{A,\alpha}}{2\pi} F(C)$$
$$= \sum_{(n,\mu)\in C} \ell_{\alpha,\mu}(n) - 2q_{A,\alpha} N(C) \in \mathbb{Z}.$$

- For an excised region,

$$m_\alpha = m_\alpha(\partial\mathcal{D}).$$

- Gauge invariant by construction.
- Topological under deformations of C that do not cross holes.

Dual lattice near the hole



- Away from $\mathcal{D}$, identify original and dual gauge fields.

$$U(\tilde{n},\mu) = U(n,\mu),$$
$$\tilde{n} = n + \frac{1}{2}\hat{1} + \frac{1}{2}\hat{2}.$$

- Dual links crossing $\partial\mathcal{D}$ are treated as independent variables.
- The dual site $\widetilde{n}^*$ inside $\mathcal{D}$ carries an independent gauge parameter.

$$\Lambda(\widetilde{n}^*) \quad \text{is independent.}$$

Gauge charge of a magnetic object

- The magnetic object transforms as

$$M_{\{m_\alpha\}}(\mathcal{D}) \longrightarrow \exp\left[\mathrm{i}\sum_\alpha q_{V,\alpha} m_\alpha \Lambda(\widetilde{n}^*)\right] M_{\{m_\alpha\}}(\mathcal{D}).$$

- This is precisely the vector charge carried by the dual vertex operator.
- It is made gauge invariant by attaching an open dual line ending at the hole:

$$\exp\left[-\mathrm{i}\sum_\alpha q_{V,\alpha} m_\alpha \sum_{(\widetilde{n},\mu)\in\widetilde{P}}^{\widetilde{n}^*} A_\mu(\widetilde{n})\right].$$

The lattice scalar action

- The lattice action is a compact transcription of the bosonized action:

$$
\begin{aligned}
S_B = \sum_{\alpha} \sum_{n\in\Gamma} \Bigg[\frac{R^2}{4\pi} \sum_{\mu} D_\mu \phi_\alpha(n) D_\mu \phi_\alpha(n) \\
+ \frac{\mathrm{i}}{2\pi} q_{V,\alpha} \sum_{\mu,\nu} \epsilon_{\mu\nu} A_\mu(\tilde{n}) D_\nu \phi_\alpha(n+\hat{\mu}) \\
+ \frac{\mathrm{i}}{2} q_{V,\alpha} \sum_{\mu,\nu} \epsilon_{\mu\nu} N_{\mu\nu}(\tilde{n}) \phi_\alpha(n+\hat{\mu}+\hat{\nu}) \Bigg].
\end{aligned}
$$

- The last term is the lattice counterterm that gives a simple anomaly variation.

Where the anomaly appears

- Under a lattice gauge transformation, the action varies as

$$S_B \longrightarrow S_B + \mathrm{i} \left(\sum_\alpha q_{V,\alpha} q_{A,\alpha} \right) \mathcal{A}_{\mathrm{lat}}[\Lambda, U],$$

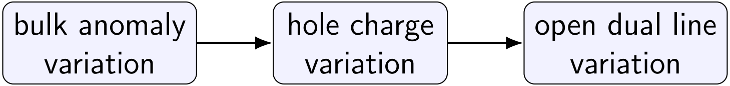
  up to integer multiples of $2\pi\mathrm{i}$.

- Important distinction:
  - For anomalous fermion contents, this variation is the regulated anomaly.
  - For anomaly-free contents, the coefficient vanishes exactly at finite lattice spacing.

  The anomaly is not postponed to the continuum limit.

Anomaly cancellation, exact gauge invariance

- If
$$\sum_{\alpha} q_{V,\alpha} q_{A,\alpha} = 0,$$
then the finite-lattice bosonic partition function is gauge invariant.
- With holes, the same cancellation leaves only the physical transformation of the magnetic object, which is canceled by the attached open dual line.



Anomalous gauge theory: what not to impose

For $\sum q_{V,\alpha} q_{A,\alpha} \neq 0$:

- A gauge-invariant local fermion measure for a dynamical gauge field is not expected.
- Trying to impose exact gauge invariance would erase the anomaly rather than reproduce it.
- The correct finite-lattice object is a gauge-covariant background-field functional with the anomalous variation.

- No-go results do not say that the anomaly cannot be represented on the lattice; they say that an anomalous theory cannot be promoted to a consistent gauge theory without additional anomaly-canceling ingredients.

Topological charge and electric selection rule

- The dual-lattice first Chern number is

$$\widetilde{Q} = \frac{1}{2\pi} \sum_{\widetilde{p} \in \widetilde{\Gamma}} F_{12}(\widetilde{p}) = \sum_{\widetilde{p} \in \widetilde{\Gamma}} N_{12}(\widetilde{p}) \in \mathbb{Z}.$$

- The electric vertex operator

$$V_{\{n_\alpha\}}(n) = \exp\left[\mathrm{i} \sum_{\alpha} n_\alpha \phi_\alpha(n)\right]$$

is selected by the constant-shift Ward identity:

$$\sum_{I} n_{I,\alpha} = q_{V,\alpha} \widetilde{Q}.$$

- This is the bosonized form of zero-mode saturation.

Magnetic selection rule

- For holes $D_{\widetilde{I}}$, define

$$Q = \frac{1}{2\pi} \sum_{p \in \Gamma - \sum_{\widetilde{I}} D_{\widetilde{I}}} F_{12}(p) + \frac{1}{2\pi} \sum_{\widetilde{I}} F(\partial D_{\widetilde{I}})$$
$$\in \mathbb{Z}.$$

- Summing the finite-lattice Bianchi identity gives

$$\sum_{\widetilde{I}} m_{\widetilde{I},\alpha} = -2 q_{A,\alpha} Q.$$

- For sufficiently small admissibility bounds, the two integers agree:

$$\widetilde{Q} = Q.$$

  - Difference is a finite number of plaquettes.

Fermion operators as electric plus magnetic vertices

- The elementary fermion fields are represented by superposing an electric vertex and a magnetic hole:

$$
\begin{aligned}
P_R\psi_\alpha : &\quad \mathrm{e}^{+\mathrm{i}\phi_\alpha/2} M_{m_\alpha=-1}(D), \\
\bar{\psi}_\alpha P_L : &\quad \mathrm{e}^{-\mathrm{i}\phi_\alpha/2} M_{m_\alpha=+1}(D), \\
P_L\psi_\alpha : &\quad \mathrm{e}^{-\mathrm{i}\phi_\alpha/2} M_{m_\alpha=-1}(D), \\
\bar{\psi}_\alpha P_R : &\quad \mathrm{e}^{+\mathrm{i}\phi_\alpha/2} M_{m_\alpha=+1}(D).
\end{aligned}
$$

- Electric part: axial charge.
- Magnetic part: vector charge.
- Dressing lines make gauge-invariant composite operators.
- These representations may be used to compute correlation functions containing fermion fields.
- Fermion number anomaly is given by

$$
\partial_\mu J_\mu^{L,R}(x) = \mp \frac{q_{L,R}}{2\pi} F_{12}(x)
$$

Dynamical gauge fields

- After constructing gauge-invariant observables, integrate over gauge fields:

$$\langle O \rangle = \frac{1}{Z} \int \prod_{\ell} \mathrm{d} U_{\ell} \prod_{\widetilde{\ell}} U_{\widetilde{\ell}} \mathrm{e}^{-S_G} \left\langle O \right\rangle_B .$$

- A convenient gauge action dynamically enforces admissibility:

$$S_G = \frac{1}{2g_0^2} \sum_{p} L_{12}(p) + \frac{1}{2g_0^2} \sum_{\widetilde{p}} L_{12}(\widetilde{p}),$$

with

$$L_{12}(p) = \begin{cases} F_{12}(p)^2 \left[1 - q_*^2 F_{12}(p)^2/\delta^2\right]^{-1}, & |q_* F_{12}(p)| < \delta, \\ \infty, & \text{otherwise.} \end{cases}$$

A Wilson-type statement

- Start from ordinary compact link variables rather than independent integer flux variables.
- Use admissibility to make branch data and topology well-defined.
- Define magnetic objects by changing the lattice geometry, not by inserting a singular plaquette.
- Demand exact gauge invariance precisely when the anomaly coefficient vanishes.
- The finite-lattice obstruction is the anomaly.
- This is the sense in which the construction supports the Wilson/overlap viewpoint.

What should be exact at finite cutoff

Should be exact:

- gauge invariance for anomaly-free fermion contents;
- anomalous variation for anomalous background-field functionals;
- integer magnetic charges and selection rules;
- locality away from excised operator insertions.

May be continuum matching:

- the precise fermion operator algebra;
- scaling dimensions and the scalar radius;
- spin-structure dependence;
- universality beyond the 2D Abelian bosonized setting.

Summary

- The central finite-lattice criterion is: gauge theory if and only if the anomaly cancellation condition is met.
- Bosonization turns the anomaly into a finite-lattice variation controlled by $\sum_{\alpha} q_{V,\alpha} q_{A,\alpha}$.
- Admissibility gives Bianchi identities, so magnetic vertex operators require excision.
- The hole carries vector charge; an open dual line provides gauge-invariant dressing.
- Modified Villain and Hamiltonian routes are complementary; the present construction keeps the Wilson/overlap question in focus.

- Recent developments
  - Magnetic objects in 4D Maxwell [Onoda '25].
  - Non-Abelian bosonization in 2D [Onoda '26].

Backup: Villain and Wilson as two organizations of compactness

Wilson-type:

$$U_\ell \in U(1), \qquad F_p = \frac{1}{\mathrm{i}} \ln U_p \quad \text{inside an admissible region.}$$

Villain-type:

$$F_p = (\delta a)_p + 2\pi n_p, \qquad n_p \in \mathbb{Z}.$$

- Wilson hides integer data until branch choices are made.
- Villain exposes integer data and therefore exposes its constraints.
- Magnetic objects are natural in both languages, but the bookkeeping differs.