Chiral Gauge Theories from Symmetry Disentanglers

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Chiral Gauge Theories

Consistency of gauge theories requires anomaly-vanishing

eg. 3+1d Weyl fermion charges must satisfy $\sum_n q_n^3 = 0$

the Standard model $SU(3) \times SU(2) \times U(1)$ satisfies this in each generation (3x15 Weyls)

does this mean the theory is consistent? how do we define it non-perturbatively?

free fermions on the lattice with on-site symmetries have fermion doubling

Weyl fermions come in charge $q, -q$ pairs, each satisfying anomaly-vanishing trivially

quark hypercharges (×3)	-4	2	1	1
lepton hypercharges	6	-3	-3	

how do we define this theory on the lattice?

Chiral Gauge Theories via symmetry disentanglers

We will relax on-siteness of symmetry => no fermion doubling theorem

define "building block" model with massless fermions & anomalous global symmetry

take decoupled stack of several building blocks, identify anomaly-free subgroup

seek "symmetry disentangler" making the anomaly-free subgroup on-site

gauge the symmetry as usual - building blocks only coupled via gauge field

In this talk: abelian symmetries only, solution in 1+1d, progress in 3+1d, open problems

Anomalies in the continuum

Eg. 2π-periodic scalar

$$\mathscr{L} = K\partial_\mu \phi \partial^\mu \phi$$

obstruction to gauging

$$\boxed{j^\mu_{\text{shift}} = \partial^\mu \phi} \longrightarrow A_{\text{shift}}$$

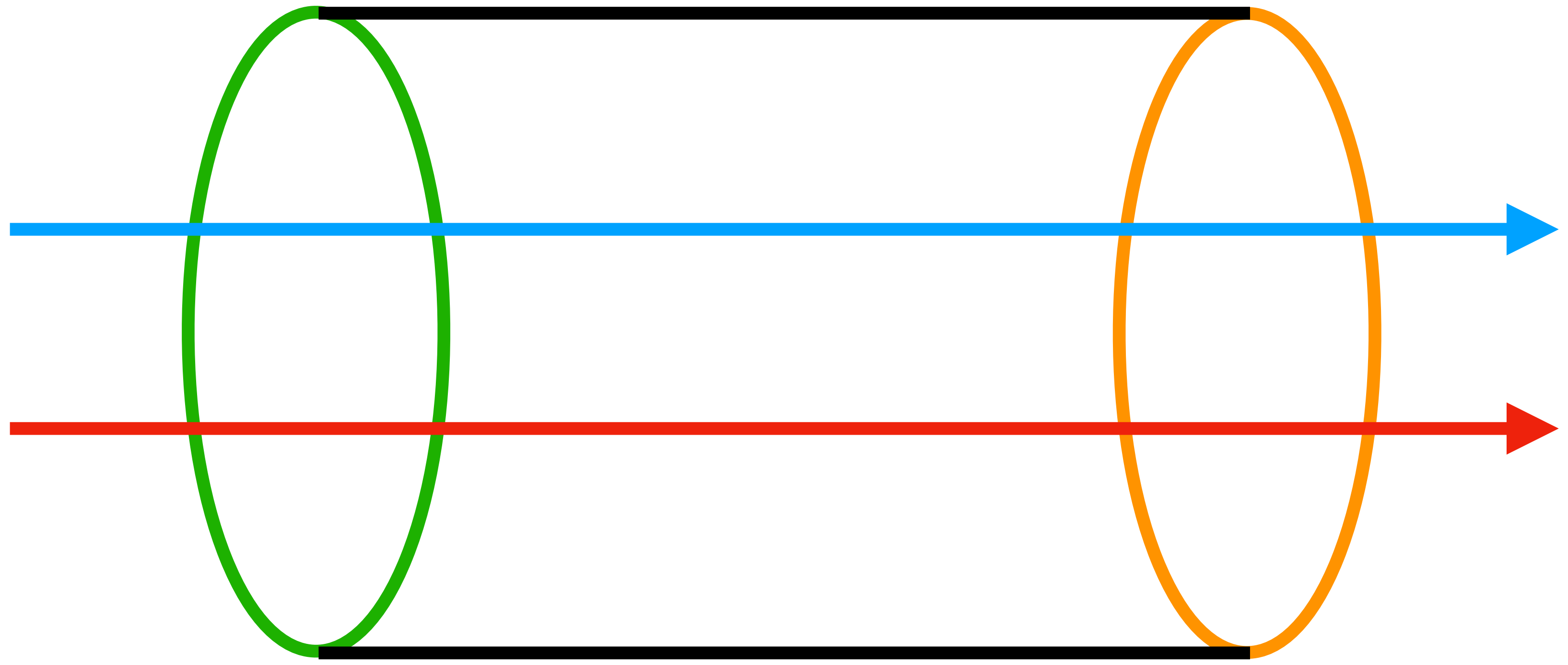
$$\boxed{j^\mu_{\text{winding}} = \epsilon^{\mu\nu}\partial_\nu \phi} \longrightarrow j^\mu_{\text{winding}} = \epsilon^{\mu\nu}\partial_\nu \phi - \epsilon^{\mu\nu} A_{\text{shift},\nu}$$

shift flux => winding charge

$$\boxed{\partial_\mu j^\mu_{\text{winding}} = -\epsilon^{\mu\nu}\partial_\mu A_{\text{shift},\nu}}$$

not conserved!

"charge pump"



A_{shift} flux

A_{winding} Hall current

2+1d bulk topological theory (SPT)

$$\frac{1}{2\pi}\int_3 A_{\text{winding}} dA_{\text{shift}}$$

well-developed theory of SPTs!

Anomalies on the lattice: gauging

introduce gauge field E, A on bonds, impose local constraint:

$$\sum_{i=1}^{d} E_{x+\hat{i}} - E_{x-\hat{i}} = \rho_x \qquad \approx \text{ Gauss law } \nabla \cdot \vec{E} = \rho$$

consistency requires:

each ρ_x defines a representation of the gauge group

for $x \neq y$, ρ_x, ρ_y commute

simplest case: on-site symmetry, ρ_x acts only at x

not-on-site symmetries may be anomalous

Anomalies on the lattice: example

consider 1d lattice of rotors $|\phi_n\rangle \in L^2(S^1)$, $n \in \mathbb{Z}$, write $\phi_n \in \mathbb{R}/\mathbb{Z}$

$$U(1)_V \qquad \phi_n \mapsto \phi_n + \alpha$$

$$U(1)_A \qquad \exp\left(2\pi i\alpha \sum_n \underbrace{(\phi_{n+1} - \phi_n) - [\phi_{n+1} - \phi_n]}_{\text{locally shift invariant! => locality preserving}}\right)$$

[x] = nearest integer to x

$$\exp\left(2\pi i \sum_{n=0}^{L} (\phi_{n+1} - \phi_n) - [\phi_{n+1} - \phi_n]\right) = e^{2\pi i\phi_{L+1}} e^{-2\pi i\phi_0}$$

A flux => V charge

nb. anomaly is a property only of the symmetry action

Anomalies on the lattice: definition

say a symmetry $U(g)$ is on-siteable if there is a locality-preserving transformation W st

$WU(g)W^\dagger$ is on-site => gaugeable by usual methods

W: symmetry disentangler

nb. $U(g) = \prod_x U_x(g)$ => on-siteable w/ ancillas

charge pump is an obstruction to on-siteability, since on-site symmetries do not pump

if symmetry is not on-siteable, we will say it is anomalous

Chiral gauge theory: a general approach

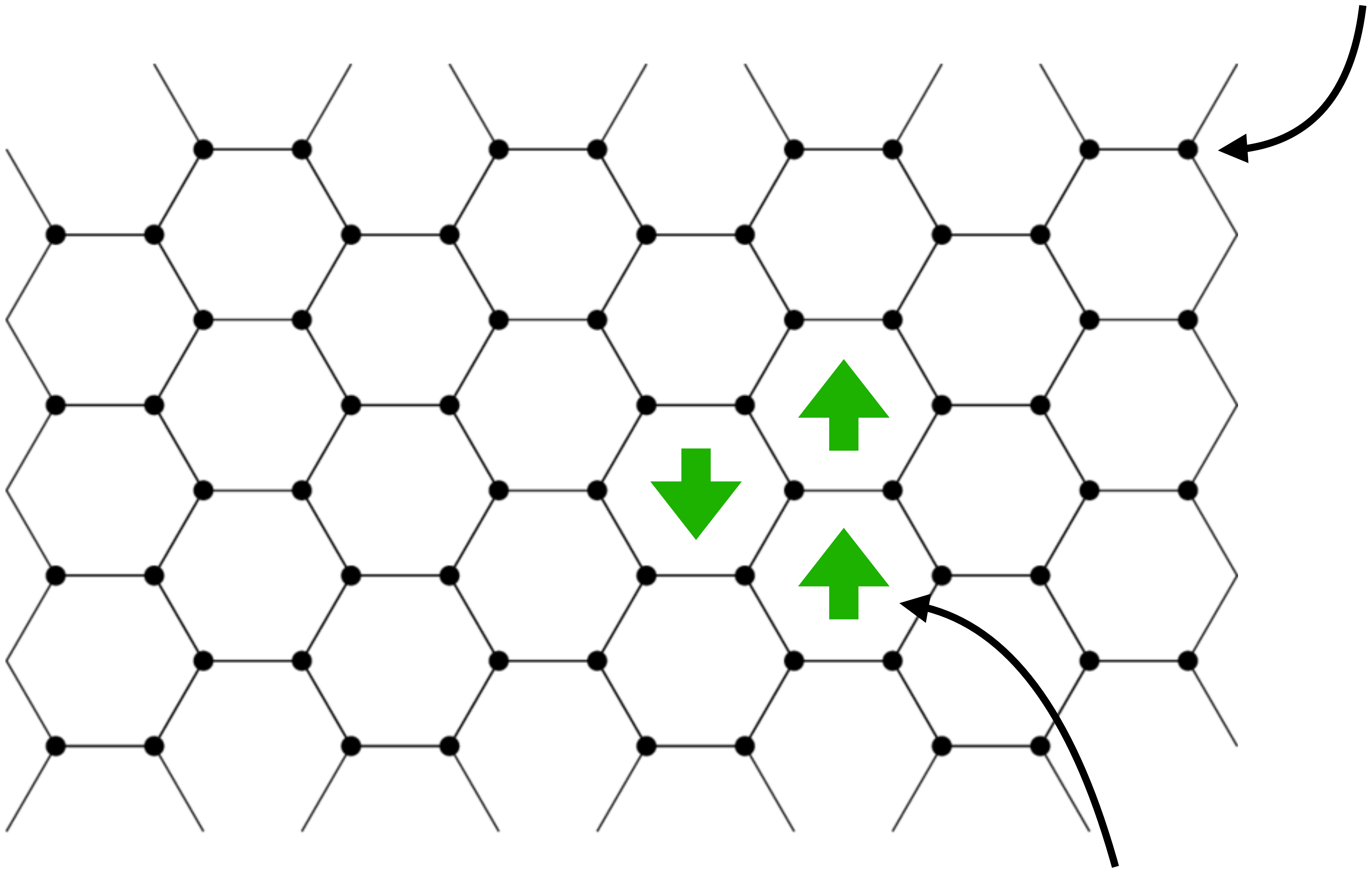
1. define a lattice model for a single chiral fermion with not-on-site symmetry G
2. take several copies, with not-on-site $G_1 \times \cdots \times G_n$ symmetry
3. identify continuum anomaly-free subgroup $G \subset G_1 \times \cdots \times G_n$
4. find symmetry disentangler making anomaly-free subgroup on-site
5. gauge as usual

Central question: does continuum anomaly-free => on-siteable?

Answer: no*

*sometimes yes

Counterexample $\mathbb{Z}_2$ symmetry in 2+1d



vertex spins

plaquette spins

step 1: flip plaquette spins

step 2: cycle vertices around down plaquettes

symmetry^2 = bdy translation

symmetric product state => no continuum anomaly
but not on-siteable/gaugeable => pure lattice anomaly!

Shirley, Zhang, Ji, Levin 25
Tu, Long, Else 25

Rotation and phase symmetries

$$U(1)_V \qquad \phi_n \mapsto \phi_n + \alpha$$

$$U(1)_A \qquad \exp\left(2\pi i\alpha \sum_n (\phi_{n+1} - \phi_n) - [\phi_{n+1} - \phi_n]\right)$$

This symmetry is a combination of on-site rotation & not-on-site phase operator

it turns out such symmetries form a well-behaved class of lattice symmetries:

a U(1) subgroup of $\prod_{\alpha=1}^{m} U(1)_{V^\alpha} \times U(1)_{A^\alpha}$ **with** $\sum_\alpha q_V^\alpha q_A^\alpha = 0$ **=> symmetry disentangler**

Disentangler for 1d rotor symmetries

$$\begin{array}{ll} U(1)_V & \underbrace{\begin{array}{c} \phi_n \mapsto \phi_n + \alpha \\ \exp\left(2\pi i\alpha \sum_n (\phi_{n+1} - \phi_n) - [\phi_{n+1} - \phi_n]\right) \end{array}}_{\text{take m copies}} \\ U(1)_A & \end{array}$$

want to construct symmetry disentangler for $Q = \sum_{\alpha=1}^{m} q_V^\alpha Q_V^\alpha + q_A^\alpha Q_A^\alpha$ st. $\sum_\alpha q_V^\alpha q_A^\alpha = 0$

following Seifnashri-Shirley, introduce ancilla rotor θ, consider $Q' = i\frac{d}{d\theta}$ and

$$W_0 |\phi^\alpha, \theta\rangle = |\phi^\alpha + q_V^\alpha \theta, \theta\rangle$$

$$W_0 e^{i\beta Q'} W_0^\dagger |\phi^\alpha, \theta\rangle = |\phi^\alpha + q_V^\alpha \beta, \theta + \beta\rangle$$

seek appropriate phase operator W_1 st. $W_0 W_1 e^{i\beta Q'} W_1^\dagger W_0^\dagger = e^{i\beta(Q+Q')}$

Disentangler for 1d rotor symmetries

$$W_0 W_1 e^{i\beta Q'} W_1^\dagger W_0^\dagger = e^{i\beta(Q+Q')}$$

$$W_1(\phi^\alpha - q_V^\alpha \theta, \theta + \beta) W_1(\phi^\alpha - q_V^\alpha \theta, \theta)^* = \exp\left(i\beta \sum_{\alpha=1}^{m} q_A^\alpha Q_A^\alpha \right)$$

solution:

$$W_1(\phi^\alpha, \theta) = \exp\left(2\pi i \int \theta \cup \left(\underbrace{\sum_\alpha q_A^\alpha [d\phi^\alpha + q_V^\alpha d\theta]}_{\text{local shift invariance requires } \sum_\alpha q_\alpha^V q_\alpha^A = 0 \text{ <= continuum anomaly-vanishing}} - d[\sum_\alpha q_A^\alpha \phi^\alpha] \right) \right)$$

Building 1+1d chiral gauge theories from "building blocks"

lattice anomaly of $U(1)_V \times U(1)_A$ matches one c=1 compact boson/Dirac fermion

$\exists$symmetric, solvable lattice Hamiltonian $H_{\text{Luttinger}}$ flowing to this theory in IR

eg. take m=2 copies, choose $Q = 4Q_V^1 - Q_A^1 + 2Q_V^2 + 2Q_A^2$: 4*(-1)+2*2=0

bosonization of 3450 fermion charges:

$$(3,5) = 4(1,1) - (1,-1)$$
$$(4,0) = 2(1,1) + 2(1,-1)$$

anomaly vanishes: apply disentangler, gauge with minimal coupling!

Cheng, Seiberg 22

Building 3+1d chiral gauge theories from "building blocks"

lattice anomaly $U(1)_V \times U(1)_A$ matching 4 LH Weyl fermions with

$U(1)_V$	$+1$	-1	0	0
$U(1)_A$	$+1$	$+1$	-1	-1

one Standard model* generation:

quark hypercharges $(\times 3)$	-4	2	1	1
lepton hypercharges	6	0	-3	-3

"sterile neutrino"

eg. take m=4x4 LH Weyls with $Q_{\text{quark}} = -4Q_V - Q_A, \;\; Q_{\text{lepton}} = 3Q_V + 3Q_A$

anomaly vanishes: apply disentangler, gauge with minimal coupling!

TODO: 1. lattice chiral symmetry ✓ 2. symmetry disentangler ✓ 3. Hamiltonian (?)

3+1d chiral symmetry on the lattice: bosons

want to stick with general form: $U(1)_V$ on-site rotation, $U(1)_A$ not-on-site phase

$$U(1)_A : \qquad \exp\left(2\pi i\alpha \underbrace{\int_3 (d\phi - [d\phi]) \cup d[d\phi]}_{\text{self-linking of vortex loop}}\right)$$

local shift invariance ✓

1-periodicity ✓

$$\exp\left(2\pi i \int_{M_3} (d\phi - [d\phi]) \cup d[d\phi]\right) = \underbrace{\exp\left(2\pi i \int_{\partial M_3} \phi \cup d[d\phi]\right)}_{\text{2+1d } U(1)_V \text{ SPT wavefunction}}$$

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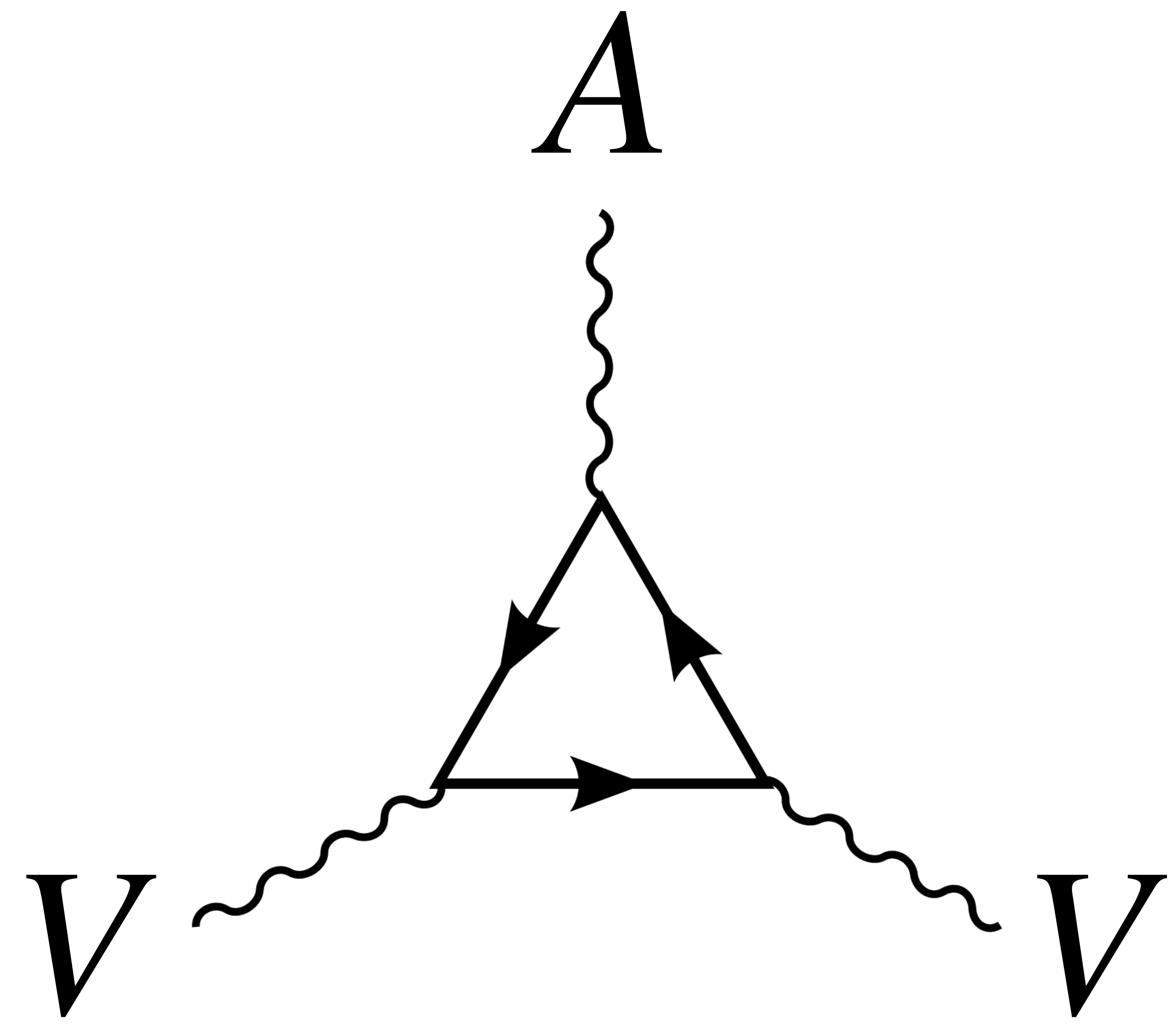
Corresponding continuum anomaly

$$e^{iS_{\text{anom}}} = \exp\left(i \int_5 A_A \wedge \underbrace{\left(\frac{dA_V}{2\pi}\right)}_{\text{2+1d } U(1)_V \text{ SPT response}}{}^2 \right)$$

2+1d $U(1)_V$ SPT response



would be desirable to have a fermionic version...

four LH Weyls

$U(1)_V$	$+1$	-1	0	0
$U(1)_A$	$+1$	$+1$	-1	-1

$$e^{i\pi Q_A} = (-1)^F$$

$$e^{iS_{\text{anom}}} = \exp\left(i \int_5 A_A \wedge \left(\frac{dA_V}{2\pi}\right)^2 \right)$$

A_A: Spin^c structure

cf. Shu-Heng's talk

3+1d chiral symmetry on the lattice: fermions

A loop in a directed graph $G = (V, E)$ is Kastelyn oriented if it encounters an odd number of anti-oriented edges (picture)

Given a dimer cover $p \subset E$, define $S_p = \prod_{\langle xy \rangle \in p} i\gamma_x\gamma_y$

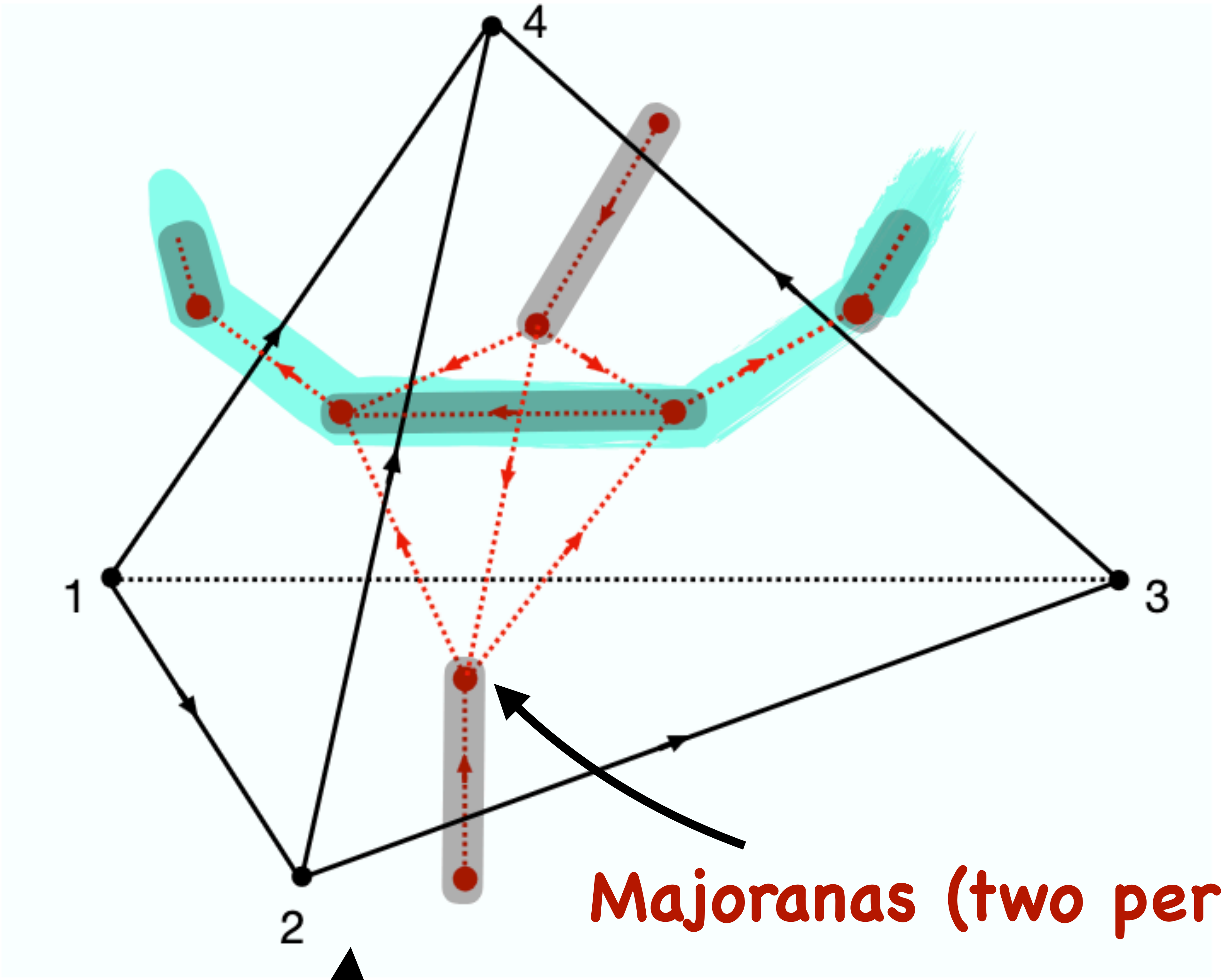
Given two dimer covers p_1, p_2, $p_1 \oplus p_2$ is a collection of closed loops, and $S_{p_1}S_{p_2} = (-1)^n$

where $n =$ number of non-Kastelyn oriented components of $p_1 \oplus p_2$

Kastelyn formula: if A is the signed adjacency matrix and G planar, #dimer covers = |Pf(A)|

Kastelyn 61
Cimsaoni, Reshetikhin 07
Tata 20
...

3+1d chiral symmetry on the lattice: fermions



Majoranas (two per face)

vertex rotors

For "resolved triangulation" ∃special edge orientation st

each loop is Kastelyn oriented iff it has trivial self-linking

let p_ϕ = dimer cover defined by vortex config

$$K = \frac{1}{2}(1 - \sum_{\langle xy \rangle \in p_\phi} i\gamma_x \gamma_y)$$

$$e^{i\pi K} = \prod_{\langle xy \rangle \in p_\phi} i\gamma_x \gamma_y = (-1)^F (-1)^{\int_3 [d\phi] d[d\phi]}$$

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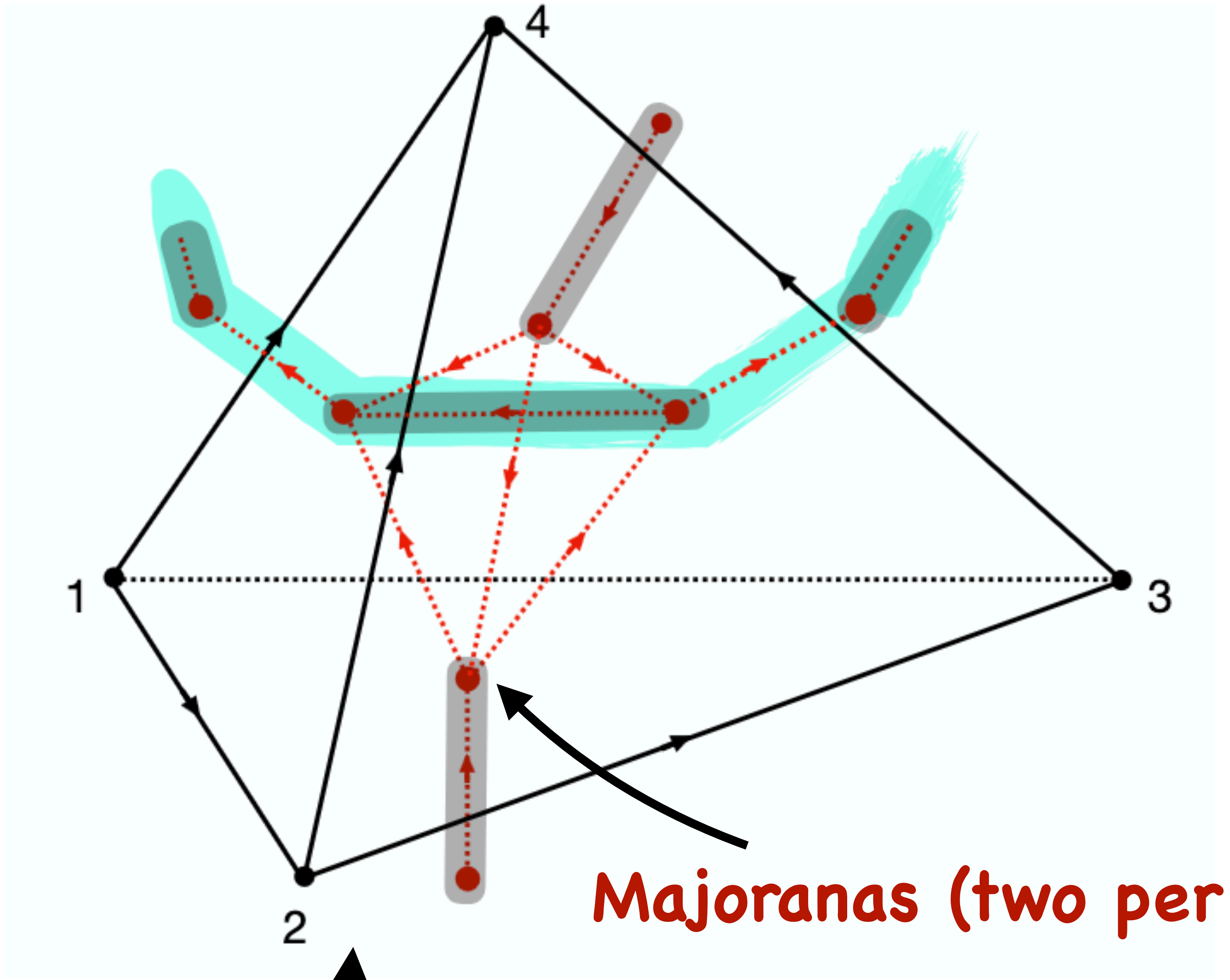
3+1d chiral symmetry on the lattice: fermions



Majoranas (two per face)

vertex rotors

$$e^{i\pi K} = \prod_{\langle xy\rangle \in p_\phi} i\gamma_x\gamma_y = (-1)^F(-1)^{\int_3 [d\phi]d[d\phi]}$$

$$Q_A = K + \int_3 [d\phi]d[d\phi]$$

$$e^{i\pi Q_A} = (-1)^F$$

2π axial rotation pumps $U(1)_V$ SPT
=> chiral anomaly

symmetry disentanglers may be constructed as in 1d!

Summary so far

Constructed anomalous 3+1d symmetry matching four LH Weyls

Possible building blocks for Standard model + sterile neutrino

Symmetry disentanglers exist for anomaly-free combinations, eg. SM hypercharges

Just need a symmetric Hamiltonian with the four LH Weyls in the IR…

Sketch of Hamiltonian Construction

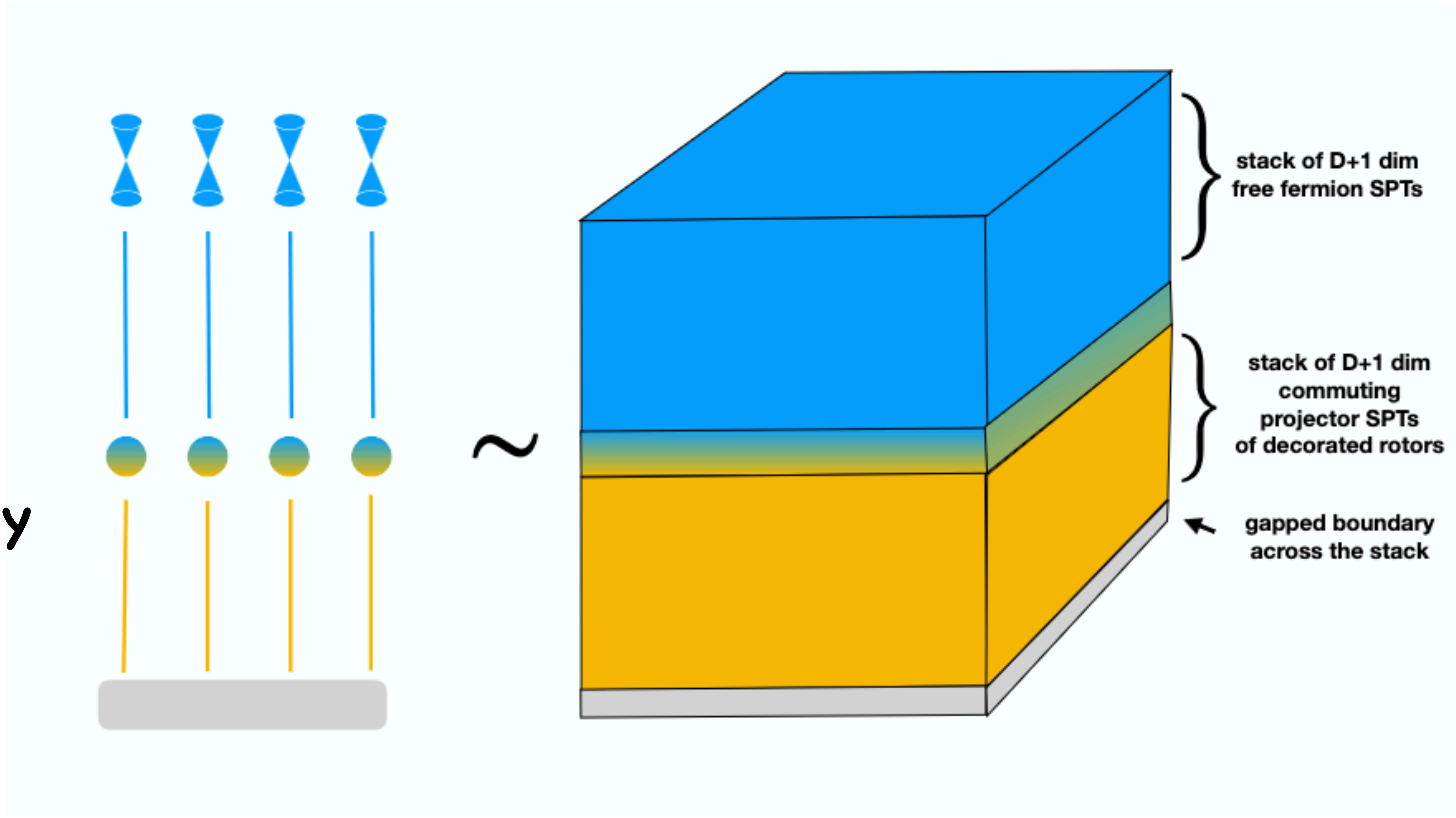
Symmetry disentangler gives commuting projector model of 4+1d $U(1)_V \times U(1)_A$ SPT

Want to define an interface
with a 4+1d top. insulator

Get free Weyls on one side

Other side has a trivial boundary
also from disentangler

Solvable SMG!



Sketch of Hamiltonian Construction

Can define a $U(1)_V$ breaking boundary Hamiltonian $H = \sum_{\langle xy \rangle} \cos(\phi_i - \phi_j) + P(\phi)$

$$Q_A = K + \int_3 [d\phi] d[d\phi]$$

Axial symmetry => $U(1)_V$ vortex carries axial charge = self-linking, cannot proliferate trivially

Couple to four Weyls* via
$U(1)_V$ charged mass

$U(1)_V$	$+1$	-1	0	0
$U(1)_A$	-1	-1	$+1$	$+1$

Index theorem calculation shows fermions contribute opposite axial charge to vortex!

In principle, can proliferate vortices into a trivial interface!

Summary

In general lattice and continuum anomalies do not match

In a setting of on-site rotations x not-on-site phases they seem to in 1d and 3d

Maybe enough for Standard model hypercharge, just lacking a Hamiltonian

Infinite dimensional local spaces get around many no-go theorems (ask me more later)

Thank you!