Violation of Parity and Flavor Symmetries in a Nambu–Jona-Lasinio Model

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Outline of the Talk

1. Motivation and Model: Aoki phase and a Nambu–Jona-Lasinio type lattice Hamiltonian.
2. Symmetries and order parameters of the model.
3. Main Theorem: Simultaneous Spontaneous Breaking of Parity and Flavor Symmetry.
4. Strategy of Proof: Reflection Positivity $\Rightarrow$ Infrared Bound $\Rightarrow$ Long-Range Order.

Aoki Phase and Nambu–Jona-Lasinio Model

Physical Background and Motivation

In lattice QCD with Wilson fermions, one expects that flavor symmetry and parity are spontaneously broken in a region. This is called the Aoki phase.

Nambu–Jona-Lasinio (NJL) effective Lagrangian is given by

$$\mathcal{L} = \bar{\Psi}\gamma_\mu\partial_\mu\Psi + \tilde{g}\left[(\bar{\Psi}\Psi)^2 + (\bar{\Psi}i\gamma_5\Psi)^2 + \sum_{j=1}^{3}\{(\bar{\Psi}\tau_j\Psi)^2 + (\bar{\Psi}i\gamma_5\tau_j\Psi)^2\}\right]$$

- $\Psi = {}^t(\Psi_u, \Psi_d)$: quark field, τ_j: Pauli matrices on the flavor space.
- NJL Lagrangian is invariant under the chiral transformation $\Psi \to e^{i\theta\gamma_5}\Psi$ and SU(2) flavor rotation $\Psi \to e^{\frac{i}{2}\theta_j\tau_j}\Psi$.

Model I: Lattice NJL Model

Consider $\Lambda = [-L+1, L]^3 \subset \mathbb{Z}^3$ with periodic boundary conditions.
The Dirac field is $\Psi_f(x) = {}^t(\psi_f^{(1)}, \psi_f^{(2)}, \psi_f^{(3)}, \psi_f^{(4)})$ obeying the anti-commutation relations $\{\psi_f^{(i)}(x), \psi_{f'}^{(j)}(y)^\dagger\} = \delta_{xy}\delta_{ij}\delta_{ff'}$ etc. with $f = u, d$.
For $\Psi = {}^t(\Psi_u, \Psi_d)$, our Hamiltonian is

$$H^{(\Lambda)} = H_K^{(\Lambda)} + H_{\mathrm{int}}^{(\Lambda)}, \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \alpha_i \Psi(x) := \begin{pmatrix} \alpha_i \Psi_u(x) \\ \alpha_i \Psi_d(x) \end{pmatrix}$$

$$H_K^{(\Lambda)} = i\kappa \sum_{x\in\Lambda} \sum_{j=1}^{3} \Big[\Psi^\dagger(x)\alpha_j \Psi(x+e_j) - \Psi^\dagger(x+e_j)\alpha_j\Psi(x) \Big].$$

- We adopt the Weyl type fermions.
- $\kappa \in \mathbb{R}$ is the hopping parameter.
- $H_{\mathrm{int}}^{(\Lambda)}$ is a four-fermion interaction as follows.

Model II: Four-Fermion Interaction

Define

$$\Gamma^{(1)}(x) = \Psi^\dagger(x)\gamma_0\Psi(x), \qquad \Gamma^{(2)}(x) = \Psi^\dagger(x)i\gamma_0\gamma_5\Psi(x),$$
$$S^{(j)}(x) = \Psi^\dagger(x)\gamma_0\tau_j\Psi(x), \qquad S_5^{(j)}(x) = \Psi^\dagger(x)i\gamma_0\gamma_5\tau_j\Psi(x) \quad (j = 1, 2, 3).$$

Then the four-fermion interaction is

$$\begin{aligned} H_{\text{int}}^{(\Lambda)} = & - g\sum_{x\in\Lambda}\sum_{\mu=1}^{3}\left[\Gamma^{(1)}(x)\Gamma^{(1)}(x+e_\mu) + \Gamma^{(2)}(x)\Gamma^{(2)}(x+e_\mu)\right] \\ & - g\sum_{x\in\Lambda}\sum_{\mu=1}^{3}\sum_{j=1}^{3}\left[S^{(j)}(x)S^{(j)}(x+e_\mu) + S_5^{(j)}(x)S_5^{(j)}(x+e_\mu)\right], \qquad g > 0. \end{aligned}$$

- Here $i\gamma_0\gamma_5\tau_j := i\gamma_0\gamma_5 \otimes \tau_j$ on $\mathbb{C}^4 \otimes \mathbb{C}^2$: τ_j act only on the flavor indices.
- Recall $\mathcal{L} = \bar{\Psi}\gamma_\mu\partial_\mu\Psi + \tilde{g}[(\bar{\Psi}\Psi)^2 + (\bar{\Psi}i\gamma_5\Psi)^2 + \sum_{j=1}^{3}\{(\bar{\Psi}\tau_j\Psi)^2 + (\bar{\Psi}i\gamma_5\tau_j\Psi)^2\}]$

Properties of Lattice NJL Hamiltonian

The model is invariant under the

- two chiral transformation $U_{\pm}$: $\Psi_u(x) \to e^{i\theta\gamma_5}\Psi_u(x)$, $\Psi_d(x) \to e^{\pm i\theta\gamma_5}\Psi_d(x)$
- SU(2) flavor rotation $\Psi \to \exp(\frac{i}{2}\theta_j\tau_j)\Psi$
- parity $\Psi(x) \to \gamma_0\Psi(-x)$, $\Psi^\dagger(x) \to \Psi^\dagger(-x)\gamma_0$.

We introduce the two order parameters:

$$O_\Lambda^{(1)} = \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\Psi^\dagger(x)\gamma_0\tau_1\Psi(x), \quad O_{5,\Lambda}^{(2)} = \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\Psi^\dagger(x)i\gamma_0\gamma_5\tau_2\Psi(x).$$

By symmetries, $\Psi^\dagger\gamma_0\tau_j\Psi \leftrightarrow \Psi^\dagger\gamma_0 i\gamma_5\tau_j\Psi \leftrightarrow \Psi^\dagger\gamma_0\Psi$.

Remark: Introducing a symmetry-breaking field does not work for proving the parity and flavor symmetry breaking.

Horsch and von der Linden State

Let $\omega_0^{(\Lambda)}(\cdots)$ be the ground state without symmetry-breaking field. By symmetries,

$$\omega_0^{(\Lambda)}(O_\Lambda^{(1)}) = \omega_0^{(\Lambda)}(O_{5,\Lambda}^{(2)}) = 0, \qquad \omega_0^{(\Lambda)}([O_\Lambda^{(1)}]^2) = \omega_0^{(\Lambda)}([O_{5,\Lambda}^{(2)}]^2).$$

Consider a Horsch–von der Linden state

$$\omega^{(\Lambda)}(A) := \mathcal{N}\omega_0^{(\Lambda)}((1+O_\Lambda)A(1+O_\Lambda)), \quad O_\Lambda := O_\Lambda^{(1)} + O_{5,\Lambda}^{(2)}.$$

In the infinite-volume limit $|\Lambda| \to \infty$, this $\omega^{(\Lambda)}$ gives a ground state since

$$\omega^{(\Lambda)}(H^{(\Lambda)} - E_{\mathrm{GS}}^{(\Lambda)}) \le \mathrm{const.}|\Lambda|^{-1}.$$

By flavor and chiral symmetries, we see

$$\omega^{(\Lambda)}(O_\Lambda^{(1)}) = \omega^{(\Lambda)}(O_{5,\Lambda}^{(2)}) = 2\mathcal{N}\omega_0^{(\Lambda)}([O_\Lambda^{(1)}]^2) = 2\mathcal{N}\omega_0^{(\Lambda)}([O_{5,\Lambda}^{(2)}]^2)$$

Namely, LRO $m_{\mathrm{LRO}}^{(\Lambda)} := \omega_0^{(\Lambda)}([O_\Lambda^{(1)}]^2) \ge c_0 > 0$ implies SSB.

Main Result: Violation of Parity and Flavor Symmetries

Theorem (Simultaneously Spontaneous Breaking at Strong Coupling)

$\exists \alpha_0 > 0$: if $\frac{|\kappa|}{g} \leq \alpha_0$, then we have LRO $m_{\mathrm{LRO}}^{(\Lambda)} \geq c_0 > 0$;

$$\lim_{\Lambda \nearrow \mathbb{Z}^3} \omega^{(\Lambda)}(O_\Lambda^{(1)}) > 0, \qquad \lim_{\Lambda \nearrow \mathbb{Z}^3} \omega^{(\Lambda)}(O_{5,\Lambda}^{(2)}) > 0.$$

i.e. SSB in the infinite-volume G.S. $\omega(\cdots) := \text{weak}^*\text{-}\lim_{\Lambda \nearrow \mathbb{Z}^3} \omega^{(\Lambda)}(\cdots)$.

- $O_\Lambda^{(1)}$: flavor symmetry breaking.
- $O_{5,\Lambda}^{(2)}$: parity (& flavor) breaking.
- Chiral symmetry is also broken.

- We use the Weyl type fermions.
- fermion doubling occurs.
- Originally, the Wilson fermion is used for Aoki phase.

We may not have the desired continuum limit due to fermion doubling.

Remark of Result: Why Two Observables?

- The pseudoscalar operator $o_j^{(2)} := \Psi^\dagger \gamma_0 i \gamma_5 \tau_j \Psi$ detects parity breaking; under the parity transformation $P : o_j^{(2)} \to -o_j^{(2)}$. Thus

  $$\omega(P^{-1}\Psi^\dagger \gamma_0 i \gamma_5 \tau_2 \Psi P) = -\omega(\Psi^\dagger \gamma_0 i \gamma_5 \tau_2 \Psi) \neq \omega(\Psi^\dagger \gamma_0 i \gamma_5 \tau_2 \Psi),$$

  parity is broken.
- Also, under the chiral transformation $U : \Psi^\dagger \gamma_0 i \gamma_5 \tau_j \Psi \to \Psi^\dagger \gamma_0 \tau_j \Psi =: o_j^{(1)}$. Note $P : o_j^{(1)} \to o_j^{(1)}$, but the chiral U rotates $(o_1^{(1)}, o_2^{(2)}) \to (o_1^{(2)}, o_2^{(1)})$.
- Since $o_1^{(1)}, o_2^{(2)}$ have nonzero expectation simultaneously, the pseudoscalar condensate is not removed by a chiral rotation.
- Hence the use of two observables is crucial.

Strategy of Proof: Reflection Positivity and Gaussian Domination

General approach to proving a phase transition

Reflection Positivity → Gaussian Domination → Infrared Bound→ LRO

- $H^{(\Lambda)}$ has reflection positivity (RP), thus the usual procedure does work.
- Let $H^{(\Lambda)}(h) := H_K^{(\Lambda)} + H_{\text{int}}^{(\Lambda)}(h)$, where $H_{\text{int}}^{(\Lambda)}(h)$ is defined by replacing

$$-g\sum_{x,\mu}\Gamma^{(1)}(x)\Gamma^{(1)}(x+e_\mu) \to \frac{g}{2}\sum_{x,\mu}[\Gamma^{(1)}(x) - \Gamma^{(1)}(x+e_\mu) + h_\mu(x)]^2 + C$$

  with functions h_μ.

  Then RP implies the gaussian domination (GD) bound :

$$\operatorname{Tr}\exp[-\beta H^{(\Lambda)}(h)] \leq \operatorname{Tr}\exp[-\beta H^{(\Lambda)}(0)].$$

- Adding ‘field’ $h = (h_1, h_2, h_3)$ rises the energy.
- For the infrared bound (IB), we choose $h_\mu(x) = |\Lambda|^{-1/2}\left[e^{ip(x+e_\mu)} - e^{ipx}\right]$.

Strategy of Proof: Gaussian Domination to Infrared Bound

Consider the Fourier transform $\widehat{\Gamma}_p^{(1)}$ and spin-wave energy E_p:

$$\widehat{\Gamma}_p^{(1)} = |\Lambda|^{-1/2} \sum_{x \in \Lambda} \Gamma^{(1)}(x) e^{ip \cdot x}, \quad E_p = \sum_{\mu=1}^{3} (1 - \cos p^{(\mu)}).$$

Infrared Bound

$$\left(\widehat{\Gamma}_{-p}^{(1)}, \widehat{\Gamma}_p^{(1)}\right)_\beta^{(\Lambda)} \leq \frac{1}{4\beta g E_p} \quad (p \neq 0).$$

- $(A, B)_\beta^{(\Lambda)}$ denotes the Duhamel two-point function (Kubo inner product)

$$(A, B)_\beta^{(\Lambda)} := Z^{-1} \int_0^1 ds\, \mathrm{Tr} \left(e^{-s\beta H^{(\Lambda)}} A e^{-(1-s)\beta H^{(\Lambda)}} B \right)$$

- The bound controls fluctuations away from $p = 0$.

Lower Bound on Long-Range Order

Define the long-range order parameter

$$\Sigma_{\mathrm{LRO}}^{(\Lambda)} := |\Lambda|^{-1} \langle \widehat{\Gamma}_0^{(1)} \widehat{\Gamma}_0^{(1)} \rangle_\beta^{(\Lambda)} = |\Lambda|^{-2} \sum_{x,y\in\Lambda} \langle \Gamma^{(1)}(x) \Gamma^{(1)}(y) \rangle_\beta^{(\Lambda)}.$$

For $\mathcal{E}^{(\Lambda)} := |\Lambda|^{-1} \sum_{x,\mu} \langle \Gamma^{(1)}(x) \Gamma^{(1)}(x+e_\mu) \rangle_\beta^{(\Lambda)}$, we obtain

$$\Sigma_{\mathrm{LRO}}^{(\Lambda)} \geq \frac{1}{3} \sqrt{\mathcal{E}^{(\Lambda)}} \left(\sqrt{\mathcal{E}^{(\Lambda)}} - \sqrt{6}\, I_3^{(\Lambda)} \right) - O(1/\beta) - O(\sqrt{|\kappa|/g}),$$

$$I_3^{(\Lambda)} := \sqrt{\frac{1}{3|\Lambda|} \sum_{p\neq 0} \frac{1}{E_p} \left(\sum_{\mu=1}^{3} \cos p^{(\mu)} \right)_+^2} \xrightarrow{|\Lambda|\to\infty} I_3 = 0.68\cdots$$

Since $\mathcal{E}^{(\Lambda)} \geq 6 - 3|\kappa|/g - \log 2/\beta g$, LRO exists in $|\kappa|/g \ll 1$.

Taking the zero-temperature limit $\beta \to \infty$, LRO in G.S. $\omega_0^{(\Lambda)}$ follows.

Summary

- We consider a lattice NJL model without a Wilson term.
- In a strong coupling regime, parity, flavor, and chiral symmetries are simultaneously broken.
- To construct a infinite-volume ground state, we need a Horsch and von der Linden type state, due to the symmetries.
- Fermion Doubling is not removed.
- Physical differences from the standard Aoki phase, especially doublers and the absence of a chiral anomaly.

THANK YOU FOR COMING TO MY TALK!