

Taste-splitting mass and bulk-boundary correspondence in 3+1D staggered fermions

Tatsuya Yamaoka (U. Osaka)

Based on [arXiv:2509.04906](https://arxiv.org/abs/2509.04906) w/ Tetsuya Onogi (U. Osaka)

[arXiv:2604.02078](https://arxiv.org/abs/2604.02078) w/ Tatsuhiro Misumi (Kindai U) and Tetsuya Onogi (U. Osaka)



1. Introduction

UV Lattice and IR QFT

**UV
Lattice model**



RG

**IR
The QFT
(continuum)**

A regularization in the ultraviolet (UV) which concretely defines the theory at a fundamental level.

It only emerges as the low energy limit in the infrared (IR).

E.g. 1+1 D staggered fermion system

[Kogut, Susskind (1975)]

**UV
Lattice model**



RG

**IR
The QFT
(continuum)**

$$\begin{aligned} H_{\text{KS}} &= i \sum_{j=1}^{2N} \left(c_j^\dagger c_{j+1} + c_j c_{j+1}^\dagger \right) \\ &= \sum_{j=1}^N \psi_j^\dagger \gamma^0 \left[\gamma^1 \frac{1}{2} (\nabla + \nabla^\dagger) - \frac{1}{2} \nabla^\dagger \nabla \right] \psi_j \end{aligned}$$

where $\psi_j = (c_{2j} \ c_{2j+1})^T$

$$H = i \int dx [\psi_L^\dagger \partial_x \psi_L - \psi_R^\dagger \partial_x \psi_R] ,$$

$$G_{\text{classical}}^{(1+1D)} = U(1)_V \times \underline{U(1)_A}$$

E.g. 1+1 D staggered fermion system

[Kogut, Susskind (1975)]

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Lattice model

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RG

The chiral symmetry does not exist on the lattice.

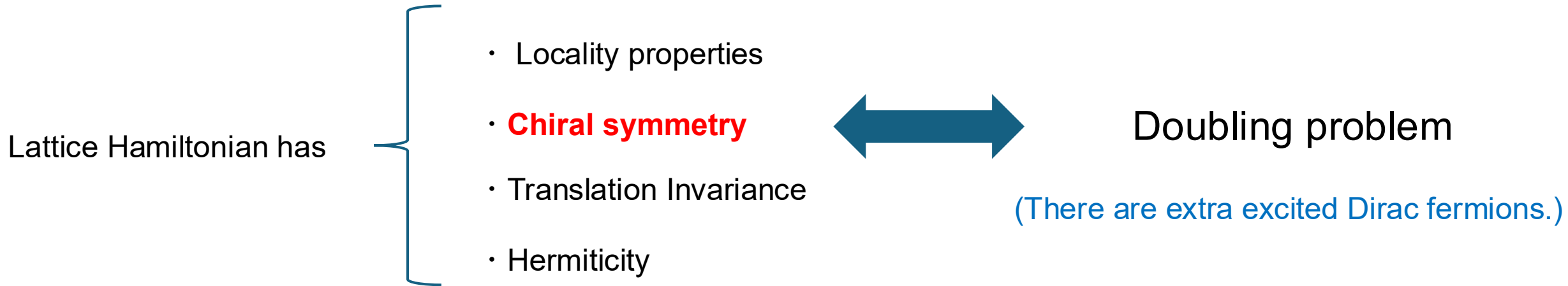
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Nielsen-Ninomiya's theorem

[Nielsen, Ninomiya (1981)] [D. Friedan (1982)] etc.
[See also all talks in the week]



It is highly non-trivial to realize the chiral (flavor) symmetries on the lattice.

Recent advances on Lattice chiral theories

There are a lot of interesting advances, as discussed in this workshop.

- Domain-wall / Overlap fermion systems Kikukawa's, Sen's, Singh's and Yasunaga's talks
- Modified Villain systems Aoki's, Shao's, Morikawa's and Furukawa's talks
- Disentangled Thorngren's talk
- Lattice anomalies, topology Kobayashi's, Fukaya's, Thorngren's and Numasawa's talks
- Tumbling hypothesis Murayama's talk
- Symmetric Mass Generation Shamir's, Xu's, Araki's, Zakharov's and Ueda's talks
- Quantum simulations Yamamoto's and Singh's talks
- RG flows/ Phase structure Gutierrez's and Goto's talks
- Graph theory Ohta's talk
- Kahler-Dirac fermion systems Vaibhav's talk
- Onsager charges Shao's, Gioia's and my talks

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- **Onsager charges** Shao's, Gioia's and my talks

Recent remarkable development

[Chatterjee, D. Pace, Shao (2024)] realized the chiral symmetry and its 't Hooft anomaly in a [1+1 D staggered fermion system](#).

[see also Shao's and Gioia's talk]

see a quick review in the next slides (2 pages).

Conserved charges in 1+1 D Staggered fermion

[Chatterjee, Pace, Shao (2024)]

• The Hamiltonian has a manifest conserved charge : $Q_V = \sum_{j=1}^{2N} \left(c_j^\dagger c_j - \frac{1}{2} \right)$

• It is quantized, i.e. $Q_V \in \mathbb{Z}$

• It generate a vector U(1) symmetry

$$c_j = \frac{1}{2}(a_j + ib_j) \quad \{a_j, a_{j'}\} = \{b_j, b_{j'}\} = 2\delta_{j,j'}$$
$$\{c_j, c_{j'}^\dagger\} = \delta_{j,j'} \quad \{a_j, b_{j'}\} = 0$$

$$T_b a_j (T_b)^{-1} = a_j, \quad T_b b_j (T_b)^{-1} = b_{j+1}$$

• There is an extra conserved charge : $Q_A = T_b Q_V (T_b)^{-1} = \frac{1}{2} \sum_{j=1}^{2N} (c_j + c_j^\dagger) (c_{j+1} - c_{j+1}^\dagger) = \frac{i}{2} \sum_{j=1}^{2N} a_j b_{j+1}$

• It is **quantized** since Q_V and Q_A are **unitary equivalent**. $Q_A \in \mathbb{Z}$

• It generates **axial U(1) symmetry** in the continuum limit.

Important properties of the conserved charges

- 1) Q_V, Q_A are part of the **Onsager algebra**. → We call the charges **Onsager charges** in this talk.
- 2) If we impose both symmetries, possible local terms δH that we can add to the Hamiltonian are strongly restricted.
 - a) Only bilinear terms are allowed.
 - b) Moreover, mass terms are prohibited.

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 - a) Only bilinear terms are allowed.
 - b) Moreover, mass terms are prohibited.

→ The presence of conserved Q_V, Q_A forces the system to be gapless. → **'t Hooft anomaly**

→ **Onsager charges are Lattice avatars of the symmetries and their anomalies in the IR continuum theory.**
= These symmetry and anomaly in IR emanate from the underlying exact lattice symmetries and anomalies.

→ **Strong constraints on the vacuum structure**

A hint to consider possible symmetric mass generation [see also Shamir's, Xu's, Araki's, Zakharov's and Ueda's talks]

Applications of the conserved Onsager charges

- 2+1D Honeycomb lattice system (describing 2-flavor Dirac fermion system in IR)
 - Onsager charges Q_n generate the flavor (valley) SU(2) that had been regarded as emergent, and its parity anomaly (with spacetime reflection) appears as the LSM anomaly of Q_n on the lattice.
 - ➔ This SU(2) and its parity anomaly are **emanant**, not emergent.
= These emanate from the underlying exact lattice symmetries and anomalies.

[Pace, Kim, Chatterjee, Shao (2025)]

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- 3+1D lattice model with the minimal fermion content (a single Weyl doublet) [Pace, Kim, Chatterjee, Shao (2025)]
- Lattice model having Fermi surface [Gioia, Thorngren (2025)]
[see also Gioia's talk]

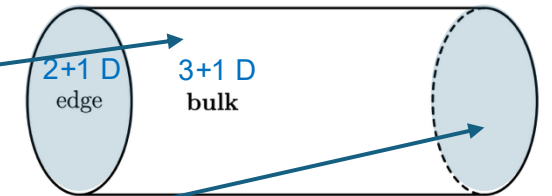
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The IR QFT information can be extracted from the Onsager charges on the lattice.

Applications of the conserved Onsager charges

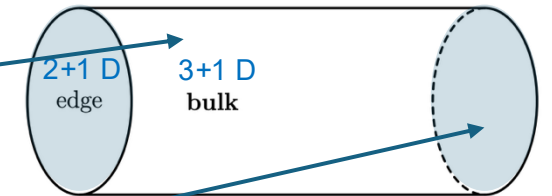
- What about in **3+1 D staggered fermion system**? [Onogi, TY (2025)]
- What about **on 2+1 D boundary** of the 3+1 D staggered fermion system? [Misumi, Onogi, TY (2026)]



Applications of the conserved Onsager charges

Today's Topic : Our work

- What about in **3+1 D staggered fermion system**? [Onogi, TY (2025)]
- What about **on 2+1 D boundary** of the 3+1 D staggered fermion system? [Misumi, Onogi, TY (2026)]



Our work 1

In 3+1 D staggered fermion Hamiltonian system,

- We find **new conserved charges** generating Onsager algebras and clarify their physical meaning — i.e. what symmetry they generate.

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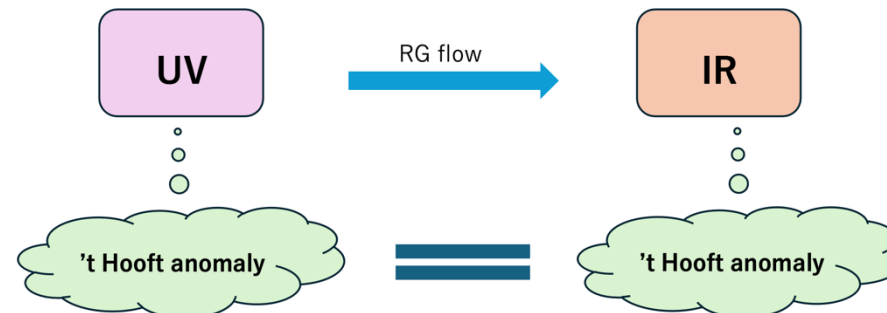
Motivation : • **useful diagnostic for identifying 't Hooft anomalies.**

The absence of symmetric mass **often** signals the presence of a corresponding 't Hooft anomaly.

Anomaly : Obstruction to a symmetric, trivially-gapped phase

[See also Kobayashi's and Thorngren's talk]

∴ **'t Hooft anomaly matching condition**



Our work 1

In 3+1 D staggered fermion Hamiltonian system,

- We find **new conserved charges** generating Onsager algebras and clarify their physical meaning — i.e. what symmetry they generate.
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Anomaly : Obstruction to a symmetric, trivially-gapped phase

- To perform numerical studies of theories such as QCD using the staggered fermion Hamiltonian formalism in the future,

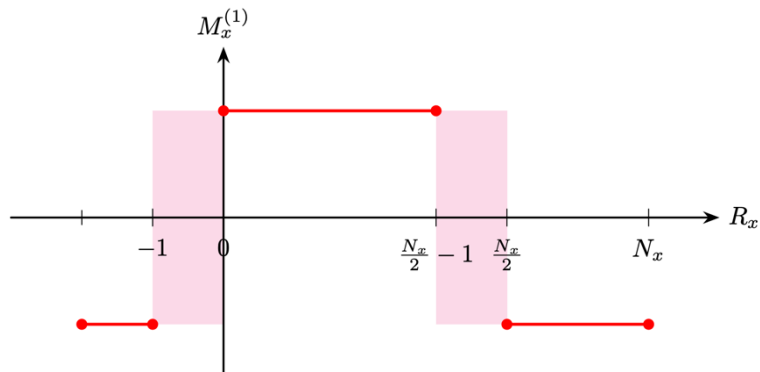
Practical motivation

Our work 2

Furthermore,

- We **introduce a kink profile for a mass = Domain-Wall**, that commutes with the Onsager charges.

[see also Kikukawa's and Sen's talks]



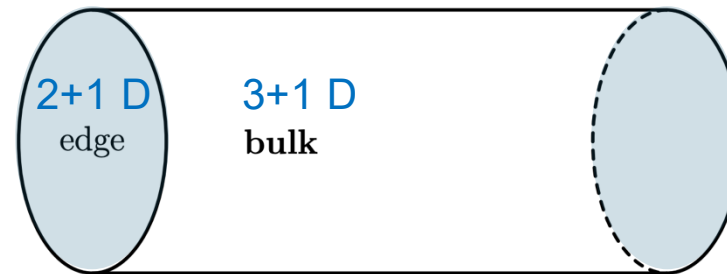
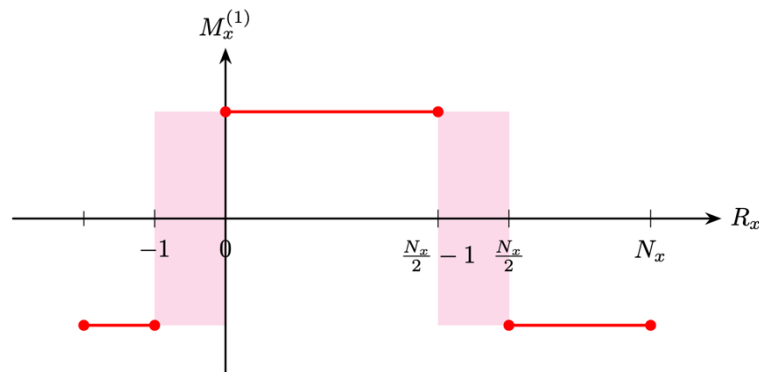
Our work 2

Furthermore,

- We **introduce a kink profile for a mass = Domain-Wall**, that commutes with the Onsager charges.

[see also Kikukawa's and Sen's talks]

- We clarify how the bulk conserved Onsager charges act **on the boundary**.



Main message

In both the bulk and the edge,

the **Onsager charges** are **the lattice avatars** of symmetries

previously thought to emerge only in the IR

— and, when those symmetries are anomalous, **avatars of** their anomalies as well.



Such symmetries and anomalies are not emergent but "**emanant.**"

= These emanate from the underlying exact lattice symmetries and anomalies.



The IR QFT information can be extracted from the Onsager charges on the lattice.

Outline



1. Introduction

2. 3+1 D Staggered fermion system

2-1. Set up

2-2. Discrete symmetries

2-3. New Conserved Charges = Onsager charges

4. Taste Splitting Mass

5. 2+1 D Boundary system

6. Summary

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Hamiltonian

[Kogut, Susskind (1975)]

[Catterall, Pradhan, Samlodia 2024]

$$H = \sum_{\mathbf{r}} \sum_{i=1}^3 \frac{i}{2} \eta_i(\mathbf{r}) \left(\chi(\mathbf{r})^\dagger \chi(\mathbf{r} + \hat{i}) + \chi(\mathbf{r}) \chi^\dagger(\mathbf{r} + \hat{i}) \right) = \sum_{\mathbf{r}} \sum_{i=1}^3 \frac{1}{4} \eta_i(\mathbf{r}) \left(a(\mathbf{r}) a(\mathbf{r} + \hat{i}) + b(\mathbf{r}) b(\mathbf{r} + \hat{i}) \right)$$

(We assume that the number of lattice sites on each direction is even L and the periodic boundary condition.)

$$\chi(\mathbf{r}) = \frac{1}{2} (a(\mathbf{r}) + ib(\mathbf{r}))$$

- staggered phase $\eta_i(\mathbf{r}) = (-1)^{r_1 + \dots + r_{i-1}}$ corresponds to inserting a π flux in the unit cell that gives the **eight** zeromodes at two momentum points in the continuum limit.



Two-flavor Dirac fermions

(We will see this in the next slides. (3pages))

From staggered fermions to Dirac fermions

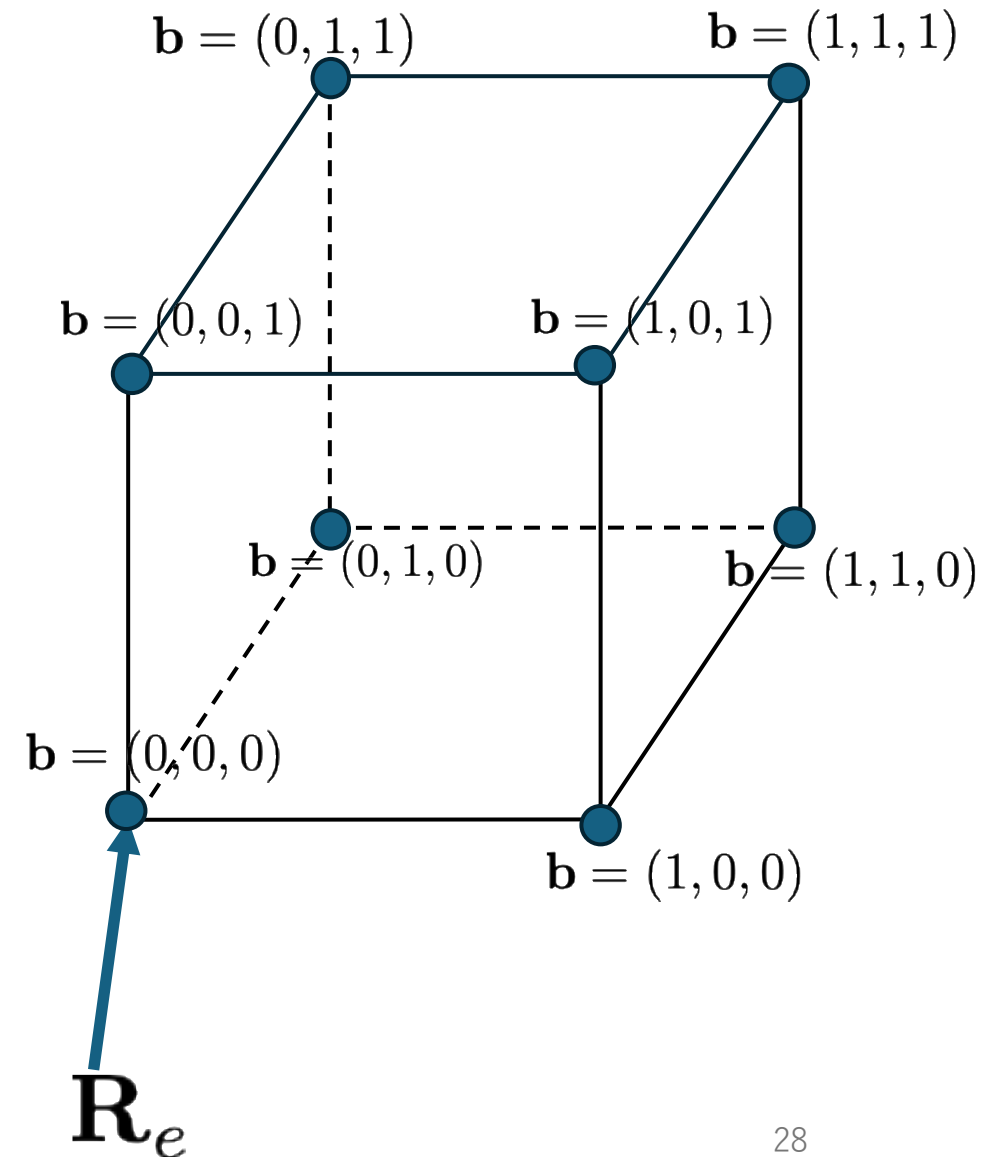
We consider a unit cell with 8 points

$$\mathbf{r} = \mathbf{R}_e + \mathbf{b}$$

Vectors labelling
different cells

Vectors
within the unit cell

We interpret \mathbf{R}_e as spatial points,
and \mathbf{b} as spin-flavor degrees of freedom



From staggered fermions to Dirac fermions

This Hamiltonian can be rewritten to **2-flavor Dirac fermion** Hamiltonian,

[Kogut, Susskind (1975)]

[Catterall, Pradhan, Samlodia 2024]

$$H = i \sum_{\mathbf{R}_e} \Psi^\dagger(\mathbf{R}_e) \left[(\alpha_i \otimes I_{2 \times 2}) \frac{\nabla_i}{2} + \beta (\gamma_5 \otimes \sigma_i^T) \frac{\nabla_i^2}{2} \right] \Psi(\mathbf{R}_e)$$

Twisted Wilson mass

$$\alpha_i = \gamma_0 \gamma_i \quad (i = 1, 2, 3), \quad \beta = \gamma_0$$

where

$$\Psi(\mathbf{R}_e) = \underbrace{\begin{pmatrix} \psi_{1+}(\mathbf{R}_e) & \psi_{2+}(\mathbf{R}_e) \\ \psi_{3-}(\mathbf{R}_e) & \psi_{4-}(\mathbf{R}_e) \end{pmatrix}}_{\text{Flavor 1,2}} \Bigg] \text{4-component spinor}$$

3+1 D staggered fermion Hamiltonian

[Kogut, Susskind (1975)]

[Catterall, Pradhan, Samlodia 2024]

In the continuum,

$$H = i \sum_{\mathbf{R}_e} \Psi^\dagger(\mathbf{R}_e) \left[(\alpha_i \otimes I_{2 \times 2}) \frac{\nabla_i}{2} + \beta \underbrace{(\gamma_5 \otimes \sigma_i^T)}_{\text{Twisted Wilson mass}} \frac{\nabla_i^2}{2} \right] \Psi(\mathbf{R}_e)$$

Describing Two-flavor massless Dirac fermion system.

$$G_{\text{classical}}^{(3+1D)} = \underline{U(1)_V \times U(1)_A \times SU(2)_L \times SU(2)_R}$$

These symmetries will be restored in the continuum limit.

3+1 D staggered fermion Hamiltonian

On the lattice,

$$H = i \sum_{\mathbf{R}_e} \Psi^\dagger(\mathbf{R}_e) \left[(\alpha_i \otimes I_{2 \times 2}) \frac{\nabla_i}{2} + \beta (\gamma_5 \otimes \sigma_i^T) \frac{\nabla_i^2}{2} \right] \Psi(\mathbf{R}_e)$$

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Discrete symmetries

[Li, Wang, You (2024)] [Aoki, Kikukawa, Takemoto (2025)]

3+1 D staggered fermion system possesses the following discrete symmetries :

- Parity symmetry \mathbb{Z}_2^P
- Reflection symmetry $\mathbb{Z}_2^{R_i}$
- Time-reversal symmetry \mathbb{Z}_2^T
- Charge conjugation symmetry \mathbb{Z}_2^C
- Shift symmetry $\mathbb{Z}_{L_i}^{S_i}$

Lattice size along the i th direction

Discrete symmetries

In the continuum limit,

- Parity symmetry \mathbb{Z}_2^P
- Reflection symmetry $\mathbb{Z}_2^{R_i}$
- Time-reversal symmetry \mathbb{Z}_2^T
- Charge conjugation symmetry \mathbb{Z}_2^C
- Shift symmetry $\mathbb{Z}_{L_i}^{S_i}$
- \mathbb{Z}_4^T
- \mathbb{Z}_2^C

where $\mathcal{T} := -S_z S_x \mathcal{T}$: $\mathcal{T}\chi(\mathbf{r})\mathcal{T}^{-1} = -(-1)^x \chi(\mathbf{r} + \hat{x} + \hat{z})$, $\mathcal{T}i\mathcal{T}^{-1} = -i$,
 $\mathcal{C} := S_y \mathcal{C}$: $\mathcal{C}\chi(\mathbf{r})\mathcal{C}^{-1} = (-1)^z \chi^\dagger(\mathbf{r} + \hat{y})$.

$$\mathcal{T}: \Psi(\mathbf{R}_e) \rightarrow (-\alpha_1 \alpha_3 \otimes \sigma_2^T) \Psi(\mathbf{R}_e),$$

$$\mathcal{C}: \Psi(\mathbf{R}_e) \rightarrow -(\beta \alpha_3 \alpha_1 \otimes i \sigma_2^T) \Psi^*(\mathbf{R}_e),$$

$$\mathcal{T}: \Psi(\mathbf{R}_e) \rightarrow (\alpha_1 \alpha_3 \otimes \mathbb{1}) \Psi(\mathbf{R}_e),$$

$$\mathcal{C}: \Psi(\mathbf{R}_e) \rightarrow i(\beta \alpha_2 \otimes \mathbb{1}) \Psi^*(\mathbf{R}_e),$$

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Translational symmetry

[Kogut, Susskind (1975)]

[Catterall, Pradhan, Samlodia 2024]

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(We assume that the number of lattice sites on each direction is even L and the periodic boundary condition.) $\chi(\mathbf{r}) = \frac{1}{2} (a(\mathbf{r}) + ib(\mathbf{r}))$

- Majorana fermions a and b are decoupled from each other.

➡ Translational symmetries acting on each Majorana fermion.

e.g. $T_{\hat{i}}^{(b)} a(\mathbf{r}) \left(T_{\hat{i}}^{(b)} \right)^{-1} = a(\mathbf{r})$, $T_{\hat{i}}^{(b)} b(\mathbf{r}) \left(T_{\hat{i}}^{(b)} \right)^{-1} = \zeta_i(\mathbf{r}) b(\mathbf{r} + \hat{i})$, $[T_{\hat{i}}^{(b)}, H] = 0$.

$$\zeta_i(\mathbf{r}) := (-1)^{\sum_{k=i+1}^3 r_k}$$

Conserved charges on the lattice

[Onogi, TY (2025)]

There are two types of conserved charges :

$$Q_0 = \sum_{\mathbf{r}} \left(\chi^\dagger(\mathbf{r})\chi(\mathbf{r}) - \frac{1}{2} \right) = \sum_{\mathbf{r}} \frac{i}{2} a(\mathbf{r})b(\mathbf{r})$$

$$Q_i = T_{\hat{i}}^{(b)} Q_0 \left(T_{\hat{i}}^{(b)} \right)^{-1} = \sum_{\mathbf{r}} \frac{i}{2} \zeta_i(\mathbf{r}) a(\mathbf{r}) b(\mathbf{r} + \hat{i}), \quad i = x, y, z$$

New Conserved Charges

$$\because [Q_0, H] = [T_{\hat{i}}^{(b)}, H] = 0$$

Onsager charges in the system

These conserved charges do Not commute with each other.

$$[Q_0, Q_i] \neq 0, [Q_i, Q_j] \neq 0,$$

This non-commutativity generates the **Onsager algebras**. [Aoki, Kikukawa, Takemoto (2025)]

Thus Q_0, Q_i are Onsager charges !!

Onsager charges in the system

Q_0, Q_i are Onsager charges !!

Our question

Are the Onsager charges the Lattice avatar of symmetries and anomalies in the continuum?

- What symmetries do the Onsager charges generate in the continuum?? (Discuss in the next slide)
- Is there any 't Hooft mixed anomaly ??



Clarifying the mass terms is helpful for diagnosing the existences of 't Hooft anomaly.

∴ 't Hooft anomaly matching condition

(Discuss later)

Physical interpretation

[Onogi, TY (2025)]

$$\Psi(\mathbf{R}_e) = \begin{pmatrix} \psi_{1+} & \psi_{2+} \\ \psi_{3-} & \psi_{4-} \end{pmatrix}$$

Taking **continuum** limit, we find

$$\lim_{L \rightarrow \infty} [Q_i, \Psi(\mathbf{R}_e)] = (\gamma_5 \otimes \sigma_i^T) \Psi(\mathbf{R}_e).$$

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Taking **continuum** limit, we find

$$\lim_{L \rightarrow \infty} [Q_i, \Psi(\mathbf{R}_e)] = (\gamma_5 \otimes \sigma_i^T) \Psi(\mathbf{R}_e).$$

We find that the quantized conserved charges generate

$$\underline{U(1)_{F_x}, U(1)_{F_y}, U(1)_{F_z} \subset SU(2)_L \times SU(2)_R}$$

— thought to be absent on the lattice by the twisted-Wilson mass, yet realized exactly (emanant).

= These symmetries emanate from the underlying exact lattice symmetries.

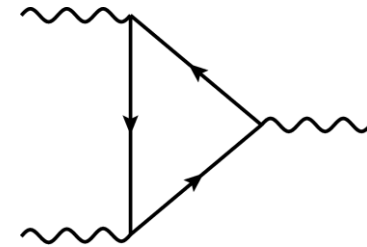
$$H = i \sum_{\mathbf{R}_e} \Psi^\dagger(\mathbf{R}_e) \left[(\alpha_i \otimes I_{2 \times 2}) \frac{\nabla_i}{2} + \beta (\gamma_5 \otimes \sigma_i^T) \frac{\nabla_i^2}{2} \right] \Psi(\mathbf{R}_e)$$

Anomalies of flavor U(1) symmetry in the continuum

$$\underline{U(1)_{F_x}, U(1)_{F_y}, U(1)_{F_z}} \subset SU(2)_L \times SU(2)_R$$

- Giving up one $U(1)_{F_i}$ symmetry \longrightarrow Twisted masses $\Psi^\dagger(\beta \otimes \sigma_{j \neq i})\Psi$ are allowed.
 $\Psi^\dagger(\beta \gamma_5 \otimes \sigma_{j \neq i})\Psi$ [Aoki(1984)][Shindler(2007)]

- Indeed, the anomaly coefficient obtained from the triangle diagram is zero.



\longrightarrow Each $U(1)_{F_i}$ can be regarded as a **vector-like symmetry**.

\longrightarrow There is **No** mixed anomaly between $U(1)_V$ and $U(1)_{F_i}$, $i = x, y, z$

Short summary

3+1 D staggered fermion Hamiltonian possesses conserved Onsager charges.

Are the Onsager charges the Lattice avatar of symmetries and anomalies in the continuum?

- What symmetries do the Onsager charges generate in the continuum??

$$\longrightarrow U(1)_{F_x}, U(1)_{F_y}, U(1)_{F_z} \subset SU(2)_L \times SU(2)_R$$

- Is there any 't Hooft mixed anomaly ??

\longrightarrow We can **expect** that there is no 't Hooft anomaly related to Q_0 and Q_i .

\longrightarrow Let's Clarify **the mass terms on the lattice** to diagnose existence of 't Hooft anomaly.

∴ 't Hooft anomaly matching condition

(Discuss next)

What is the anomaly?

[’t Hooft (1980)] [Seiberg, Shao, Zhang (2025)]

[See also Kobayashi’s and Thorngren’s talk]

- The standard definition of ’t Hooft anomaly in **continuum QFT** involves placing the system on a general Euclidean spacetime with background gauge fields for the **internal** symmetries and finding that the partition function is not well-defined.

$$Z[A] \rightarrow Z[A] e^{\underbrace{2\pi i \int_X \alpha}_{\text{Anomaly}}}$$

Anomaly



Anomaly = Obstruction to gauging the symmetry

What is the anomaly?

[Chang, Lin, Shao, Wang, Ying (2018)] [Thorngren and Wang (2019)]
[Seiberg, Shao, Zhang (2025)]

[See also Kobayashi's and Thorngren's talk]

However, when conserved charges are highly non-local,
it is not clear if there is a sensible prescription for gauging it.

(e.g.) Conserved charges which generate the Onsager algebra.

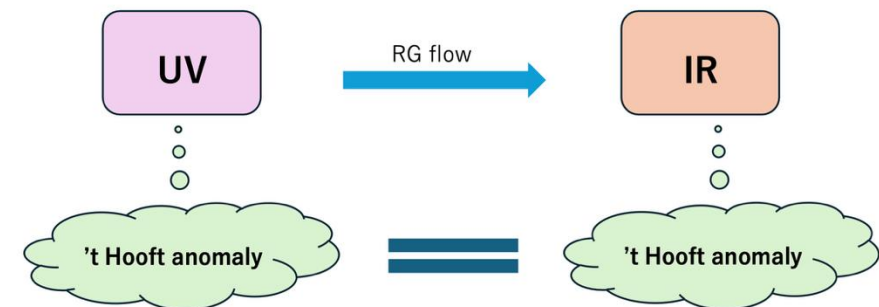
Another definition of anomaly :

Anomaly = Obstruction to a symmetric, trivially-gapped phase

• • 't Hooft anomaly matching condition

(e.g.) Lieb-Schultz-Mattis (LSM)-like constraints

[Lieb, Schultz, Mattis (1961)]



Outline



1. Introduction

2. 3+1 D Staggered fermion system



2-1. Set up



2-2. Discrete symmetries



2-3. New Conserved Charges = Onsager charges

4. Taste Splitting Mass

5. 2+1 D Boundary system

6. Summary

Classification of Dirac mass terms

A Dirac fermion is reconstructed from 8 one-component staggered fermions in a unit cube.

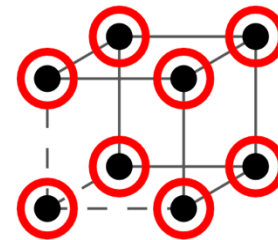


mass terms need not be on-site

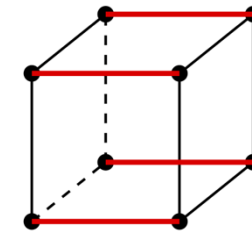
— they may arise from one-, two-, or three-link hoppings within the cube.

We classify **all bilinear Dirac mass terms** that are

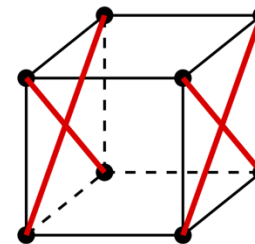
- Hermitian
- particle-number-preserving
- localized within a unit cube
- gap out the continuum Dirac fermions



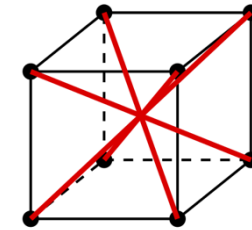
On-site



One-link



Two-link

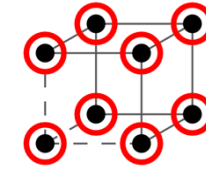


Three-link

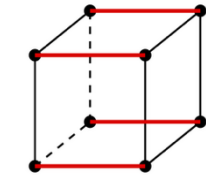
Mass terms in the unit cell

[Misumi, Onogi, TY (2026)]

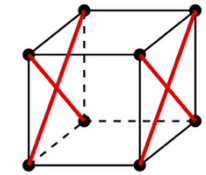
On-site mass :
$$\delta H^{(0)} = \sum_{\mathbf{R}_e} m \Psi^\dagger(\mathbf{R}_e) (\gamma_0 \otimes \mathbb{1}) \Psi(\mathbf{R}_e)$$



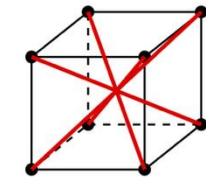
One-link mass :
$$\delta H_i^{(1)} = \sum_{\mathbf{R}_e} m \Psi^\dagger(\mathbf{R}_e) (i\gamma_0 \gamma_5 \otimes \sigma_i) \Psi(\mathbf{R}_e)$$



Two-link mass :
$$\delta H_i^{(2)} = \sum_{\mathbf{R}_e} m \Psi^\dagger(\mathbf{R}_e) (\gamma_0 \otimes \sigma_i) \Psi(\mathbf{R}_e)$$

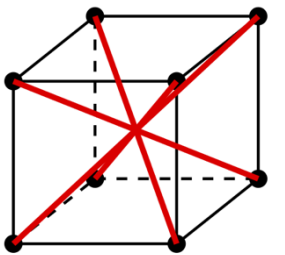
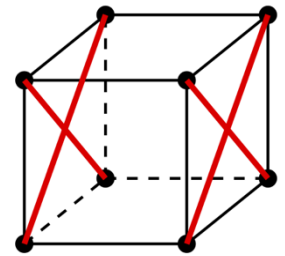
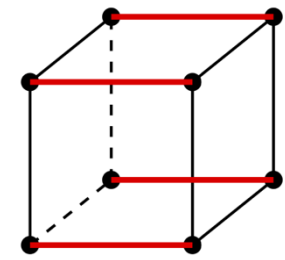


Three-link mass :
$$\delta H^{(3)} = \sum_{\mathbf{R}_e} m \Psi^\dagger(\mathbf{R}_e) (i\gamma_0 \gamma_5 \otimes \mathbb{1}) \Psi(\mathbf{R}_e)$$



Masses and symmetries

[Misumi, Onogi, TY (2026)]



	P	R_x	R_y	R_z	T	C	\mathcal{T}	\mathcal{C}	S_x	S_y	S_z	Q_0	Q_x	Q_y	Q_z
$\delta H^{(0)}$	○	○	○	○	○	×	○	○	×	×	×	○	×	×	×
$\delta H_x^{(1)}$	×	×	○	○	○	○	×	○	×	○	○	○	×	○	○
$\delta H_y^{(1)}$	×	○	×	○	○	○	×	×	×	×	○	○	×	×	○
$\delta H_z^{(1)}$	×	○	○	×	○	○	×	×	×	×	×	○	×	×	×
$\delta H_x^{(2)}$	○	○	×	×	×	○	○	×	×	×	×	○	×	×	×
$\delta H_y^{(2)}$	○	×	○	×	×	○	○	×	×	×	×	○	×	×	×
$\delta H_z^{(2)}$	○	×	×	○	×	○	○	×	×	×	×	○	×	×	×
$\delta H^{(3)}$	×	×	×	○	○	×	○	×	×	×	×	○	×	×	×

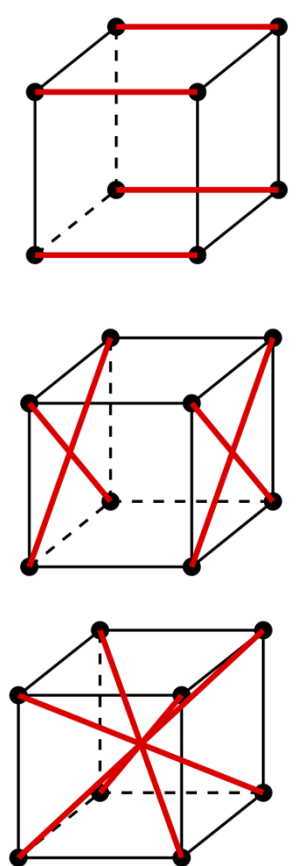
Table 1. Commutation relation between masses and symmetries

○ : commutative , × : non-commutative.

Masses and symmetries

[Misumi, Onogi, TY (2026)]

$$\delta H^{(0)} = \sum_{\mathbf{R}_e} m \Psi^\dagger(\mathbf{R}_e) (\gamma_0 \otimes \mathbb{1}) \Psi(\mathbf{R}_e)$$



	P	R_x	R_y	R_z	T	C	\mathcal{T}	\mathcal{C}	S_x	S_y	S_z	Q_0	Q_x	Q_y	Q_z
$\delta H^{(0)}$	○	○	○	○	○	×	○	○	×	×	×	○	×	×	×
$\delta H_x^{(1)}$	×	×	○	○	○	○	×	○	×	○	○	○	×	○	○
$\delta H_y^{(1)}$	×	○	×	○	○	○	×	×	×	×	○	○	×	×	○
$\delta H_z^{(1)}$	×	○	○	×	○	○	×	×	×	×	×	○	×	×	×
$\delta H_x^{(2)}$	○	○	×	×	×	○	○	×	×	×	×	○	×	×	×
$\delta H_y^{(2)}$	○	×	○	×	×	○	○	×	×	×	×	○	×	×	×
$\delta H_z^{(2)}$	○	×	×	○	×	○	○	×	×	×	×	○	×	×	×
$\delta H^{(3)}$	×	×	×	○	○	×	○	×	×	×	×	○	×	×	×

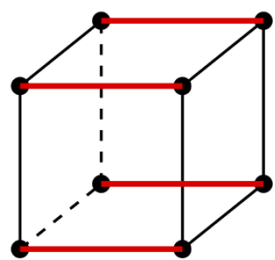
Table 1. Commutation relation between masses and symmetries

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Masses and symmetries

[Misumi, Onogi, TY (2026)]

$$\delta H_i^{(1)} = \sum_{\mathbf{R}_e} m \Psi^\dagger(\mathbf{R}_e) (i\gamma_0 \gamma_5 \otimes \sigma_i) \Psi(\mathbf{R}_e)$$



	P	R_x	R_y	R_z	T	C	\mathcal{T}	\mathcal{C}	S_x	S_y	S_z	Q_0	Q_x	Q_y	Q_z
$\delta H^{(0)}$	○	○	○	○	○	×	○	○	×	×	×	○	×	×	×
$\delta H_x^{(1)}$	×	×	○	○	○	○	×	○	×	○	○	○	×	○	○
$\delta H_y^{(1)}$	×	○	×	○	○	○	×	×	×	×	○	○	×	×	○
$\delta H_z^{(1)}$	×	○	○	×	○	○	×	×	×	×	×	○	×	×	×

- $\delta H_x^{(1)}$ respects **many** of the symmetries of the staggered fermion Hamiltonian, and implies the **absence** of a mixed 't Hooft anomaly between $U(1)_V$ and $U(1)_{F_{i \neq x}}$

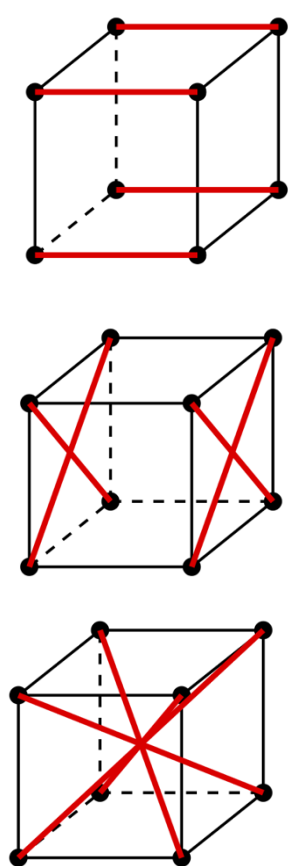
- This anisotropy originates from the staggered phase $\eta_i(\mathbf{r}) = (-1)^{r_1 + \dots + r_{i-1}}$, which singles out the x-direction.

Under a cyclic \mathbb{Z}_3 relabeling of $\{x, y, z\}$,
a different one-link mass would preserve the largest residual symmetry.

Masses and symmetries

[Misumi, Onogi, TY (2026)]

$$\delta H_i^{(2)} = \sum_{\mathbf{R}_e} m \Psi^\dagger(\mathbf{R}_e) (\gamma_0 \otimes \sigma_i) \Psi(\mathbf{R}_e)$$



	P	R_x	R_y	R_z	T	C	\mathcal{T}	\mathcal{C}	S_x	S_y	S_z	Q_0	Q_x	Q_y	Q_z
$\delta H^{(0)}$	○	○	○	○	○	×	○	○	×	×	×	○	×	×	×
$\delta H_x^{(1)}$	×	×	○	○	○	○	×	○	×	○	○	○	×	○	○
$\delta H_y^{(1)}$	×	○	×	○	○	○	×	×	×	×	○	○	×	×	○
$\delta H_z^{(1)}$	×	○	○	×	○	○	×	×	×	×	×	○	×	×	×
$\delta H_x^{(2)}$	○	○	×	×	×	○	○	×	×	×	×	○	×	×	×
$\delta H_y^{(2)}$	○	×	○	×	×	○	○	×	×	×	×	○	×	×	×
$\delta H_z^{(2)}$	○	×	×	○	×	○	○	×	×	×	×	○	×	×	×
$\delta H^{(3)}$	×	×	×	○	○	×	○	×	×	×	×	○	×	×	×

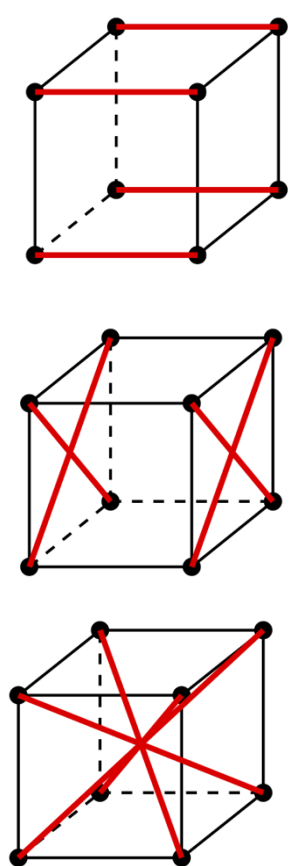
Table 1. Commutation relation between masses and symmetries

○ : commutative , × : non-commutative.

Masses and symmetries

[Misumi, Onogi, TY (2026)]

$$\delta H^{(3)} = \sum_{\mathbf{R}_e} m \Psi^\dagger(\mathbf{R}_e) (i\gamma_0 \gamma_5 \otimes \mathbb{1}) \Psi(\mathbf{R}_e)$$



	P	R_x	R_y	R_z	T	C	\mathcal{T}	\mathcal{C}	S_x	S_y	S_z	Q_0	Q_x	Q_y	Q_z
$\delta H^{(0)}$	○	○	○	○	○	×	○	○	×	×	×	○	×	×	×
$\delta H_x^{(1)}$	×	×	○	○	○	○	×	○	×	○	○	○	×	○	○
$\delta H_y^{(1)}$	×	○	×	○	○	○	×	×	×	×	○	○	×	×	○
$\delta H_z^{(1)}$	×	○	○	×	○	○	×	×	×	×	×	○	×	×	×
$\delta H_x^{(2)}$	○	○	×	×	×	○	○	×	×	×	×	○	×	×	×
$\delta H_y^{(2)}$	○	×	○	×	×	○	○	×	×	×	×	○	×	×	×
$\delta H_z^{(2)}$	○	×	×	○	×	○	○	×	×	×	×	○	×	×	×
$\delta H^{(3)}$	×	×	×	○	○	×	○	×	×	×	×	○	×	×	×

Table 1. Commutation relation between masses and symmetries

○ : commutative , × : non-commutative.

Short summary

3+1 D staggered fermion Hamiltonian possesses conserved Onsager charges.

Are the Onsager charges the Lattice avatar of symmetries and anomalies in the continuum?

- What symmetries do the Onsager charges generate in the continuum??

$$\longrightarrow U(1)_{F_x}, U(1)_{F_y}, U(1)_{F_z} \subset SU(2)_L \times SU(2)_R$$

- Is there some 't Hooft mixed anomaly ??

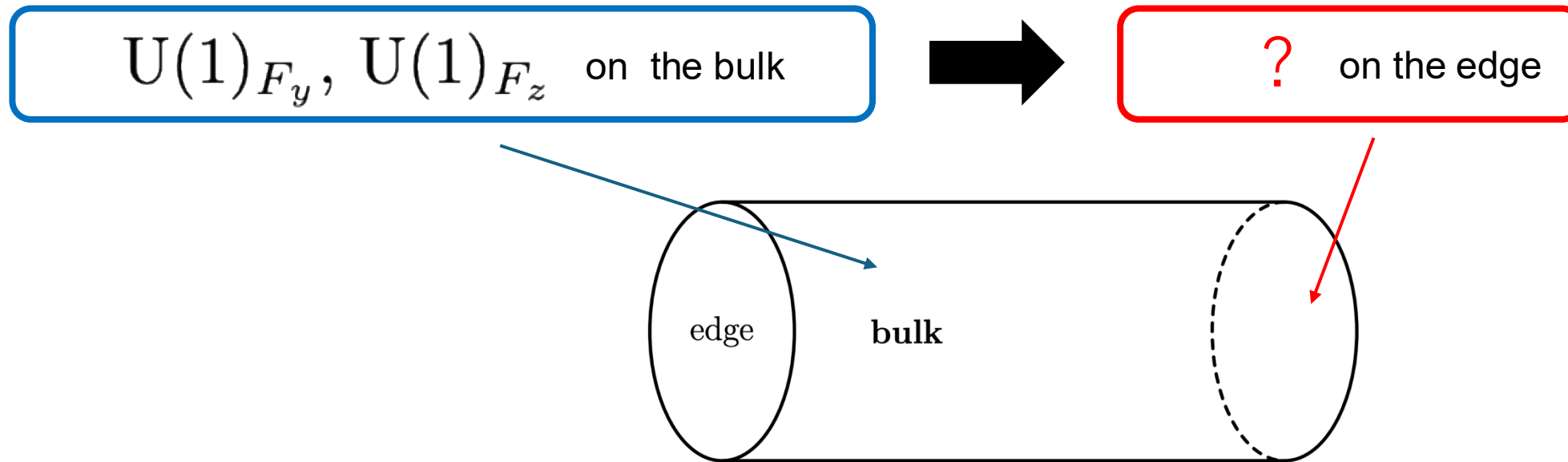
\longrightarrow There is **No** 't Hooft anomaly between Q_0 and Q_y, Q_z .

\longrightarrow consistent with the continuum theory.

What about Q_0 and Q_x ? (future work)

Next Question

In the 3+1D staggered fermion system with a boundary, what physical role do the Onsager charges play on **the 2+1D boundary**?



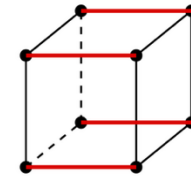
Outline

- ✓ 1. Introduction
- 2. 3+1 D Staggered fermion system
 - ✓ 2-1. Set up
 - ✓ 2-2. Discrete symmetries
 - ✓ 2-3. New Conserved Charges = Onsager charges
- ✓ 4. Taste Splitting Mass
- 5. 2+1 D Boundary system
- 6. Summary

From 3+1 D bulk to 2+1 D edge [Misumi, Onogi, TY (2026)]

The one-link mass in the x-direction preserves larger symmetries than the other mass terms.

$$\delta H_x^{(1)} = \frac{m}{2} \sum_{\mathbf{R}_e} \Psi^\dagger(\mathbf{R}_e) (i\gamma_0 \gamma_5 \otimes \sigma_1) \Psi(\mathbf{R}_e)$$

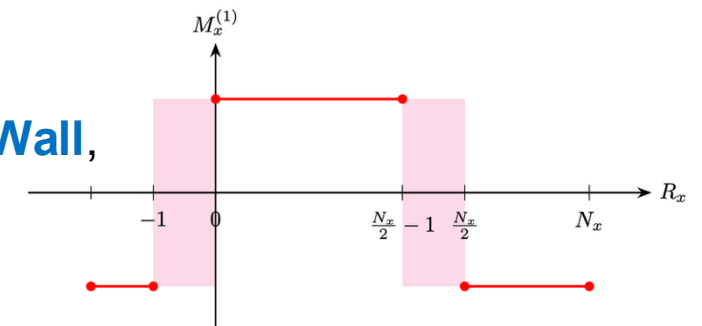


	P	R_x	R_y	R_z	T	C	\mathcal{T}	\mathcal{C}	S_x	S_y	S_z	Q_0	Q_x	Q_y	Q_z
$\delta H_x^{(1)}$	×	×	○	○	○	○	×	○	×	○	○	○	×	○	○

Our questions

If we introduce a kink profile of the x-directed one-link mass = Domain-Wall,

- What 2+1 D edge theory is derived ?
- What is the physical role of Onsager conserved charges Q_y, Q_z on the 2+1 D edge ?



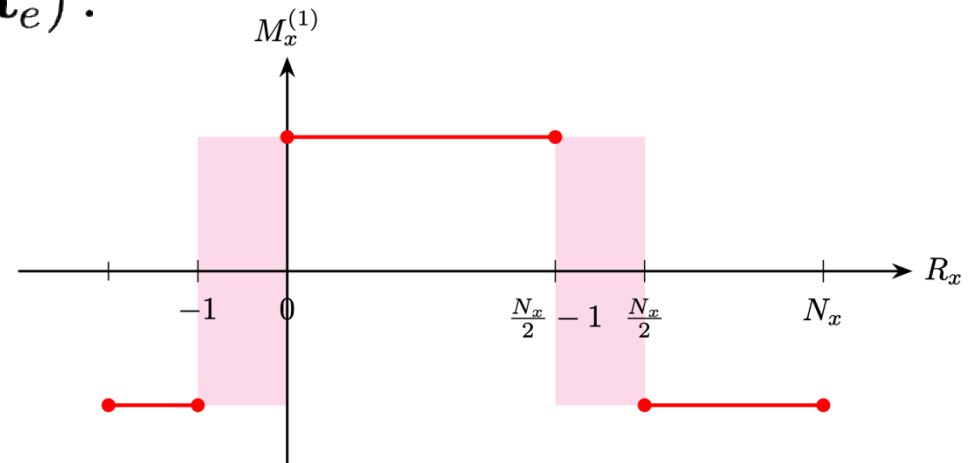
Set up : Introducing a kink mass

$$H_{DW} = H + M_x^{(1)}$$

$$H = i \sum_{\mathbf{R}_e} \Psi^\dagger(\mathbf{R}_e) \left[(\alpha_i \otimes \mathbb{1}_{2 \times 2}) \frac{\nabla_i}{2} + (\beta \gamma_5 \otimes \sigma_i^T) \frac{\nabla_i^2}{2} \right] \Psi(\mathbf{R}_e)$$

$$M_x^{(1)} = i \sum_{\mathbf{R}} \frac{m_0}{2} \theta(R_x) \Psi^\dagger(\mathbf{R}_e) (\gamma_0 \gamma_5 \otimes \sigma_1) \Psi(\mathbf{R}_e).$$

$$\theta(R_x) = \begin{cases} 1 & 0 \leq R_x \leq \frac{N_x}{2} - 1, \\ -1 & \frac{N_x}{2} \leq R_x \leq N_x. \end{cases}$$

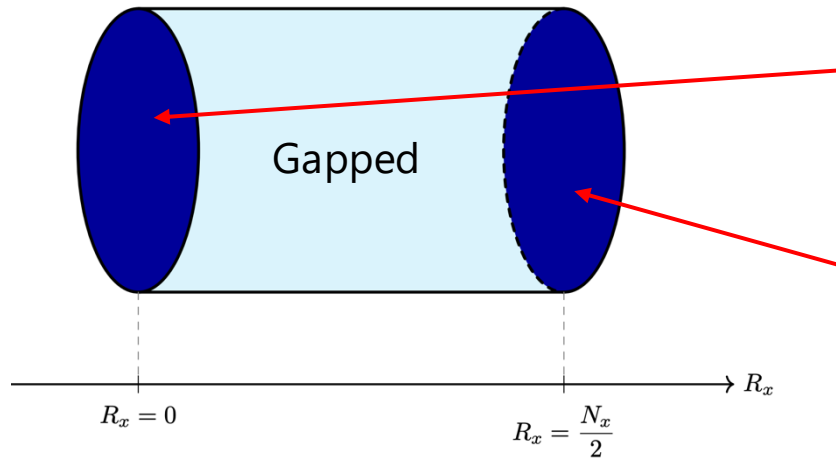


On the 2+1 D edge

$$i = 1 = y \text{ and } i = 2 = z \quad \hat{\beta} = \hat{\gamma}_0 = \sigma_3, \hat{\alpha}_1 = \sigma_2, \hat{\alpha}_2 = -\sigma_1, \\ \hat{\gamma}_1 = -i\sigma_1, \hat{\gamma}_2 = -i\sigma_2.$$

Taking the low energy limit,

two-flavor free massless fermions are localized on the 2 + 1 D domain walls.



$$H_{(x=0)}^{2D} = i \sum_{\mathbf{R}} \sum_{f=1,2} \hat{\Psi}_{-}^{(f)\dagger} \hat{\alpha}_i \partial_i \hat{\Psi}_{-}^{(f)},$$

$$H_{(x=\frac{N_x}{2})}^{2D} = i \sum_{\mathbf{R}} \sum_{f=1,2} \hat{\Psi}_{+}^{(f)\dagger} \hat{\alpha}_i \partial_i \hat{\Psi}_{+}^{(f)}.$$

The fermions can be identified as eigenstates of the reflection symmetry generated by R_x :

$$R_x : \begin{pmatrix} \hat{\Psi}_{-}(x=0, t, \mathbf{R}) \\ \hat{\Psi}_{+}(x=\frac{N_x}{2}, t, \mathbf{R}) \end{pmatrix} \rightarrow \sigma_3 \begin{pmatrix} \hat{\Psi}_{-}(x=0, t, \mathbf{R}) \\ \hat{\Psi}_{+}(x=-\frac{N_x}{2}, t, \mathbf{R}) \end{pmatrix}$$

The action of the discrete symmetries

	P	R _x	R _y	R _z	T	C	\mathcal{T}	\mathcal{C}	S _x	S _y	S _z	Q ₀	Q _x	Q _y	Q _z
$\delta H_x^{(1)}$	×	×	○	○	○	○	×	○	×	○	○	○	×	○	○

The discrete symmetries that are exact in the bulk act on the edge modes as follows,

spin flavor Edge (i.e. $x = 0$ or $\frac{N_x}{2}$)

$$\begin{aligned}
 P_{\perp} := R_y R_z : & \begin{pmatrix} \hat{\Psi}_{-}(x=0, t, \mathbf{R}) \\ \hat{\Psi}_{+}(x=\frac{N_x}{2}, t, \mathbf{R}) \end{pmatrix} \rightarrow (\hat{\beta} \otimes \sigma_3 \otimes \mathbb{1}) \begin{pmatrix} \hat{\Psi}_{-}(x=0, t, -\mathbf{R}) \\ \hat{\Psi}_{+}(x=\frac{N_x}{2}, t, -\mathbf{R}) \end{pmatrix}, \\
 R_y : & \begin{pmatrix} \hat{\Psi}_{-}(x=0, t, \mathbf{R}) \\ \hat{\Psi}_{+}(x=\frac{N_x}{2}, t, \mathbf{R}) \end{pmatrix} \rightarrow (\hat{\gamma}_1 \otimes i\sigma_1 \otimes \sigma_3) \begin{pmatrix} \hat{\Psi}_{-}(x=0, t, \widetilde{R}_y(\mathbf{R})) \\ \hat{\Psi}_{+}(x=\frac{N_x}{2}, t, \widetilde{R}_y(\mathbf{R})) \end{pmatrix}, \\
 R_z : & \begin{pmatrix} \hat{\Psi}_{-}(x=0, t, \mathbf{R}) \\ \hat{\Psi}_{+}(x=\frac{N_x}{2}, t, \mathbf{R}) \end{pmatrix} \rightarrow -(\hat{\gamma}_2 \otimes i\sigma_2 \otimes \sigma_3) \begin{pmatrix} \hat{\Psi}_{-}(x=0, t, \widetilde{R}_z(\mathbf{R})) \\ \hat{\Psi}_{+}(x=\frac{N_x}{2}, t, \widetilde{R}_z(\mathbf{R})) \end{pmatrix}, \\
 T : & \begin{pmatrix} \hat{\Psi}_{-}(x=0, t, \mathbf{R}) \\ \hat{\Psi}_{+}(x=\frac{N_x}{2}, t, \mathbf{R}) \end{pmatrix} \rightarrow (\hat{\gamma}_2 \otimes \sigma_2 \otimes \mathbb{1}) \begin{pmatrix} \hat{\Psi}_{-}(x=0, -t, \mathbf{R}) \\ \hat{\Psi}_{+}(x=\frac{N_x}{2}, -t, \mathbf{R}) \end{pmatrix}, \\
 C : & \begin{pmatrix} \hat{\Psi}_{-}(x=0, t, \mathbf{R}) \\ \hat{\Psi}_{+}(x=\frac{N_x}{2}, t, \mathbf{R}) \end{pmatrix} \rightarrow (i\hat{\gamma}_1 \otimes \sigma_1 \otimes \mathbb{1}) \begin{pmatrix} \hat{\Psi}_{-}^*(x=0, t, \mathbf{R}) \\ \hat{\Psi}_{+}^*(x=\frac{N_x}{2}, t, \mathbf{R}) \end{pmatrix}, \\
 \mathcal{C} : & \begin{pmatrix} \hat{\Psi}_{-}(x=0, t, \mathbf{R}) \\ \hat{\Psi}_{+}(x=\frac{N_x}{2}, t, \mathbf{R}) \end{pmatrix} \rightarrow (\hat{\gamma}_1 \otimes \sigma_3 \otimes \sigma_3) \begin{pmatrix} \hat{\Psi}_{-}^*(x=0, t, \mathbf{R}) \\ \hat{\Psi}_{+}^*(x=\frac{N_x}{2}, t, \mathbf{R}) \end{pmatrix}
 \end{aligned}$$

Physical interpretation of Q_y, Q_z on the edge

	P	R_x	R_y	R_z	T	C	\mathcal{T}	\mathcal{C}	S_x	S_y	S_z	Q_0	Q_x	Q_y	Q_z
$\delta H_x^{(1)}$	×	×	○	○	○	○	×	○	×	○	○	○	×	○	○

In the low energy limit,

$$\begin{aligned}
 \lim_{N \rightarrow \infty} [Q_y, \begin{pmatrix} \hat{\Psi}_-(x=0, t, \mathbf{R}) \\ \hat{\Psi}_+(x=\frac{N_x}{2}, t, \mathbf{R}) \end{pmatrix}] &= (\overset{\text{spin}}{\mathbb{1}} \otimes \overset{\text{flavor}}{\sigma_2} \otimes \overset{\text{Edge (i.e. } x=0 \text{ or } \frac{N_x}{2})}{\mathbb{1}}) \begin{pmatrix} \hat{\Psi}_-(x=0, t, \mathbf{R}) \\ \hat{\Psi}_+(x=\frac{N_x}{2}, t, \mathbf{R}) \end{pmatrix}, \\
 \lim_{N \rightarrow \infty} [Q_z, \begin{pmatrix} \hat{\Psi}_-(x=0, t, \mathbf{R}) \\ \hat{\Psi}_+(x=\frac{N_x}{2}, t, \mathbf{R}) \end{pmatrix}] &= -(\mathbb{1} \otimes \sigma_1 \otimes \mathbb{1}) \begin{pmatrix} \hat{\Psi}_-(x=0, t, \mathbf{R}) \\ \hat{\Psi}_+(x=\frac{N_x}{2}, t, \mathbf{R}) \end{pmatrix}.
 \end{aligned}$$

Q_y, Q_z generate a flavor $SU(2)$ symmetry on the Domain wall.

Parity anomaly b/w $SU(2)$ and $\mathbb{Z}_2^{\text{R}_i}$ on the edge

Comment: It is known that 2+1 D two-flavor Dirac fermion (**continuum**) system has parity anomaly associated w/ flavor $SU(2)$ symmetry and spacetime reflection symmetry.

[Seiberg and Witten (2016)]

[Pace, Kim, Chatterjee, Shao (2025)]

Parity anomaly b/w $SU(2)$ and $\mathbb{Z}_2^{R_i}$ on the edge

Comment: It is known that 2+1 D two-flavor Dirac fermion (**continuum**) system has parity anomaly associated w/ flavor $SU(2)$ symmetry and spacetime reflection symmetry.

[Seiberg and Witten (2016)]

[Pace, Kim, Chatterjee, Shao (2025)]

In our system,

Imposing $\left\{ \begin{array}{l} \text{Flavor } SU(2) \text{ symmetry generated by } Q_y, Q_z \\ \text{Reflection symmetry } \mathbb{Z}_2^{R_i}, i = y, z \end{array} \right.$



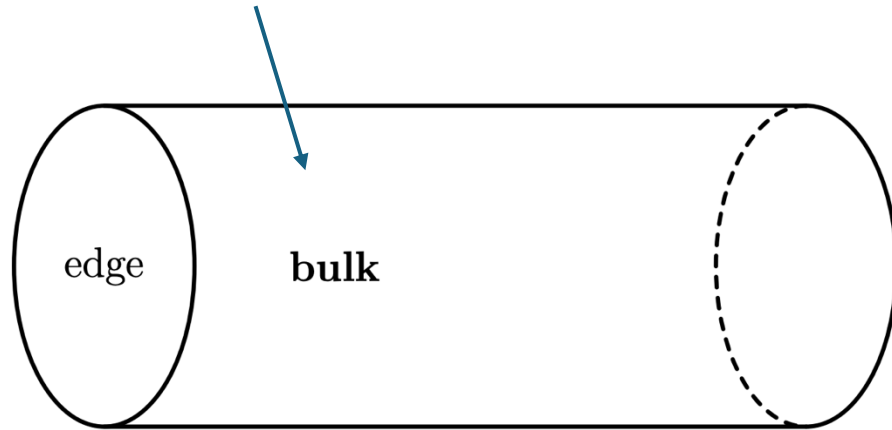
No mass term is allowed.

This establishes the **parity anomaly** associated with $SU(2)$ and $\mathbb{Z}_2^{R_i}$.

Emanant symmetries

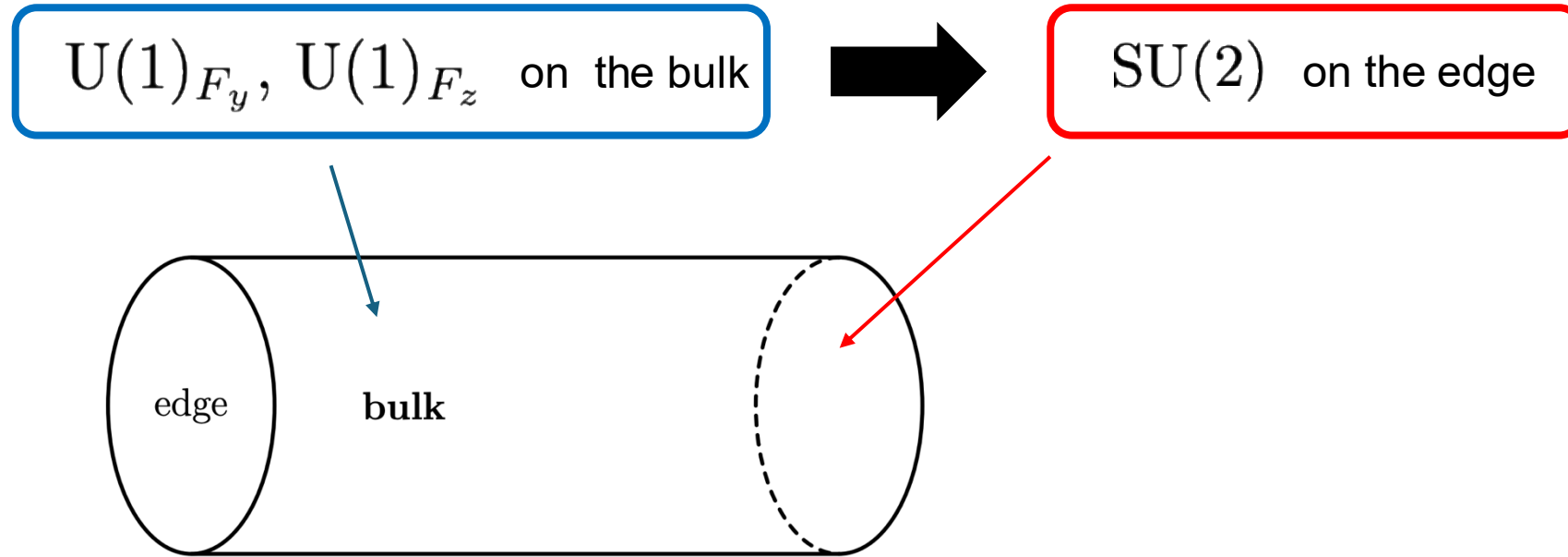
In the low energy limit, Q_y, Q_z conserved **on the lattice** generate

$U(1)_{F_y}, U(1)_{F_z}$ on the bulk



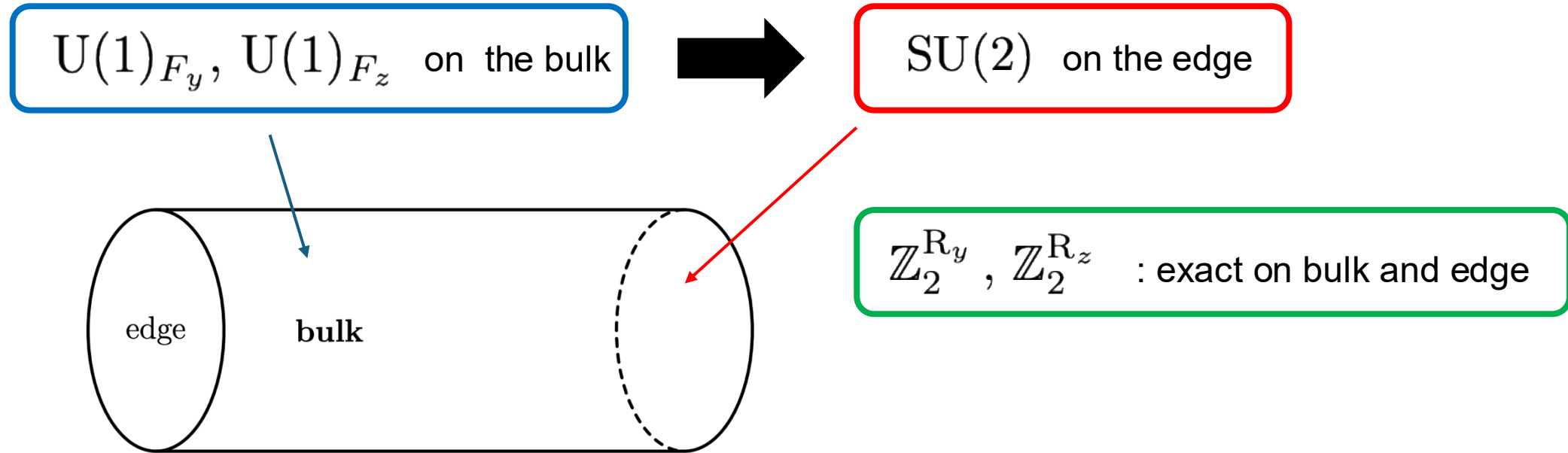
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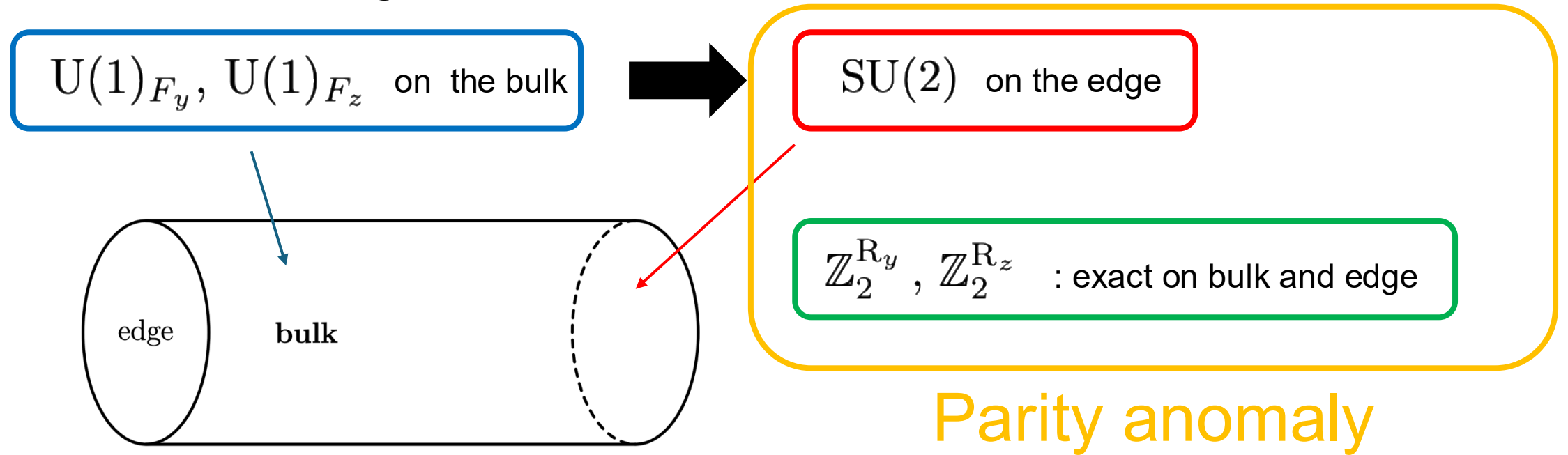
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Therefore

This flavor $SU(2)$ symmetry and the parity anomaly are not emergent but **Emanant**.

Comment: Our result is consistent with the 2+1D honeycomb lattice system [Pace, Kim, Chatterjee, Shao (2025)].

Outline

- ✓ 1. Introduction
- 2. 3+1 D Staggered fermion system
 - ✓ 2-1. Set up
 - ✓ 2-2. Discrete symmetries
 - ✓ 2-3. New Conserved Charges = Onsager charges
- ✓ 4. Taste Splitting Mass
- ✓ 5. 2+1 D Boundary system
- 6. Summary

Summary

The 3+1D staggered fermion system has conserved charges generating the Onsager algebra — the Onsager charges.

On the 3+1 D bulk

- Continuum: generate $U(1)_{F_x}, U(1)_{F_y}, U(1)_{F_z} \subset SU(2)_L \times SU(2)_R$
— thought to be violated on the lattice by the twisted-Wilson mass, yet realized exactly (emanant).
- No mixed anomaly with $Q_0 (U(1)_V)$ — consistent with the IR.

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On the 2+1 D edge

- Q_y, Q_z generate a flavor $SU(2)$ symmetry in the low-energy limit.
- No mass term symmetric under flavor $SU(2)$ symmetry and $\mathbb{Z}_2^{\mathbb{R}_i}$ reflection symmetry. → **Parity anomaly**

Main message

In both the bulk and the edge,

the **Onsager charges** are **the lattice avatars** of symmetries

previously thought to emerge only in the IR

— and, when those symmetries are anomalous, **avatars of** their anomalies as well.



Such symmetries and anomalies are not emergent but "**emanant.**"

= These emanate from the underlying exact lattice symmetries and anomalies.



The IR QFT information can be extracted from the Onsager charges on the lattice.

Back Up

From staggered fermions to Dirac fermions

Using 4x4 matrices

$$\alpha_i = \gamma_0 \gamma_i \quad (i = 1, 2, 3), \quad \beta = \gamma_0$$

$$\alpha^{\mathbf{b}} = \alpha_1^{b_1} \alpha_2^{b_2} \alpha_3^{b_3}$$

In chiral basis,

$$\alpha_i = \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$$

We define matrix valued fermion from staggered fermion χ as

$$\Phi(\mathbf{R}_e) = N_0 \sum_{\{\mathbf{b}\}} \chi(\mathbf{R}_e + \mathbf{b}) \alpha^{\mathbf{b}} = \begin{pmatrix} \psi_{1+}(\mathbf{R}_e) & \psi_{2+}(\mathbf{R}_e) & 0 & 0 \\ 0 & 0 & \psi_{3-}(\mathbf{R}_e) & \psi_{4-}(\mathbf{R}_e) \end{pmatrix}$$

Combining Weyl fermions as

$$\Psi(\mathbf{R}_e) = \underbrace{\begin{pmatrix} \psi_{1+}(\mathbf{R}_e) & \psi_{2+}(\mathbf{R}_e) \\ \psi_{3-}(\mathbf{R}_e) & \psi_{4-}(\mathbf{R}_e) \end{pmatrix}}_{\text{Flavor 1,2}} \Bigg] \text{4-component spinor}$$

we obtain 2-flavor Dirac fermions.

Discrete symmetries

[Li, Wang, You (2024)]

[Aoki, Kikukawa, Takemoto (2025)]

3+1 D staggered fermion system possesses the following discrete symmetries :

- Parity symmetry \mathbb{Z}_2^P
- Reflection symmetry $\mathbb{Z}_2^{R_i}$
- Time-reversal symmetry \mathbb{Z}_2^T
- Charge conjugation symmetry \mathbb{Z}_2^C
- Shift symmetry $\mathbb{Z}_{L_i}^{S_i}$

Lattice size along the i th direction

Action on the staggered fermions

$$P : P\chi(\mathbf{r})P^{-1} = \epsilon(\mathbf{r})\chi(-\mathbf{r}), \quad \text{with } P = R_x R_y R_z,$$

$$R_i : R_i\chi(\mathbf{r})R_i^{-1} = (-1)^{r_i}\chi(\tilde{R}_i(\mathbf{r})) = \epsilon(\mathbf{r})\eta_i(\mathbf{r})\zeta_i(\mathbf{r})\chi(\tilde{R}_i(\mathbf{r})), \quad \tilde{R}_i(r_1, \dots, r_i, \dots, r_3) = (r_1, \dots, -r_i, \dots, r_3)$$

$$T : T\chi(\mathbf{r})T^{-1} = \epsilon(\mathbf{r})\chi(\mathbf{r}), \quad T i T^{-1} = -i,$$

$$\epsilon(\mathbf{r}) = (-1)^{x+y+z}$$

$$C : C\chi(\mathbf{r})C^{-1} = \chi^\dagger(\mathbf{r}),$$

$$\zeta_i(\mathbf{r}) := (-1)^{\sum_{k=i+1}^3 r_k}$$

$$S_i : S_i\chi(\mathbf{r})S_i^{-1} = \zeta_i(\mathbf{r})\chi(\mathbf{r} + \hat{i}), \quad S_i = T_{\hat{i}}^{(a)}T_{\hat{i}}^{(b)},$$

Discrete symmetries

In the continuum limit,

- Parity symmetry \mathbb{Z}_2^P
- Reflection symmetry $\mathbb{Z}_2^{R_i}$
- Time-reversal symmetry \mathbb{Z}_4^T
- Charge conjugation symmetry \mathbb{Z}_2^C
- Shift symmetry $\mathbb{Z}_{L_i}^{S_i}$

where $\mathcal{T} := -S_z S_x T$: $\mathcal{T}\chi(\mathbf{r})\mathcal{T}^{-1} = -(-1)^x \chi(\mathbf{r} + \hat{x} + \hat{z})$, $\mathcal{T}i\mathcal{T}^{-1} = -i$,
 $\mathcal{C} := S_y C$: $\mathcal{C}\chi(\mathbf{r})\mathcal{C}^{-1} = (-1)^z \chi^\dagger(\mathbf{r} + \hat{y})$.

$$\begin{aligned}
 P: \quad \Psi(\mathbf{R}_e) &\rightarrow (\beta \otimes \mathbb{1}) \Psi(-\mathbf{R}_e), & \text{with } P = R_x R_y R_z, \\
 R_x: \quad \Psi(\mathbf{R}_e) &\rightarrow (-\beta \gamma_5 \alpha_1 \otimes \sigma_1^T) \Psi(\tilde{R}_x(\mathbf{R}_e)) = (-i\beta \alpha_2 \alpha_3 \otimes \sigma_1^T) \Psi(\tilde{R}_x(\mathbf{R}_e)), \\
 R_y: \quad \Psi(\mathbf{R}_e) &\rightarrow (\beta \gamma_5 \alpha_2 \otimes \sigma_2^T) \Psi(\tilde{R}_y(\mathbf{R}_e)) = (i\beta \alpha_3 \alpha_1 \otimes \sigma_2^T) \Psi(\tilde{R}_y(\mathbf{R}_e)), \\
 R_z: \quad \Psi(\mathbf{R}_e) &\rightarrow (-\beta \gamma_5 \alpha_3 \otimes \sigma_3^T) \Psi(\tilde{R}_z(\mathbf{R}_e)) = (-i\beta \alpha_1 \alpha_2 \otimes \sigma_3^T) \Psi(\tilde{R}_z(\mathbf{R}_e)),
 \end{aligned}$$

$$\begin{aligned}
 T: \quad \Psi(\mathbf{R}_e) &\rightarrow (-\alpha_1 \alpha_3 \otimes \sigma_2^T) \Psi(\mathbf{R}_e), \\
 C: \quad \Psi(\mathbf{R}_e) &\rightarrow -(\beta \alpha_3 \alpha_1 \otimes i\sigma_2^T) \Psi^*(\mathbf{R}_e), \\
 \mathcal{T}: \quad \Psi(\mathbf{R}_e) &\rightarrow (\alpha_1 \alpha_3 \otimes \mathbb{1}) \Psi(\mathbf{R}_e), \\
 \mathcal{C}: \quad \Psi(\mathbf{R}_e) &\rightarrow i(\beta \alpha_2 \otimes \mathbb{1}) \Psi^*(\mathbf{R}_e),
 \end{aligned}$$

$$S_i: \quad \Psi(\mathbf{R}_e) \rightarrow (\gamma_5 \otimes \sigma_i^T) \Psi(\mathbf{R}_e), \quad \text{with rotation angle } \frac{\pi}{2}.$$



Onsager algebra (Ons3)

[Aoki, Kikukawa, Takemoto (2025)]

$$T_g^{(b)} := (T_x^{(b)})^{n_x} (T_y^{(b)})^{n_y} (T_z^{(b)})^{n_z} \quad g(\mathbf{r}) = \mathbf{r} + n_x \hat{x} + n_y \hat{y} + n_z \hat{z}$$

$$Q_g = T_g^{(b)} Q_0 (T_g^{(b)})^{-1} = \frac{i}{2} \sum \zeta_g(\mathbf{r}) a_{\mathbf{r}} b_{g(\mathbf{r})},$$

$$G_{g,h} = \frac{i}{2} \sum (\zeta_g a_{\mathbf{r}} a_{g(\mathbf{r})} + \zeta_h b_{\mathbf{r}} b_{g(\mathbf{r})})$$

$$[Q_g, Q_h] = \sqrt{-1} G_{g^{-1}h, hg^{-1}},$$

$$[Q_g, G_{h,k}] = \sqrt{-1} (Q_{gh^{-1}} + Q_{k^{-1}g} - Q_{gh} - Q_{kg}),$$

$$[G_{g_1, h_1}, G_{g_2, h_2}] = \sqrt{-1} (G_{g_2 g_1, h_1 h_2} - G_{g_2^{-1} g_1, h_1 h_2^{-1}} - G_{g_1 g_2, h_2 h_1} + G_{g_1 g_2^{-1}, h_2^{-1} h_1}).$$