Lieb-Schultz-Mattis anomalies in dissipative Majorana chains

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Based on joint work with Hung-Hsuan Teh (Nagoya U.) (to appear as arXiv:2607.*****).

2026/6/30

Question and Main Answer

Question

Can we extend the notion of anomalies to open quantum systems?

Main answer

After applying the operator-state map, the anomaly becomes a projective algebra between weak and strong symmetries:

$$\boxed{(-1)^{F_{\mathrm{w}}} T^{\pm}_{\mathrm{RR}} = -T^{\pm}_{\mathrm{RR}} (-1)^{F_{\mathrm{w}}}.}$$

Continuum limit

dissipative lattice Majorana anomaly $\longrightarrow$ anomaly in a Majorana CFT with dissipation.

Outline

Majorana Chain and Symmetries

We consider Kitaev's Majorana-chain Hamiltonian.

Kitaev's Majorana chain [Kitaev, 02]

$$H_{\mathrm{K}} = \frac{i}{2}\sum_{\ell=1}^{L-1}\chi_{\ell+1}\chi_\ell \pm \frac{i}{2}\chi_1\chi_L, \qquad \{\chi_\ell, \chi_{\ell'}\} = 2\delta_{\ell\ell'}.$$

This is the staggered Majorana-fermion Hamiltonian.

This Hamiltonian is invariant under fermion parity $(-1)^F$ and one-site translation T.

Fermion parity

$$(-1)^F : \chi_\ell \mapsto -\chi_\ell.$$

One-site translation

$$T : \chi_\ell \mapsto \chi_{\ell+1}.$$

Translation symmetry can be broken, for example, by the staggered mass term $\sum_\ell (-1)^\ell \chi_{\ell+1}\chi_\ell$.

Continuum Majorana CFT

The continuum counterpart is the $1+1$-dimensional massless Majorana fermion theory.

$$\mathcal{L} = i\chi_{\mathrm{L}}(\partial_t - \partial_x)\chi_{\mathrm{L}} + i\chi_{\mathrm{R}}(\partial_t + \partial_x)\chi_{\mathrm{R}}.$$

$$H = i\int dx\,\left(\chi_{\mathrm{L}}\partial_x\chi_{\mathrm{L}} - \chi_{\mathrm{R}}\partial_x\chi_{\mathrm{R}}\right).$$

This is the continuum Majorana Hamiltonian.

This system is invariant under fermion parity $(-1)^F$ and chiral fermion parity $(-1)^{F_L}$.

Fermion parity

$$(-1)^F : \chi_{\mathrm{L}}, \chi_{\mathrm{R}} \mapsto -\chi_{\mathrm{L}}, -\chi_{\mathrm{R}}.$$

Chiral fermion parity

$$(-1)^{F_{\mathrm{L}}} : \chi_{\mathrm{L}} \mapsto -\chi_{\mathrm{L}}, \quad \chi_{\mathrm{R}} \mapsto \chi_{\mathrm{R}}.$$

Chiral fermion parity can be broken, for example, by the Majorana mass term $\chi_L(x)\chi_R(x)$.

Translation and Chiral Fermion Parity

The momentum-space creation/annihilation operator can be written as

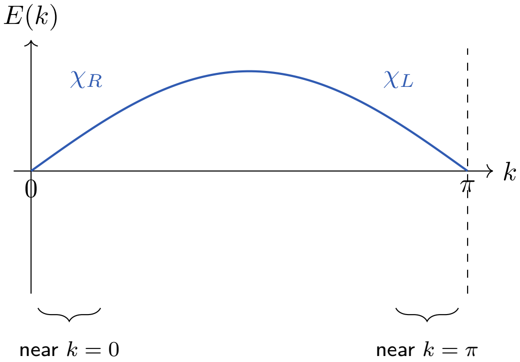
$$d_k \propto \sum_\ell e^{\frac{2\pi i k \ell}{L}} \chi_\ell.$$

Translation acts on momentum modes as

$$T d_k T^{-1} = e^{2\pi i k/L} d_k.$$

Near $k = 0$, this gives the usual momentum phase.

Near $k = \pi$, the same operation gives an additional -1, which realizes $(-1)^{F_{\mathrm{L}}}$ [Shao–Seiberg, 23].



't Hooft Anomalies in QFT

We start with the continuum theory, where the notion of an anomaly is well established, especially for internal symmetries.

A global symmetry has a 't Hooft anomaly if it cannot be consistently gauged.

- In quantum mechanics, gauging a finite symmetry is essentially projection onto singlets.
- In quantum field theory, gauging also requires summing over twisted sectors.
- Anomalies sometimes appear as unavoidable phases in the algebra of symmetry defects/operators.

Anomaly in the Majorana CFT

The $(-1)^F \times (-1)^{F_L}$ symmetry of the Majorana CFT has a mixed anomaly. With anti-periodic boundary conditions (the NSNS sector), $(-1)^F$ and $(-1)^{F_L}$ commute. However, with periodic boundary conditions (the RR sector), the two operators anticommute:

$$(-1)^F(-1)^{F_\mathrm{L}} = -(-1)^{F_\mathrm{L}}(-1)^F.$$

- Therefore, in the RR sector, one cannot simultaneously project onto states invariant under both symmetries.

Lattice Origin of the Anomaly

To write the translation operator explicitly in the RR sector, define the adjacent-Majorana swap operator by

$$S_{\ell,\ell+1} = \exp\left(\frac{\pi}{4}\chi_\ell\chi_{\ell+1}\right).$$

Then the one-site translation can be represented schematically as

$$T_{\mathrm{RR}} \;\propto\; \chi_1\, S_{12}S_{23}\cdots S_{L-1,L}.$$

Because T_{RR} contains an odd number of Majorana operators,

$$(-1)^F T_{\mathrm{RR}} = -T_{\mathrm{RR}}(-1)^F.$$

A projective algebra in a twisted sector enforces degeneracy or nontrivial IR behavior. This is the $\mathbb{Z}_2$ analogue of the Lieb-Schultz-Mattis-type anomaly.

Strategy for Open Systems

The theory of open quantum systems is most developed for finite-dimensional quantum systems, whereas anomalies are best established in continuum QFT. Therefore, using the matching

$$\text{lattice LSM anomaly} \quad \longleftrightarrow \quad \text{chiral fermion-parity anomaly in the CFT.}$$

in Hamiltonian systems, we extend the notion of an anomaly to open quantum systems by first considering the Majorana chain with dissipation and then taking the continuum limit.

Open-system strategy

$$\begin{array}{ccc}
\text{closed Majorana chain} & \longrightarrow & \text{Majorana CFT anomaly} \\
\downarrow \text{ add dissipation} & & \downarrow \text{ dissipation} \\
\text{open Majorana chain} & \longrightarrow & \text{open-system anomaly}
\end{array}$$

Now we introduce open dynamics through Lindblad generators.

Lindblad Master Equation

For a closed system,

$$\frac{d\rho}{dt} = -i[H, \rho].$$

For a Markovian open system, the evolution is described by

$$\frac{d\rho}{dt} = \mathcal{L}(\rho) = -i[H, \rho] + \sum_m \left(L_m \rho L_m^\dagger - \frac{1}{2}\{L_m^\dagger L_m, \rho\} \right).$$

- H: Hamiltonian part.
- L_m: jump operators describing dissipation.
- $L_m \rho L_m^\dagger$: quantum jump term.
- $-\frac{1}{2}\{L_m^\dagger L_m, \rho\}$: trace-preserving correction.

The finite-time evolution is

$$\rho(t) = e^{t\mathcal{L}}(\rho(0)),$$

and $e^{t\mathcal{L}}$ is a quantum channel.

Operator-State Map

We use vectorization:

$$\rho = \sum_{ij} \rho_{ij} |i\rangle\langle j| \quad \longmapsto \quad |\rho\rangle = \sum_{ij} \rho_{ij} |i\rangle_+ |j\rangle_- .$$

More precisely, we choose a maximally entangled state $|I\rangle \in \mathcal{H}_+ \otimes \mathcal{H}_-$ and map

$$\rho \longmapsto \rho^+ |I\rangle$$

Then the Lindblad equation becomes

$$\frac{d}{dt}|\rho\rangle = \mathcal{L}|\rho\rangle.$$

For parity-even jump operators, schematically

$$\mathcal{L} = -i(H^+ - H^-) + \sum_m \left(L_m^+ L_m^- - \frac{1}{2} L_m^{+\dagger} L_m^+ - \frac{1}{2} L_m^{-\dagger} L_m^- \right).$$

This is a non-Hermitian Hamiltonian on the doubled Hilbert space, so we can use the same techniques as in Hamiltonian systems to study anomalies.

Strong and Weak Symmetries

In open quantum systems, there are two notions of symmetry.

Strong Symmetry

$$U^+\mathcal{L}(U^+)^{-1} = \mathcal{L}, \qquad U^-\mathcal{L}(U^-)^{-1} = \mathcal{L}.$$

The symmetry acts independently on the two contours. This is the symmetry familiar from Hamiltonian systems.

Weak Symmetry

$$U_{\mathrm{w}} \propto U^+U^-, \qquad U_{\mathrm{w}}\mathcal{L}U_{\mathrm{w}}^{-1} = \mathcal{L}.$$

New feature: open systems allow projective algebras between weak and strong symmetries. Such an LSM anomaly is peculiar to open quantum systems.

Modular Conjugation Symmetry

There is another symmetry that is intrinsic to the doubled formulation. Physical density matrices satisfy $\rho^\dagger = \rho$. In the doubled Hilbert space $\mathcal{H}_+ \otimes \mathcal{H}_-$, this becomes an antiunitary symmetry called modular conjugation [cf: Kawabata Kulkarni Li TN Ryu, 22]:

Modular conjugation

$$\mathcal{J}\chi_\ell^+\mathcal{J}^{-1} = \chi_\ell^-, \qquad \mathcal{J}\chi_\ell^-\mathcal{J}^{-1} = \chi_\ell^+, \qquad \mathcal{J}z\mathcal{J}^{-1} = z^*.$$
$$\mathcal{J}^2 = 1, \qquad \mathcal{J}\mathcal{L}\mathcal{J}^{-1} = \mathcal{L}.$$

- This symmetry is built into any Hermiticity-preserving evolution.
- It exchanges ket and bra contours.
- Spectrally, it pairs complex eigenvalues by complex conjugation.

Open Majorana Chain in doubled Hilbert Space

Doubled lattice fermions

The operator-state map turns the open chain into two Majorana chains:

$$\chi_\ell^+ \quad \text{and} \quad \chi_\ell^-, \qquad \{\chi_\ell^s, \chi_{\ell'}^{s'}\} = 2\delta_{\ell\ell'}\delta_{ss'}.$$

Lindbladian as a doubled lattice Hamiltonian

$$\mathcal{L} = -i(H^+ - H^-) + \mathcal{D},$$

where $\mathcal{D}$ contains the jump terms coupling the $+$ and $-$ contours.

- From the lattice viewpoint, this is a non-Hermitian doubled Majorana lattice model.
- The important question is how the closed-chain translation anomaly appears in this doubled system.

Symmetry Operators on the Doubled Lattice

As we have seen, open quantum systems have weak and strong symmetries. For fermion parity $(-1)^F$ and Majorana translations T, we have weak and strong versions:

Fermion parities

Strong fermion parities:

$$(-1)^{F^+} : \chi^+_\ell \mapsto -\chi^+_\ell, \qquad (-1)^{F^-} : \chi^-_\ell \mapsto -\chi^-_\ell .$$

Weak fermion parity:

$$(-1)^{F_{\mathrm{w}}} \propto (-1)^{F^+}(-1)^{F^-} .$$

Translations

Strong translations:

$$T^+ : \chi^+_\ell \mapsto \chi^+_{\ell+1}, T^- : \chi^-_\ell \mapsto \chi^-_{\ell+1}.$$

Weak translation:

$$T_{\mathrm{w}} \propto T^+T^- .$$

The doubled system has both strong and weak symmetry algebras, and both can carry projective phases.

Examples: Strong Translation-Symmetric Jumps

A jump operator L_m has strong translation symmetry if

$$TL_mT^{-1} = L_m$$

so that the Lindbladian has strong translation symmetry. Examples of strong translation-symmetric jump terms are:

Linear zero-mode jump

$$L_0 = \sqrt{\Gamma}\sum_{\ell=1}^{L}\chi_\ell.$$

- This is not invariant under strong fermion parity!
- It couples only to zero modes.

Even translation-invariant jumps

$$L_H = \sqrt{\Gamma}\,H_{\mathrm{K}}, \qquad L_{\mathrm{H}} = \sqrt{\gamma}\sum_{\ell} i\chi_{\ell+1}\chi_\ell.$$

- Energy-dephasing/bond-dephasing type.
- Invariant under strong fermion parity $(-1)^{F^+}$.

Main Lattice Results: Even L with Periodic Boundary Conditions

$\mathrm{RR}^{+}\mathrm{RR}^{-}$ sector

For $L = 2N$ with periodic boundary conditions on both contours, phases can be chosen so that

$$(T^{+}_{\mathrm{RR}})^{L} = (T^{-}_{\mathrm{RR}})^{L} = (T^{w}_{\mathrm{RR}})^{L} = 1.$$

Open-system LSM anomaly

$$\boxed{(-1)^{F_{\mathrm{w}}} T^{\pm}_{\mathrm{RR}} = -T^{\pm}_{\mathrm{RR}} (-1)^{F_{\mathrm{w}}}}$$

- Open-system anomaly between weak fermion parity and strong translation.
- This has no direct analogue in ordinary closed Hamiltonian systems.

Lattice Origin of the Sign

For the $+$ contour, the RR translation operator has the same fermionic structure as in the closed chain:

$$T^{+}_{\mathrm{RR}} \;\propto\; \chi^{+}_{1}\,(1+\chi^{+}_{1}\chi^{+}_{2})\cdots(1+\chi^{+}_{L-1}\chi^{+}_{L})(-1)^{F_{-}}.$$

Thus it is odd under $(-1)^{F^{+}}$:

$$(-1)^{F^{+}}T^{+}_{\mathrm{RR}} = -T^{+}_{\mathrm{RR}}(-1)^{F^{+}}.$$

This is essentially the same as in Hamiltonian systems.

Weak parity contains the single-contour parity

$$(-1)^{F_{\mathrm{w}}} = (-1)^{F^{+}}(-1)^{F^{-}}, \qquad [(-1)^{F^{-}}, T^{+}_{\mathrm{RR}}] = 0.$$

Therefore

$$(-1)^{F_{\mathrm{w}}}T^{+}_{\mathrm{RR}} = -T^{+}_{\mathrm{RR}}(-1)^{F_{\mathrm{w}}}.$$

The same argument gives the relation for T^{-}_{RR}.

Continuum limit : Majorana CFT with dissipation

Open Majorana chain $\longrightarrow$ doubled Majorana CFT

The continuum theory has independent $+$ and $-$ contours, with momenta $P^{\pm}$ and chiral parities $(-1)^{F_L^{\pm}}$.

lattice	continuum CFT
$T_{\mathrm{RR}}^{\pm}$	$(-1)^{F_L^{\pm}}$
$(-1)^{F_{\mathrm{w}}}$	$(-1)^{F^+}(-1)^{F^-}$
weak twisted boundary conditions	simultaneous change of sectors

Strong twisted boundary conditions can be considered [Kawabata, Sohal, and Ryu, 23], but they do not give a physical Lindbladian.

RR Zero Modes in the Doubled CFT

We now see the same commutation relation in the RR sector of the dissipative Majorana CFT.

$\mathrm{RR}^{+}\mathrm{RR}^{-}$ sector

Each contour has left- and right-moving zero modes:

$$\chi^{+}_{L,0},\ \chi^{+}_{R,0}, \qquad \chi^{-}_{L,0},\ \chi^{-}_{R,0}.$$

Chiral parity is fermionic in the RR sector

Schematically,

$$(-1)^{F_L^{+}} \propto \sqrt{2}\,\chi^{+}_{L,0}, \qquad (-1)^{F_L^{-}} \propto \sqrt{2}\,\chi^{-}_{L,0}.$$

Therefore the two chiral parities anticommute:

$$(-1)^{F_L^{+}}(-1)^{F_L^{-}} = -(-1)^{F_L^{-}}(-1)^{F_L^{+}}.$$

Continuum Reproduction of the Dissipative Majorana Chain Anomaly

Define the diagonal weak chiral parity by

$$(-1)^{F_L}_{\mathrm{w}} = -i(-1)^{F_L^+}(-1)^{F_L^-}.$$

Using the RR zero-mode algebra,

$$(-1)^{F_L}_{\mathrm{w}}(-1)^{F^+} = -(-1)^{F^+}(-1)^{F_L}_{\mathrm{w}}.$$

Using the lattice-continuum relation

$$T^+_{\text{lattice}} \longrightarrow (-1)^{F_L^+},$$

we reproduce

$$\boxed{(-1)^{F_{\mathrm{w}}} T^+_{\mathrm{RR}} = -T^+_{\mathrm{RR}}(-1)^{F_{\mathrm{w}}}}.$$

The same argument applies to the $-$ contour.

Lindbladian for the Zero-Mode Space

To see how the anomaly constrains the Lindblad spectrum, consider a zero-mode dissipator. The zero-mode part of the Lindbladian is

$$\mathcal{L}_0 = \frac{m}{2}\chi^+_{R,0}\chi^+_{L,0} + \frac{m}{2}\chi^-_{R,0}\chi^-_{L,0} - i\Gamma\chi^+_{R,0}\chi^-_{R,0} - \Gamma$$

where $i\chi^+_{R,0}\chi^+_{L,0}$ is the mass term. The spectrum is

$$\lambda = 0,\ -2\Gamma,\ -\Gamma - \sqrt{\Gamma^2 - m^2},\ -\Gamma + \sqrt{\Gamma^2 - m^2}$$

Imposing strong fermion parity gives $m = 0$.

$$\longrightarrow \lambda = 0,\ -2\Gamma,\ -2\Gamma,\ 0$$

The spectrum has two degeneracies protected by the anomaly.

Odd Chains and the Mod-8 Phase

Strong-sector fractional momentum

For an odd Majorana chain, the low-energy sector is NSR or RNS. Strong translations retain the fractional anomaly:

$$(T_{\mathrm{odd}}^{\pm})^{L} \sim e^{\pm 2\pi i/16}.$$

Weak diagonal cancellation

The weak translation is diagonal, $T_{\mathrm{w}} = T^{+}T^{-}$. Ket and bra phases can cancel:

$$P_{\mathrm{w}} = P^{+} - P^{-} \in \frac{1}{2}\mathbb{Z}.$$

Strong translations remember the mod-8 anomaly, while weak diagonal translation can become anomaly-free.

Summary

1. The closed Majorana chain has the lattice translation anomaly

$$(-1)^F T_{\mathrm{RR}} = -T_{\mathrm{RR}}(-1)^F.$$

2. Lindblad dynamics maps to a doubled lattice-fermion problem on $\mathcal{H}_+ \otimes \mathcal{H}_-$. Open systems distinguish strong and weak symmetries.
3. The main open-chain algebra is

$$\boxed{(-1)^{F_{\mathrm{w}}} T^{\pm}_{\mathrm{RR}} = -T^{\pm}_{\mathrm{RR}}(-1)^{F_{\mathrm{w}}}}.$$

4. The doubled Majorana CFT reproduces this lattice algebra.

Thank you.