Exotic theta terms in 2+1d fractonic field theory [1]

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[1] This talk discusses neither fermions nor chiral symmetries. But lattice systems do appear.

Fracton phases and exotic field theories

Fracton phases are characterized by

- ground-state degeneracy that depends nontrivially on the system size
- excitations with restricted mobility (fractons)

Motivated by this, people have studied a new class of QFTs. [Pretko, Slagle, . . .]

Such theories exhibit unusual continuum behavior:

- discontinuous field configurations and UV/IR mixing [Gorantla-Lam-Seiberg-Shao, '21]
  - Discontinuous correlation functions in the continuum limit
  - Non-commutativity of continuum and thermodynamic limits

Question

What happens to topological terms in the presence of such discontinuities?

Answer

Discontinuities activate new topological terms!

We study this phenomenon in the simplest setting: the 2+1d ϕ-theory.

The XY-plaquette model

- 2d square lattice
- ϕ_s : $\mathrm{U}(1)$-valued site variable, $\phi_s \sim \phi_s + 2\pi$
- π_s : the momentum conjugate to ϕ_s, $[\phi_s, \pi_{s'}] = i\delta_{s,s'}$

The 2+1d XY-plaquette model $(U \gg K > 0)$ [Paramekanti-Balents-Fisher,'02]

$$H = \frac{K}{2}\sum_s \pi_s^2 - U\sum_s \cos(\phi_{s+\hat{x}+\hat{y}} - \phi_{s+\hat{x}} - \phi_{s+\hat{y}} + \phi_s).$$

- $(\Delta_x\Delta_y\phi)_s = \phi_{s+\hat{x}+\hat{y}} - \phi_{s+\hat{x}} - \phi_{s+\hat{y}} + \phi_s \sim 0 \bmod 2\pi$ is energetically enforced.
  - weaker than $(\Delta_\mu\phi)_s = \phi_{s+\hat{\mu}} - \phi_s \sim 0 \bmod 2\pi$, which guarantees the smoothness of field configurations.
- Discontinuous field configurations still survive in the low-energy physics.
- This feature is not captured by the conventional compact boson.

The ϕ-theory

Taking the continuum limit carefully, we obtain the ϕ-theory [Seiberg-Shao,'20]

$$S_0 = \int d\tau dx dy \left\{ \frac{\mu_0}{2}(\partial_\tau \phi)^2 + \frac{1}{2\mu}(\partial_x \partial_y \phi)^2 \right\}.$$

Exotic global symmetries:

- Momentum subsystem symmetry

$$\phi(\tau, x, y) \to \phi(\tau, x, y) + c_x(x) + c_y(y).$$

- Winding subsystem symmetry

$$\partial_\tau(\partial_x \partial_y \phi) = \partial_x \partial_y(\partial_\tau \phi) \longrightarrow \partial_\tau J_w^{xy} - \partial_x \partial_y J_w^\tau = 0$$

$$J_w^\tau = \frac{1}{2\pi}\partial_\tau \phi, \ \ J_w^{xy} = \frac{1}{2\pi}\partial_x \partial_y \phi$$

$$\begin{aligned} &\partial_\tau J_w^{xy} - \partial_x \partial_y J_w^\tau = 0 \\ \longrightarrow\ &\partial_\tau J_w^{xy} - \partial_y(\partial_x J_w^\tau) = 0 \\ \longrightarrow\ &\text{the continuity equation in the } \tau y\text{-plane} \end{aligned}$$

This means that one can define a winding charge for a closed loop C_{x_0} lying in the plane $x = x_0$

$$\begin{aligned} Q_w^x(x_0; C_{x_0}) &= \oint_{C_{x_0}} (\partial_x J_w^\tau \, d\tau + J_w^{xy} \, dy) \\ &= \frac{1}{2\pi} \oint_{C_{x_0}} (\partial_x \partial_\tau \phi \, d\tau + \partial_x \partial_y \phi \, dy) \end{aligned}$$

The charge operator is restricted on a plane $x = x_0$.
Winding charges on different τy-planes are independent due to field discontinuities.

Modified Villain lattice model

The modified Villain formulation of the ϕ-theory [Gorantla-Lam-Seiberg-Shao,'21]
3d Euclidean lattice model (τ, x, y)

- ϕ : real-valued site variable
- ϕ^{xy} : real-valued cube variable
- n_τ : integer-valued, living on τ-links
- n_{xy} : integer-valued, living on xy-plaquettes

$$
\begin{aligned}
S_0[\phi, n_\tau, n_{xy}] = & \frac{\beta_0}{2} \sum_n (\Delta_\tau \phi + 2\pi n_\tau)^2(n) + \frac{\beta}{2} \sum_n (\Delta_x \Delta_y \phi + 2\pi n_{xy})^2(n) \\
& + i \sum_n \phi^{xy}(n)(\Delta_\tau n_{xy} - \Delta_x \Delta_y n_\tau)(n).
\end{aligned}
$$

Gauge symmetry $(k, k' \in \mathbb{Z})$

$$
\phi \to \phi + 2\pi k,\ n_\tau \to n_\tau - \Delta_\tau k,\ n_{xy} \to n_{xy} - \Delta_x \Delta_y k,\ \phi^{xy} \to \phi^{xy} + 2\pi k'
$$

$$
\begin{aligned}
S_0[\phi, n_\tau, n_{xy}] =& \frac{\beta_0}{2}\sum_n (\Delta_\tau \phi + 2\pi n_\tau)^2(n) + \frac{\beta}{2}\sum_n (\Delta_x\Delta_y\phi + 2\pi n_{xy})^2(n) \\
&+ i\sum_n \phi^{xy}(n)(\Delta_\tau n_{xy} - \Delta_x\Delta_y n_\tau)(n).
\end{aligned}
$$

The model has a momentum subsystem symmetry: $\phi \to \phi + c^x(x) + c^y(y)$.

On the other hand, unlike the XY-plaquette model before taking the continuum limit, we also have an exact winding subsystem symmetry: $\phi^{xy}(n) \to \phi^{xy}(n) + \hat{c}^x(x) + \hat{c}^y(y)$

In fact, the Lagrange multiplier ϕ^{xy} imposes

$$
\begin{aligned}
&\Delta_\tau n_{xy} - \Delta_x\Delta_y n_\tau = 0. \\
\longrightarrow &\Delta_\tau(\Delta_x\Delta_y\phi + 2\pi n_{xy}) = \Delta_x\Delta_y(\Delta_\tau\phi + 2\pi n_\tau) \\
\longrightarrow &\text{ lattice version of the continuity equation } (\partial_\tau(\partial_x\partial_y\phi) = \partial_x\partial_y(\partial_\tau\phi)) \\
\longrightarrow &\text{ exact winding subsystem symmetry}
\end{aligned}
$$

If we insert an operator $e^{i\phi^{xy}(n_0)}$,

$$
\Delta_\tau n_{xy}(n_0) - \Delta_x\Delta_y n_\tau(n_0) = +1 \longrightarrow \; e^{i\phi^{xy}(n_0)} \text{creates a vortex at } n_0
$$

Theta angles are important parameters in quantum field theory.
They do not affect the equation of motion. (definition)

In fractonic field theories, discontinuous field configurations appear to obscure the topology of the fields, obstructing the existence of topological terms.

However, the opposite effect also occur:

- Topological terms that are originally regarded as trivial, in the sense that they always vanish, can become nontrivial due to field discontinuities.

$\longrightarrow$ New theta terms

Previous works on topological terms in fractonic field theories

- [Pretko, '17] : theta terms and Witten effects in 3+1d tensor gauge theories
- [Bedogna-Mancani, '26] : introduced a theta term (the bulk theta term) in the 2+1d ϕ-theory in the continuum

Despite extensive studies of topological terms and fractonic systems independently, their interplay remains largely unexplored.

In [YF, '26] , we discuss the following two theta terms both in the continuum and on the lattice.

1. Bulk theta term [Bedogna-Mancani,'26] :

$$S^{xy}_{\text{top}}[\theta^{xy};\phi] = -\frac{i\theta^{xy}}{2\pi^2}\int d\tau dxdy\,\partial_\tau\phi\,\partial_x\partial_y\phi$$

   - $\theta^{xy} \sim \theta^{xy} + 2\pi$

2. Foliated theta term (along the x-direction)

$$S^{x}_{\text{top}}[\theta^{x};\phi] = -\frac{i\lambda^x}{(2\pi)^2}\int d\tau dxdy\,\theta^x(x)\left\{\partial_x(\partial_\tau\phi)\,\partial_x(\partial_x\partial_y\phi) - (\partial_x\partial_y\phi)\,\partial_x^2(\partial_\tau\phi)\right\}$$

   - $\theta^x(x)$: an x-dependent theta angle
   - The foliated theta term admits a spatially varying theta parameter without changing the equation of motion.

- Both of them vanish when computed for configurations without discontinuity.

Bulk theta term

Continuum [Bedogna-Mancani,'26] :

$$S^{xy}_{\rm top}[\theta^{xy};\phi] = -\frac{i\theta^{xy}}{2\pi^2}\int d\tau dxdy\, \partial_\tau\phi\, \partial_x\partial_y\phi$$

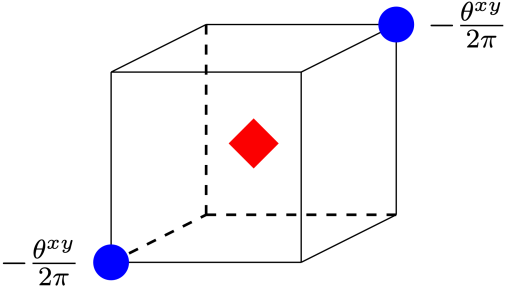
Lattice [YF, '26] :

$$\begin{aligned}
&S^{xy}_{\rm top}[\theta^{xy};\phi,n_\tau,n_{xy}]\\
=&-\frac{i\theta^{xy}}{(2\pi)^2}\sum_n \{(\Delta_\tau\phi+2\pi n_\tau)(n)(\Delta_x\Delta_y\phi+2\pi n_{xy})(x+\hat{\tau})\\
&+(\Delta_x\Delta_y\phi+2\pi n_{xy})(n)(\Delta_\tau\phi+2\pi n_\tau)(n+\hat{x}+\hat{y})\}\\
=&-i\theta^{xy}\sum_n(n_\tau(n)n_{xy}(n+\hat{\tau})+n_{xy}(n)n_\tau(n+\hat{x}+\hat{y}))\\
&+i\sum_n \underbrace{(\Delta_\tau n_{xy}-\Delta_x\Delta_y n_\tau)(n)}\left(\frac{\theta^{xy}}{2\pi}\phi(n+\hat{\tau}+\hat{x}+\hat{y})+\frac{\theta^{xy}}{2\pi}\phi(n)\right)
\end{aligned}$$

- $\theta^{xy}\sim\theta^{xy}+2\pi$.

The second term induces an interesting effect.

$$
\begin{aligned}
&S_{\text{top}}^{xy}[\theta^{xy};\phi,n_\tau,n_{xy}] \\
&= -i\theta^{xy}\sum_n (n_\tau(n)n_{xy}(n+\hat\tau)+n_{xy}(n)n_\tau(n+\hat x+\hat y)) \\
&\quad + i\sum_n (\Delta_\tau n_{xy}-\Delta_x\Delta_y n_\tau)(n)\left(\frac{\theta^{xy}}{2\pi}\phi(n+\hat\tau+\hat x+\hat y)+\frac{\theta^{xy}}{2\pi}\phi(n)\right)
\end{aligned}
$$



- Witten effect : a vortex induces momentum charges.

Foliated theta term

Continuum :

$$S^{x}_{\mathrm{top}}[\theta^x;\phi] = -\frac{i\lambda^x}{(2\pi)^2}\int d\tau dx dy\, \theta^x(x)\left\{\partial_x(\partial_\tau\phi)\,\partial_x(\partial_x\partial_y\phi) - (\partial_x\partial_y\phi)\,\partial_x^2(\partial_\tau\phi)\right\}$$

The term does not affect the classical equation of motion for any $\theta^x(x)$.

This term can be constructed on the lattice in the following way.

For the standard compact scalar fields ϕ_1,ϕ_2 with $\phi_i \sim \phi_i + 2\pi$, one can define a topological charge

$$Q = \frac{1}{(2\pi)^2}\int d\phi_1 \wedge d\phi_2 \in \mathbb{Z}.$$

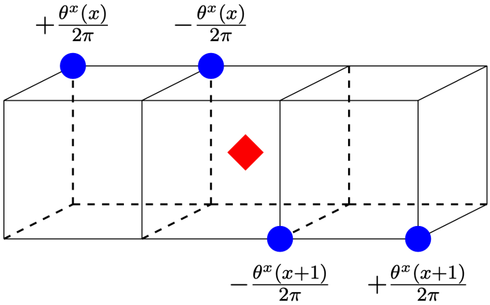
As mentioned earlier, independent winding currents can be defined on the plane $x = x_0$ and the plane $x = x_0 + a$. So, one can couple them.

Lattice :

$$
\begin{aligned}
&S^x_{\mathrm{top}}[\theta^x, \phi, n_\tau, n_{xy}] \\
&= + \frac{i}{(2\pi)^2} \sum_x \theta^x(x) \sum_{\tau, y} \{ \Delta_x(\Delta_\tau \phi + 2\pi n_\tau)(n)(\Delta_x \Delta_y \phi + 2\pi n_{xy})(n - \hat{x} + \hat{\tau}) \\
&\quad - (\Delta_x \Delta_y \phi + 2\pi n_{xy})(n) \Delta_x(\Delta_\tau \phi + 2\pi n_\tau)(n - \hat{x} + \hat{y}) \} \\
&= + i \sum_x \theta^x(x) \sum_{\tau, y} \{ \Delta_x n_\tau(n)\, n_{xy}(n - \hat{x} + \hat{\tau}) - n_{xy}(n)\, \Delta_x n_\tau(n - \hat{x} + \hat{y}) \} \\
&\quad - i \sum_{\tau, x, y} \underset{\sim\sim\sim}{(\Delta_\tau n_{xy} - \Delta_x \Delta_y n_\tau)(n)} \left(\frac{\theta^x(x+1)}{2\pi} \Delta_x \phi(n + \hat{x}) - \frac{\theta^x(x)}{2\pi} \Delta_x \phi(n + \hat{\tau} - \hat{x} + \hat{y}) \right)
\end{aligned}
$$

One can again read off the Witten effect from the second term.

$$
\begin{aligned}
& S^x_{\mathrm{top}}[\theta^x, \phi, n_\tau, n_{xy}] \\
= & + i \sum_x \theta^x(x) \sum_{\tau, y} \{\Delta_x n_\tau(n)\, n_{xy}(n - \hat{x} + \hat{\tau}) - n_{xy}(n)\, \Delta_x n_\tau(n - \hat{x} + \hat{y})\} \\
& - i \sum_{\tau, x, y} (\Delta_\tau n_{xy} - \Delta_x \Delta_y n_\tau)(n) \left(\frac{\theta^x(x+1)}{2\pi} \Delta_x \phi(n + \hat{x}) - \frac{\theta^x(x)}{2\pi} \Delta_x \phi(n + \hat{\tau} - \hat{x} + \hat{y}) \right)
\end{aligned}
$$



Witten effects

- θ^x is constant $\longrightarrow$ quadrupole$_{xx}$ $\propto \lambda^x \sim O(a_x^2)$
- θ^x has a smooth x-dependence $\longrightarrow$ dipole $\sim O(a_x^2)$

Summary

- We discussed two kinds of theta terms in the ϕ-theory.
- These terms are activated by field discontinuities, and induce the Witten effects.
- In particular, the foliated theta angle admits a spatial dependence of the theta angle.

Future direction [WIP w/ Takuya Okuda] :

- Construction in more complicated systems (continuum/lattice)
- Application to subsystem symmetry-protected topological phases