

# On the Hamiltonian formulation of Spin(10) CLGT

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# *Toward the understanding of Lattice Chiral Gauge Theories in Hamiltonian formulation*

## *In Path-Integral formalism*

- Reflection positivity of Overlap fermions
- Reflection positivity in the Spin(10) CLGT with overlap Weyl fermions

## *In Hamiltonian formalism*

- Relation between Creutz-Horvath-Neuberger fermion and DW fermion
- an implementation of the Spin(10) CLGT with CHN fermion/DW fermion
- the Ground State of chiral DW fermions  
Integrability or Bulk independence?

Thanks to

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S(hoto).Aoki, T. Takemoto, Y. Furukawa, Y. Ikeda,

for the discussions on the related subjects

- **Reflection positivity of Overlap fermions**
- **Reflection positivity in the Spin(10) CLGT with overlap Weyl fermions**

## Overlap Weyl fermions on the lattice

- overlap Dirac operator: a local, gauge-covariant solution to GW rel.

$$D = \frac{1}{2a} \left( 1 + X \frac{1}{\sqrt{X^\dagger X}} \right), \quad X = aD_w - m_0, \quad X^\dagger = \gamma_5 X \gamma_5 \quad [\text{Neuberger (1997)}]$$

$$D_w = -\gamma_\mu \frac{1}{2} (\nabla_\mu - \nabla_\mu^\dagger) + \frac{a}{2} \nabla_\mu \nabla_\mu^\dagger$$

- Low energy effective “local” lattice action of 4+1 dim. DW fermion

[Kaplan (1992), Shamir(1993)]

- Index theorem holds true on the lattice (at a finite lattice spacing “a”)

$$\text{Index}(D) = \text{Tr} \gamma_5 (1 - aD) (\mathbf{x}, \mathbf{x})$$

[Hasenfratz et. al. (1993), Luscher(1998)]

- Weyl fermion degrees of freedom

[Narayanan-Neuberger(1993), Luscher (1998)]

$$\hat{\gamma}_5 = \gamma_5 (1 - 2aD) \quad \hat{\gamma}_5^2 = \mathbb{I} \quad \hat{P}_\pm = \left( \frac{1 \pm \hat{\gamma}_5}{2} \right), \quad P_\pm = \left( \frac{1 \pm \gamma_5}{2} \right)$$

$$\hat{\gamma}_5 \psi_\pm(x) = \pm \psi_\pm(x), \quad \bar{\psi}_\pm(x) \gamma_5 = \mp \bar{\psi}_\pm(x) \quad D = P_+ \hat{P}_- + P_- \hat{P}_+$$

$$\begin{aligned} S &= a^4 \sum_x \bar{\psi}(x) D \psi(x) \\ &= a^4 \sum_x \{ \bar{\psi}(x) P_+ D \hat{P}_- \psi(x) + \bar{\psi}(x) P_- D \hat{P}_+ \psi(x) \} \end{aligned}$$

# Reflection positivity of Overlap fermions (in the weak gauge-coupling limit)

- Spectral function and Positivity

[Luscher (2000)]

$$\langle \psi(x) \bar{\psi}(y) \rangle_{x_0 > y_0} = \int_0^\infty dE \int_{-\pi/a}^{\pi/a} \frac{d^3 \mathbf{p}}{(2\pi)^3} \varrho(E, \mathbf{p}) e^{-E(x_0 - y_0) + i\mathbf{p}(\mathbf{x} - \mathbf{y})}$$

$$dE d^3 \mathbf{p} \zeta^\dagger \gamma_0 \varrho(E, \mathbf{p}) \zeta \geq 0$$

$$\varrho(E, \mathbf{p}) = (\gamma_0 \sinh E - i\gamma_k \hat{p}_k)$$

$$\cosh E_{\mathbf{p}} = 1 + \frac{1}{\hat{\mathbf{p}}^2} \left\{ 1 + \frac{1}{4} \sum_{k \neq l} \hat{p}_k^2 \hat{p}_l^2 \right\}, \quad E_{\mathbf{p}} > 0$$

$$\times \left\{ \delta(E - \omega_{\mathbf{p}}) \theta(\cosh E - \frac{1}{2} \hat{\mathbf{p}}^2) \frac{\cosh E - \frac{1}{2} \hat{\mathbf{p}}^2}{\sinh 2E} \right.$$

$$p_0 = i\omega_{\mathbf{p}}, \quad \sinh \omega_{\mathbf{p}} = |\hat{\mathbf{p}}|$$

$$\left. + \frac{1}{2\pi} \theta(E - E_{\mathbf{p}}) \frac{\{\hat{\mathbf{p}}^2 (\cosh E - \cosh E_{\mathbf{p}})\}^{1/2}}{\hat{\mathbf{p}}^2 (\cosh E - \cosh E_{\mathbf{p}}) + (\cosh E - \frac{1}{2} \hat{\mathbf{p}}^2)^2} \right\}$$

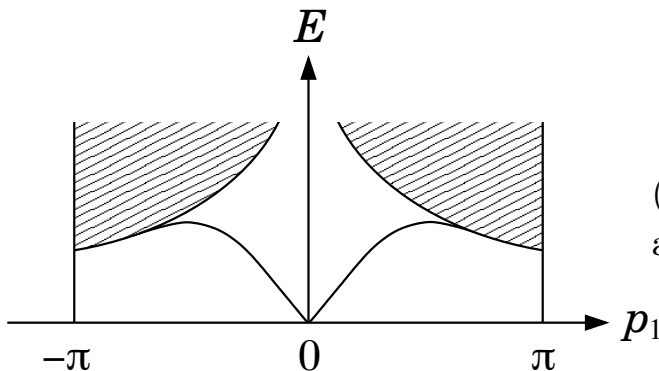


Fig. 3. Support of the spectral density (5.2) at  $p_2 = p_3 = 0$ . The physical pole (curves starting from the origin) disappears in the continuum (shaded areas) precisely at the boundary of the inner region of the Brillouin zone.

- Reflection Positivity

[Usui, YK (2010)]

$$A(\bar{\psi}, \psi) = \sum_{x \in \Lambda} \bar{\psi}(x) D_L \psi(x) \quad \Lambda = [-L + 1, L]^4 \subset \mathbb{Z}^4$$

$$\langle F \rangle := \frac{1}{Z} \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] e^{A(\bar{\psi}, \psi)} F(\bar{\psi}, \psi) \quad \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] := \prod_{x \in \Lambda; \alpha=1,2,3,4} \{d\psi_\alpha(x) d\bar{\psi}_\alpha(x)\}$$

$$\langle \theta(F_+) F_+ \rangle \geq 0 \quad \text{for } \forall F_+ \in \mathcal{A}_+$$

$$\theta(t, \mathbf{x}) = (-t + 1, \mathbf{x})$$

$$\theta(\psi(x)) = (\bar{\psi}(\theta x) \gamma_0)^T$$

$$\theta(\bar{\psi}(x)) = (\gamma_0 \psi(\theta x))^T$$

$$\theta(\alpha F + \beta G) = \alpha^* \theta(F) + \beta^* \theta(G)$$

$$\theta(FG) = \theta(G) \theta(F),$$

- R.P. in Chiral Yukawa theories (with ultra-local interactions)

$$A = \sum_{x \in \Lambda} \left\{ \bar{\psi} D \psi - \phi^* \partial_\mu^\dagger \partial_\mu \phi - m_0^2 \phi^* \phi - \frac{\lambda_0}{2} (\phi^* \phi)^2 - 2\bar{\chi} \chi \right. \\ \left. + g_0 (\bar{\psi} + \bar{\chi}) \left\{ \frac{1}{2} (1 - \gamma_5) \phi + \frac{1}{2} (1 + \gamma_5) \phi^* \right\} (\psi + \chi) \right\}$$

$$D(x, y) \Big|_{x_0 \neq y_0} = \int \frac{d^4 \mathbf{p}}{(2\pi)^4} e^{i\mathbf{p} \cdot (x-y)} \frac{X(p_0, \mathbf{p})}{2\sqrt{X^\dagger X(p_0, \mathbf{p})}}$$

$$X^\dagger X(iE_1, \mathbf{p}) = 0, \quad E_1 > 0$$

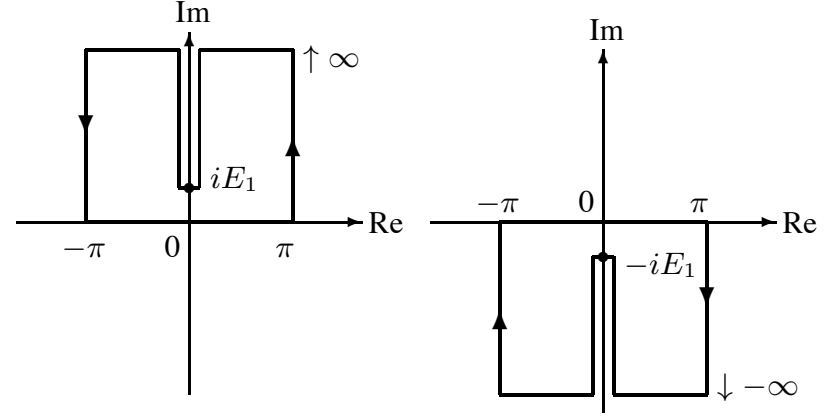


FIG. 1: Complex integration contours

$$D(x, y) \Big|_{x_0 - y_0 > 0} = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \int_{E_1}^{\infty} \frac{dE}{2\pi} e^{-E(x_0 - y_0)} e^{i\mathbf{p} \cdot (x-y)} \frac{X(iE, \mathbf{p})}{\sqrt{-X^\dagger X(iE, \mathbf{p})}}$$

$$\mp \gamma_0 X(\pm iE, \mathbf{p}) = Y_{\pm}^{\dagger} Y_{\pm}(E, \mathbf{p}) \quad (E \geq E_1)$$

$$Y_{\pm}(E, \mathbf{p}) = - \sum_{k=1}^3 \frac{l(E, \mathbf{p}) \sin p_k}{W(E, \mathbf{p})} \gamma_k \mp i \frac{W(E, \mathbf{p})}{2l(E, \mathbf{p})} \gamma_0 + il(E, \mathbf{p})$$

$$l(E, \mathbf{p}) = \left[ \frac{1}{2} \frac{\sinh E}{\sum_{k=1}^3 \sin^2 p_k / W(E, \mathbf{p})^2 + 1} \times \left( 1 + \sqrt{1 - \frac{\sum_{k=1}^3 \sin^2 p_k + W(E, \mathbf{p})^2}{\sinh^2 E}} \right) \right]^{\frac{1}{2}}$$

$$W(E, \mathbf{p}) = \sum_{k=1}^3 (1 - \cos p_k) + 1 - \cosh E - m$$

$$\begin{aligned}
D_L(x, y) \Big|_{x_0 \neq y_0} &= \sum_{\mathbf{p}} \int_{E_1}^{\infty} \frac{dE}{2\pi} \frac{1}{1 + e^{-2EL}} \frac{1}{V} \\
&\quad \times e^{-E|x_0 - y_0|} e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} \frac{X(\epsilon iE, \mathbf{p})}{\sqrt{-X^\dagger X(iE, \mathbf{p})}} \\
&+ \sum_{\mathbf{p}} \int_{E_1}^{\infty} \frac{dE}{2\pi} \frac{e^{-2EL}}{1 + e^{-2EL}} \frac{1}{V} \\
&\quad \times e^{E|x_0 - y_0|} e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} \frac{-X(-\epsilon iE, \mathbf{p})}{\sqrt{-X^\dagger X(iE, \mathbf{p})}},
\end{aligned}$$

$$\sum_{x \in \Lambda_+} \sum_{y \in \Lambda_-} \bar{\psi}(x) D_L(x, y) \psi(y)$$

$$= - \sum_{\mathbf{p}} \int_{E_1}^{\infty} \frac{dE}{2\pi} \frac{1}{V} \left[ C_{E, \mathbf{p}} \theta(C_{E, \mathbf{p}}) + D_{E, \mathbf{p}} \theta(D_{E, \mathbf{p}}) \right]$$

$$\begin{aligned}
C_{E, \mathbf{p}} &= \sqrt{\frac{1}{1 + e^{-2EL}}} \sum_{x \in \Lambda_+} \bar{\psi}(x) \gamma_0 \tilde{Y}_+(E, \mathbf{p})^\dagger e^{-Ex_0} e^{i\mathbf{p} \cdot \mathbf{x}}
\end{aligned}$$

$$\begin{aligned}
D_{E, \mathbf{p}} &= \sqrt{\frac{e^{-2EL}}{1 + e^{-2EL}}} \sum_{x \in \Lambda_+} \bar{\psi}(x) \gamma_0 \tilde{Y}_-(E, \mathbf{p})^\dagger e^{Ex_0} e^{i\mathbf{p} \cdot \mathbf{x}}
\end{aligned}$$

- Weyl fermions

$$\langle F \rangle^{(-)} := \frac{1}{Z^{(-)}} \int \mathcal{D}[\psi_-] \mathcal{D}[\bar{\psi}_-] e^{A^{(-)}(\bar{\psi}_-, \psi_-)} F(\bar{\psi}_-, \psi_-) \quad \mathcal{D}[\psi_{\pm}] \mathcal{D}[\bar{\psi}_{\pm}] = \prod_i dc_{\pm}^i \prod_{x \in \Lambda; \alpha_{\mp}} d\bar{\psi}_{\alpha_{\mp}}(x)$$

$$\begin{aligned} \langle F(\bar{\psi}_-, \psi_-) \rangle^{(-)} &= \langle F(\bar{\psi}_-, \psi_-) \rangle^{(-)} \langle 1 \rangle^{(+)} \\ &= \langle F(\bar{\psi}_-, \psi_-) \rangle. \end{aligned}$$

$$\psi_{\pm}(x) = \sum_i v_{\pm}^i(x) c_{\pm}^i$$

$$\{v_{\pm}^i(x) \mid \hat{\gamma}_5 v_{\pm}^i(x) = \pm v_{\pm}^i(x); i = 1, \dots, n_{\pm}\}$$

$$\alpha_+ = \{1, 2\}, \alpha_- = \{3, 4\}$$

$$\langle F(\psi_{-\alpha_-}, \bar{\psi}_{-\alpha_+}) \rangle^{(-)} = \langle F(\psi_{\alpha_-}, \bar{\psi}_{\alpha_+}) \rangle$$

$$\left\{ \left( \frac{1 - \hat{\gamma}_5}{2} \right) D^{-1} \right\}_{\alpha_-, \alpha_+} = \left\{ \left( \frac{1 - \gamma_5}{2} \right) D^{-1} \right\}_{\alpha_-, \alpha_+}$$

$$\langle \theta(F_+) F_+ \rangle^{(-)} \geq 0 \quad \text{for } \forall F_+ \in \mathcal{A}_+^{(-)}$$

$$\theta(t, \mathbf{x}) = (-t + 1, \mathbf{x})$$

$$\theta(\psi_{-\alpha_-}(x)) = \{\bar{\psi}_-(\theta x) \gamma_0\}_{\alpha_-} = \bar{\psi}_{-\alpha_+}(\theta x)$$

$$\theta(\bar{\psi}_{-\alpha_+}(x)) = \{\gamma_0 \psi_-(\theta x)\}_{\alpha_+} = \psi_{-\alpha_-}(\theta x)$$

$$\alpha_+ = \alpha_- - 2.$$

the spectral property and the positivity of the overlap Dirac fermion is inherited from that of the domain wall fermion (Wilson fermion)

$$dE d^3\mathbf{p} \zeta^\dagger \gamma_0 \varrho(E, \mathbf{p}) \zeta \geq 0$$

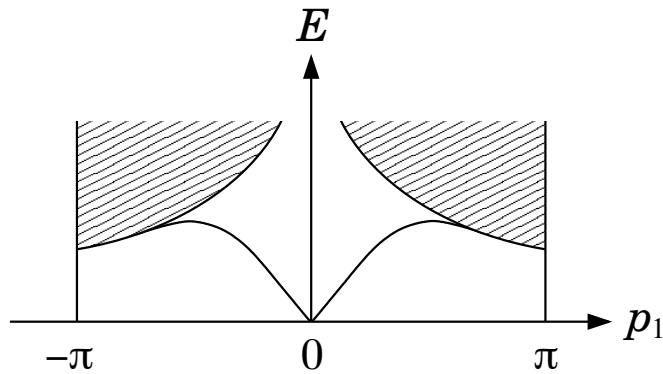


Fig. 3. Support of the spectral density (5.2) at  $p_2 = p_3 = 0$ . The physical pole (curves starting from the origin) disappears in the continuum (shaded areas) precisely at the boundary of the inner region of the Brillouin zone.

cf. Strictly Positive Transfer matrix of Wilson fermions

[Luscher (1977)]

# Lattice domain-wall fermion

[Kaplan(1992)] [Shamir(1993)]

$$S_{\text{DW}} = \sum_{x,t} \bar{\psi}(x,t) X_{\text{w}}^{(5)} \psi(x,t)$$

$$X_{\text{w}}^{(5)} = D_{\text{w}}^{(5)} - m_0 \quad m_0 \in (0, 2)$$

$$D_{\text{w}} = -\gamma_{\mu} \frac{1}{2} (\nabla_{\mu} - \nabla_{\mu}^{\dagger}) + \frac{a}{2} \nabla_{\mu} \nabla_{\mu}^{\dagger}$$

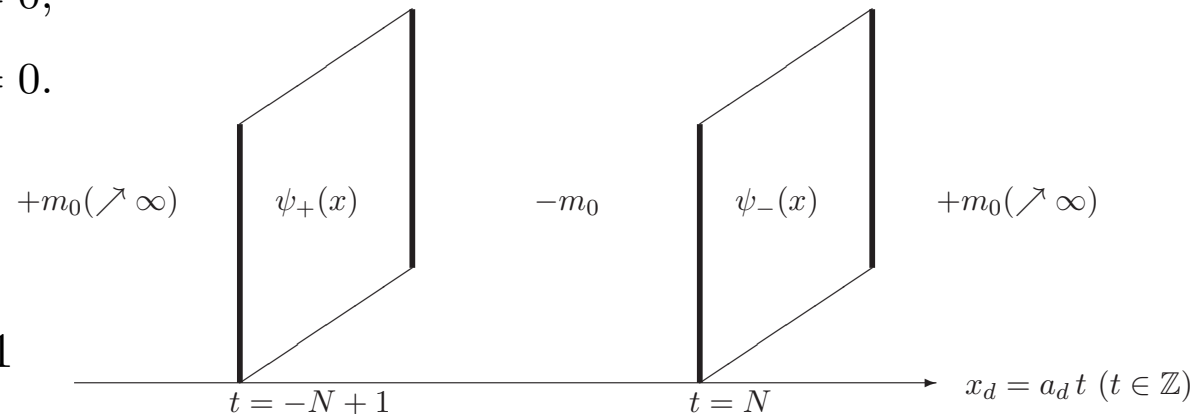
## Dirichlet b.c. (open b.c.)

$$\psi_{-}(x,t)|_{t=-N} = 0, \quad \psi_{+}(x,t)|_{t=N+1} = 0,$$

$$\bar{\psi}_{-}(x,t)|_{t=-N} = 0, \quad \bar{\psi}_{+}(x,t)|_{t=N+1} = 0.$$

## Vector-like set up

$$U(x,t;\mu) = U(x,\mu) \quad U(x,t;5) = 1$$



## Boundary field variables

$$q(x) = \psi_{-}(x, N) + \psi_{+}(x, -N + 1)$$

$$\bar{q}(x) = \bar{\psi}_{-}(x, N) + \bar{\psi}_{+}(x, -N + 1)$$

$$\langle q(x) \bar{q}(y) \rangle = D_c'^{-1}$$

no-doubling, chiral invariant, but non-local  $D_c'$

$$\det X_w^{(5)}|_{\text{Dir.}} = \det D'_{\text{ov}} \det X_w^{(5)}|_{\text{AP}} \quad (N \rightarrow \infty) \quad [\text{Neuberger}(1998)]$$

$$D'_{\text{ov}} = \frac{1}{2} \left( 1 + X \frac{1}{\sqrt{X^\dagger X}} \right) \quad X = X_w \frac{1}{1 + a_5 X_w / 2}$$

$$\langle q(x) \bar{q}(y) \rangle = D'_{\text{ov}}{}^{-1} - 1 \quad q(x) = (1 - 2D)\psi(x) \quad [\text{Noguchi-YK}(1999)]$$

$$\bar{q}(x) = \bar{\psi}(x)$$

overlap Dirac fermion as

the boundary(edge) effective theory on the lattice (not in the continuum)

rigorous result at a finite lattice spacing “ $a$ ”

# Chiral Lattice Gauge Theory with Overlap Weyl fermions / GW rel.

- Path integral measure and Chiral determinant

$$Z \equiv \int \mathcal{D}[U] e^{-S_G[U] + \Gamma_W[U]}$$

$$e^{\Gamma_W[U]} = \int \mathcal{D}[\psi_-] \mathcal{D}[\bar{\psi}_-] e^{-a^4 \sum_x \bar{\psi}_-(x) D \psi_-(x)}$$

$$= \int \prod_i d c_i \prod_j d \bar{c}_j e^{-\sum_{ij} \bar{c}_j M_{ji} c_i} = \det M_{ji}$$

$$M_{ji} = a^4 \sum_x \bar{v}_j D v_i(x)$$

( $N_- \times \bar{N}_-$  rectangular matrix)

$$\psi_-(x) = \sum_i v_i(x) c_i \quad \bar{\psi}_-(x) = \sum_i \bar{c}_i \bar{v}_i(x)$$

$$\{v_i(x) \mid \hat{\gamma}_5 v_i(x) = -v_i(x) \ (i = 1, \dots, N_-)\}$$

$$\{\bar{v}_i(x) \mid \bar{v}_i(x) \gamma_5 = +\bar{v}_i(x) \ (i = 1, \dots, \bar{N}_-)\}$$

*Single-valued, Smooth Function of Link Gauge Fields, satisfying the requirement of Locality, Gauge-invariance & Lattice Symmetry ?*

$$(\sum_R \text{Tr}_R [T^a \{T^b, T^c\}] = 0)$$

$$\mathfrak{U} = \left\{ \{U(x, \mu)\} \mid \|1 - U_{\mu\nu}(x)\| < \epsilon \ \forall (x, \mu, \nu) \right\}$$

$$\mathcal{O} \subset \mathfrak{U}[Q] \quad \tilde{v}_i(x) = v_l(x) (Q^{-1})_{li}, \quad \tilde{c}_j = \sum_l Q_{jl} c_l$$

$$\mathcal{D}[\psi_-] \rightarrow \mathcal{D}[\psi_-] \det Q \quad \det M_{ji} \rightarrow \det M_{ji} \det Q$$

*Single-valued, Smooth Function globally ?*

Determinant line bundle over  $\mathfrak{U}$

# Chiral gauge theories with Weyl fermions in an anomaly-free rep.

- Gauge anomaly cancellation

- local gauge anomalies :  $\sum_R Tr_R [ T^a \{ T^b, T^c \} ] = 0$

- global gauge anomalies :  $exp( i 2\pi \eta ) = 1$

- The standard model (SM)

quarks and leptons belong to complex reps. of the gauge groups

$$SU(3) \times SU(2) \times U(1) \times U(1)_{B-L} \quad \begin{matrix} (\underline{3}, \underline{2})_{1/6} & & (\underline{1}, \underline{2})_{-1/2} \\ (\underline{3}^*, \underline{1})_{-2/3} & (\underline{3}^*, \underline{1})_{1/3} & (\underline{1}, \underline{1})_1 & (\underline{1}, \underline{1})_0 \end{matrix}$$

- GUT models

$$SU(5) \times U(1)_Q \quad \begin{matrix} (\underline{10})_1 \\ (\underline{5}^*)_{-3} & (\underline{1})_5 \end{matrix}$$

$$SO(10) \quad (\underline{16})$$

- local gauge anomalies are cancelled non-trivially

- there is no global gauge anomalies [D. S. Freed (2008)]

$$\Omega_5^{spin}(BSpin(10)) = 0 \quad [Garcia-Etxebarria-Montero (2018)]$$

[Wang-Wen, Wang-Wen-Witten (2018) ]

# A gauge invariant path-integral measure for the overlap Weyl fermions in 16 of SO(10)

[YK (2017)]

$$\mathcal{D}[\psi_-]\mathcal{D}[\bar{\psi}_-] \equiv \mathcal{D}[\psi]\mathcal{D}[\bar{\psi}] \prod_{x \in \Lambda} F(T_+(x)) \prod_{x \in \Lambda} F(\bar{T}_+(x))$$

$$\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}] \equiv \prod_{x \in \Lambda} \prod_{\alpha=1}^4 \prod_{s=1}^{16} d\psi_{\alpha s}(x) \prod_{x \in \Lambda} \prod_{\alpha=1}^4 \prod_{s=1}^{16} d\bar{\psi}_{\alpha s}(x)$$

$$\psi_+(x) = \hat{P}_+ \psi(x) \quad \bar{\psi}(x)_+ = \bar{\psi}(x) P_-$$

$$T_+(x) = \frac{1}{2} V_+^a(x) V_+^a(x), \quad V_+^a(x) = \psi_+(x)^T i\gamma_5 C_D T^a \psi_+(x) \quad T^a = C\Gamma^a$$

$$\bar{T}_+(x) = \frac{1}{2} \bar{V}_+^a(x) \bar{V}_+^a(x), \quad \bar{V}_+^a(x) = \bar{\psi}_+(x) i\gamma_5 C_D T^a \bar{\psi}_+(x)^T \quad T^{aT} = T^a$$

cf.  $\hat{P}_+^T i\gamma_5 C_D P_+ T^a E^a(x) \hat{P}_+ = (1 - D)^T i\gamma_5 C_D P_+ T^a E^a(x) (1 - D)$

[Eichten-Preskill(1986)]

$$F(w) \equiv 4! (z/2)^{-4} I_4(z) \Big|_{(z/2)^2=w} = 4! \sum_{k=0}^{\infty} \frac{w^k}{k!(k+4)!}$$

$$F(w) \Big|_{w=(1/2)u^a u^a} = (\pi^5/12)^{-1} \int \prod_{a=1}^{10} de^a \delta(\sqrt{e^b e^b} - 1) e^{e^c u^c}$$

[Wen(2013)]

# Chiral determinant and 't Hooft vertex pfaffians

$$Z \equiv \int \mathcal{D}[U] e^{-S_G[U] + \Gamma_W[U]}$$

$$e^{\Gamma_W[U]} \equiv \int \mathcal{D}[\psi_-] \mathcal{D}[\bar{\psi}_-] e^{-S_W[\psi_-, \bar{\psi}_-]}$$

$$= \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] \prod_{x \in \Lambda} F(T_+(x)) \prod_{x \in \Lambda} F(\bar{T}_+(x)) e^{-S_W[\psi_-, \bar{\psi}_-]}$$

$$= \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] \mathcal{D}[E] \mathcal{D}[\bar{E}] e^{-S_W[\psi_-, \bar{\psi}_-] + \sum_{x \in \Lambda} \{E^a(x) V_+^a(x) + \bar{E}^a(x) \bar{V}_+^a(x)\} [\psi_+, \bar{\psi}_+]}$$

$$= \det(\bar{v} D v) \times \int \mathcal{D}[\bar{E}] \text{pf}(u^T i \gamma_5 C_D T^a E^a u) \int \mathcal{D}[\bar{E}] \text{pf}(\bar{u} i \gamma_5 C_D T^{a\dagger} \bar{E}^a \bar{u}^T)$$

$$\mathcal{D}[E] = \prod_{x \in \Lambda} (\pi^5/12)^{-1} \prod_{a=1}^{10} dE^a(x) \delta(\sqrt{E^b(x) E^b(x)} - 1)$$

$$\mathcal{D}[\bar{E}] = \prod_{x \in \Lambda} (\pi^5/12)^{-1} \prod_{a=1}^{10} d\bar{E}^a(x) \delta(\sqrt{\bar{E}^b(x) \bar{E}^b(x)} - 1)$$

$$\frac{1}{2} \sum_{x \in \Lambda} \begin{pmatrix} \psi^T & \bar{\psi} \end{pmatrix} (x) \begin{pmatrix} -\hat{P}_+^T i \gamma_5 C_D T^a E^a \hat{P}_+ & -\hat{P}_-^T D^T P_+^T \\ P_+ D \hat{P}_- & -P_- i \gamma_5 C_D T^{a\dagger} \bar{E}^a P_-^T \end{pmatrix} \begin{pmatrix} \psi \\ \bar{\psi}^T \end{pmatrix} (x)$$

$$\text{pf} \begin{pmatrix} -\hat{P}_+^T i \gamma_5 C_D T^a E^a \hat{P}_+ & -\hat{P}_-^T D^T P_+^T \\ P_+ D \hat{P}_- & -P_- i \gamma_5 C_D T^{a\dagger} \bar{E}^a P_-^T \end{pmatrix}$$

$$= \text{pf} \begin{pmatrix} -(u^T i \gamma_5 C_D T^a E^a u) & 0 & 0 & 0 \\ 0 & 0 & 0 & -(v^T D^T \bar{v}^T) \\ 0 & 0 & -(\bar{u} i \gamma_5 C_D T^{a\dagger} \bar{E}^a \bar{u}^T) & 0 \\ 0 & (\bar{v} D v) & 0 & 0 \end{pmatrix}$$

$$\psi_-(x) = \sum_j v_j(x) c_j, \quad \bar{\psi}_-(x) = \sum_k \bar{c}_k \bar{v}_k(x)$$

$$\psi_+(x) = \sum_j u_j(x) b_j, \quad \bar{\psi}_+(x) = \sum_k \bar{b}_k \bar{u}_k(x)$$

## The saturation of the Right-handed measures due to 't Hooft vertices (the anti-fields)

the part of the anti-field:  $\mathcal{D}_\star[\bar{\psi}_+]$

$$\int \mathcal{D}[\bar{E}] \text{pf}(\bar{u} i \gamma_5 C_D T^{a\dagger} \bar{E}^a \bar{u}^T) = 1 \quad \text{for all topological sectors}$$

$$(\bar{u} i \gamma_5 C_D T^{a\dagger} \bar{E}^a \bar{u}^T)_{kl} = i \epsilon_{\sigma\sigma'} \delta_{xx'} (T^{a\dagger} P_+)_{tt'} \bar{E}^a(x')$$

$$\begin{aligned} \text{pf}(\bar{u} i \gamma_5 C_D T^a \bar{E}^a \bar{u}^T) &= \prod_x \det(P_- + P_+ i T^{a\dagger} \bar{E}^a(x)) \\ &= \prod_x \det(i \check{T}^{a\dagger} \bar{E}^a(x)) \\ &= \prod_x \det(i \check{C}^\dagger [E^{10}(x) + i \check{\Gamma}^{a'} \bar{E}^{a'}(x)]) \\ &= 1. \end{aligned}$$

cf. strong coupling limit of Eichten-Prekill model

[Eichten-Prekill(1986)]

$$\begin{aligned} 1 &= \int \mathcal{D}_\star[\bar{\psi}_+] F(\bar{T}_+(x)[\bar{\psi}_+]) \\ &= \int \prod_{x \in \Lambda} \prod_{\alpha=3}^4 \prod_{s=1}^{16} d\bar{\psi}_{\alpha s}(x) \prod_{x \in \Lambda} \frac{4!}{8!12!} \left\{ \frac{1}{2} \bar{\psi}(x) P_- i \gamma_5 C_D T^a \bar{\psi}(x)^T \bar{\psi}(x) P_- i \gamma_5 C_D T^a \bar{\psi}(x)^T \right\}^8 \end{aligned}$$

# The Spin(10) CLGT with overlap Weyl fermions is R.P. in the weak gauge-coupling limit

$$S_{\text{Ov}}[\psi, \bar{\psi}, E^a, \bar{E}^a] = \sum_{x \in \Lambda} \bar{\psi}_-(x) D \psi_-(x) - \sum_{x \in \Lambda} \{ E^a(x) \psi_+^T(x) i \gamma_5 C_D T^a \psi_+(x) + \bar{E}^a(x) \bar{\psi}_+(x) i \gamma_5 C_D T^{a\dagger} \bar{\psi}_+(x)^T \}$$

$$\text{pf} (u^T i \gamma_5 C_D T^a E^a u)$$

$$S_{\text{Ov}}[\psi, \bar{\psi}, E^a] = \sum_{x \in \Lambda} \psi_-(x) D \psi(x) - \sum_{x \in \Lambda} E^a(x) \left\{ \psi_+^T(x) i \gamma_5 C_D T^a \psi_+(x) + \bar{\psi}_+(x) i \gamma_5 C_D T^{a\dagger} \bar{\psi}_+(x)^T \right\}$$

$$z_+ = 1 \quad y = 1$$




$$\text{pf} (u^T i \gamma_5 C_D [y P_+ + (z_+^2/y) P_-] T^a E^a u)$$

$$S_{\text{Ov}}[\psi, \bar{\psi}, E^a] = \sum_{x \in \Lambda} \psi_-(x) D \psi(x) + \sum_{x \in \Lambda} z_+ \psi_+(x) D \psi_+(x) - \sum_{x \in \Lambda} y E^a(x) \left\{ \psi_+^T(x) i \gamma_5 C_D P_+ T^a \psi_+(x) + \bar{\psi}_+(x) i P_- \gamma_5 C_D T^{a\dagger} \bar{\psi}_+(x)^T \right\}$$

$$\hat{P}_+^T i \gamma_5 C_D P_+ T^a E^a(x) \hat{P}_+ = (1 - D)^T i \gamma_5 C_D P_+ T^a E^a(x) (1 - D)$$

A chiral Yukawa theory with ultra-local interactions

$y = 1$   
 (maximum value) 

$$S_{\text{DW/Ov}} = \sum_{t=1}^{L_5} \sum_{x \in \Lambda} \bar{\psi}(x, t) \{ [D_{4\text{w}} + 1 - m_0] \delta_{t,t'} - P_+ \delta_{t+1,t'} - P_- \delta_{t,t'+1} \} \psi(x, t) \\
- \sum_{x \in \Lambda} y E^a(x) \left\{ \psi(x, 1)^T i \gamma_5 C_D P_+ \Gamma^a \psi(x, 1) + \bar{\psi}(x, 1) P_- i \gamma_5 C_D \Gamma^{a\dagger} \bar{\psi}(x, 1)^T \right\}$$

$$\psi_+(x, 1) = q_+(x) = (1 - D)\psi_+(x)$$

$$\bar{\psi}_+(x, 1) = \bar{q}_+(x) = \bar{\psi}_+(x)$$

DW fermion with the ultra-local boundary Eichten-Prekill term

[Wen(2013)]

[Kaplan(1992), Creutz et al (1997)]

[Wen(2013), You-BenTov-Xu(2014),

You-Xu (2015),

Morimoto et al (2015) ]

Thus, the Spin(10) CLGT with the overlap Weyl fermions is R.P. in the weak gauge-coupling limit

At a finite gauge coupling, R.P. is not maintained even in the gauge-field-action, due to the admissibility condition

$$\mathfrak{A} = \left\{ \{U(x, \mu)\} \mid \|1 - U_{\mu\nu}(x)\| < \epsilon^{\forall(x, \mu, \nu)} \right\}$$

*cf. improved gauge actions*

[Luscher-Weisz (1984)]

- **Relation between Creutz-Horvath-Neuberger fermions and DW fermions**
- **Implementation of the Spin(10) CLGT with CHN fermion/DW fermion**

$$\begin{aligned} \left[ E_k^a(x), \{U_{k'}(y)\}_{ij} \right] &= + \{T^a U_k(x)\}_{ij} \delta_{k,k'} \delta_{x,y}, & [\psi(x), \psi^\dagger(x)] &= \delta_{x,y} \\ \left[ E_k^a(x), \{U_{k'}^{-1}(y)\}_{ij} \right] &= - \{U_k^{-1}(x) T^a\}_{ij} \delta_{k,k'} \delta_{x,y}. \end{aligned}$$

$$E_k^a(x) = \{T^a U_k(x)\}_{ij} \frac{\delta}{\delta \{U_k(x)\}_{ij}} \quad G^a(x) = (\nabla_k^* E_k)^a(x) - \bar{\psi}(x) T^a \psi(x)$$

$$H = H_G + H_F$$

$$H_G = a^3 \sum_{x \in \mathbb{L}^3} \left\{ \frac{g^2}{2} \sum_{k=1}^3 \sum_a E_k^a(x) E_k^a(x) + \sum_{k,l=1}^3 \frac{1}{2g^2} \text{Tr}(1 - U_{kl}(x))(1 - U_{kl}(x))^\dagger \right\}$$

$$H_F = a^3 \sum_{x \in \mathbb{L}^3} \psi^\dagger(x) \gamma_0 (D + m_0) \psi(x) \quad D = D_w^{(3)}, D_{\text{ks}}^{(3)}, D_{\text{ov}}^{(3)}$$

# Creutz-Horvath-Neuberger fermions

[Creutz, Horvath, Neuberger (2001)]

$$D_w^{(3)} = \sum_{k=1,2,3} \left\{ \gamma_k \frac{1}{2} (\nabla_k - \nabla_k^\dagger) + \frac{1}{2} \nabla_k \nabla_k^\dagger \right\}$$

$$D_{\text{ov}}^{(3)} = \frac{1}{2} \left( 1 + X_w^{(3)} \frac{1}{\sqrt{X_w^{(3)\dagger} X_w^{(3)}}} \right) \quad X_w^{(3)} = D_w^{(3)} - m_0 \quad (0 < m_0 < 2)$$

[Neuberger, YK (1998)]

$$\gamma_5 D + D \gamma_5 = 2D \gamma_5 D$$

$$\gamma_0 D + D \gamma_0 = 2D \gamma_0 D$$

$$Q_5 = \gamma_5 (1 - D), \quad \mathcal{H} = \gamma_0 D, \quad [\mathcal{H}, Q_5] = 0, \quad Q_5^2 + \mathcal{H}^2 = \mathbb{I}$$

$$[Q_5, H_{\text{ov},F}] = 0 \quad Q_5 = \sum_x \psi^\dagger(x) \gamma_5 (1 - D) \psi(x)$$

chiral anomaly ?

[Hayata, Nakayama, Yamamoto(2024)]

[Hikida, Yamamoto(2025)]

unitary discrete chiral symmetry,  $\theta$  term & counter terms ?

cf. [Dempsey, Klebanov, Pufu, Zan 2023)] [YK (2025)]

## Weyl fermion degrees of freedom ?

$$\bar{\psi}(x) = \psi^\dagger(x)\gamma_0$$

asymmetric definition of chiral components is not allowed  
in Hamiltonian formalism ?

$$\hat{\gamma}_5 = \gamma_5(1 - 2aD) \quad \hat{\gamma}_5^2 = \mathbb{I} \quad \hat{P}_\pm = \left( \frac{1 \pm \hat{\gamma}_5}{2} \right), \quad P_\pm = \left( \frac{1 \pm \gamma_5}{2} \right)$$

$$\hat{\gamma}_5 \psi_\pm(x) = \pm \psi_\pm(x), \quad \bar{\psi}_\pm(x) \gamma_5 = \mp \bar{\psi}_\pm(x)$$

$$D = P_+ \hat{P}_- + P_- \hat{P}_+$$

## Relation to Domain wall fermion ?

$$\det X_w^{(4)}|_{\text{Dir.}} = \det D'_{\text{ov}}^{(3)} \det X_w^{(4)}|_{\text{AP}} \quad (N \rightarrow \infty)$$

$$\det \left\{ \lambda - \gamma_0 X_w^{(4)}|_{\text{Dir.}} \right\} \neq \det \left\{ \lambda - \gamma_0 D'_{\text{ov}}^{(3)} \right\} \det \left\{ \lambda - \gamma_0 X_w^{(4)}|_{\text{AP}} \right\}$$

*It's a mystery!*

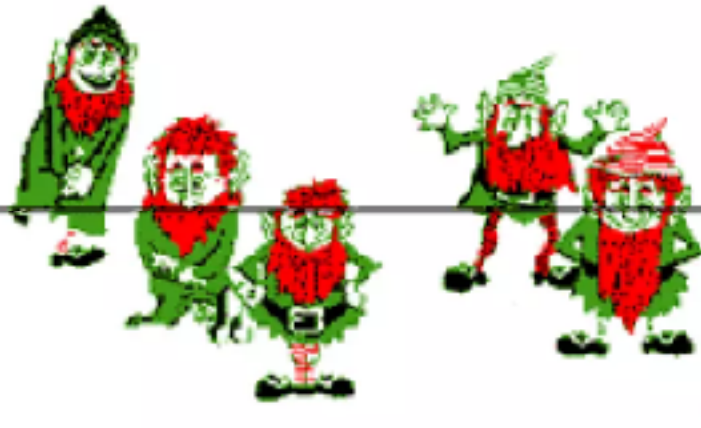
## The Vanishing

## Leprechaun



Which one vanishes? Where did he go? When he comes back, where has he been? It's a mystery!

# Leprechaun



*It's a mystery!*

# The Vanishing



Which one vanishes? Where did he go? When he comes back, where has he been? It's a mystery!

## The relation between Creutz-Horvath-Neuberger fermion and DW fermion in Hamiltonian formalism

$$\begin{aligned}
 H &= \sum_{x,t} \Psi(x,t)^\dagger \gamma_0 X_w \Psi(x,t)|_{\text{Dir.}} + \sum_x \chi(x)^\dagger \gamma_0 \chi(x) \\
 &= \sum_{x,t} \Psi'(x,t)^\dagger \gamma_0 X_w \Psi'(x,t)|_{\text{AP}} + \sum_x \psi(x)^\dagger \gamma_0 D_{\text{ov}} \psi(x) \quad (N \rightarrow \infty)
 \end{aligned}$$

### Canonical (Unitary) transformation

$$\begin{aligned}
 q(x) &= Q(x) + (1 - D_{\text{ov}})\psi(x) & \psi(x) &= q(x) + \chi(x) \\
 \chi(x) &= -Q(x) + D_{\text{ov}}\psi(x) & Q(x) &= D_{\text{ov}}q(x) - (1 - D_{\text{ov}})\chi(x)
 \end{aligned}$$

Chiral components,

but mix with the boundary variables of the bulk Wilson fermion

$$\begin{aligned}
 q_-(x) &= Q_-(x) + (1 - D_{\text{ov}})\psi_-(x) & \hat{\gamma}_5 \psi_-(x) &= -\psi_-(x) \\
 & & (1 - D_{\text{ov}}) &= P_+ \hat{P}_+ + P_- \hat{P}_-
 \end{aligned}$$

$$\gamma_0(D_{5w} - m_0)$$

$$= \gamma_0 \{(D_w - m_0 + 1)\delta_{ts} - P_- \delta_{t+1s} - P_+ \delta_{ts+1}\}$$

$$= \begin{pmatrix} \begin{pmatrix} -iC^\dagger & -iB \\ iB & iC \end{pmatrix} & \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix} & \cdot & \cdot & \begin{bmatrix} 0 & 0 \\ +i & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ -i & 0 \end{bmatrix} & \begin{pmatrix} -iC^\dagger & -iB \\ iB & iC \end{pmatrix} & \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix} & \cdot & \cdot \\ \cdot & \begin{bmatrix} 0 & 0 \\ -i & 0 \end{bmatrix} & \begin{pmatrix} -iC^\dagger & -iB \\ iB & iC \end{pmatrix} & \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix} & \cdot \\ \cdot & \cdot & \begin{bmatrix} 0 & 0 \\ -i & 0 \end{bmatrix} & \begin{pmatrix} -iC^\dagger & -iB \\ iB & iC \end{pmatrix} & \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & -i \\ 0 & 0 \end{bmatrix} & \cdot & \cdot & \begin{bmatrix} 0 & 0 \\ -i & 0 \end{bmatrix} & \begin{pmatrix} -iC^\dagger & -iB \\ iB & iC \end{pmatrix} \end{pmatrix} \begin{pmatrix} \psi_{+}(1) \\ q_- \\ \psi_{+}(2) \\ \psi_{-}(2) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ q_+ \\ \psi_{-}(N) \end{pmatrix}$$

$$\Psi(x, s) \rightarrow V\Psi(x, s)$$

$$H_{NN}|_{\text{Dir.}} = q^\dagger \gamma_0 D_c^{(N)} q$$

$$H_{NN}|_{\text{AP}} = Q^\dagger \gamma_0 (D_c^{(N)} + 1) Q$$

$$h = +\chi^\dagger \gamma_0 \chi$$

$$h_{\text{CHN}} = \psi^\dagger \gamma_0 D_{\text{ov}}^{(N)} \psi$$

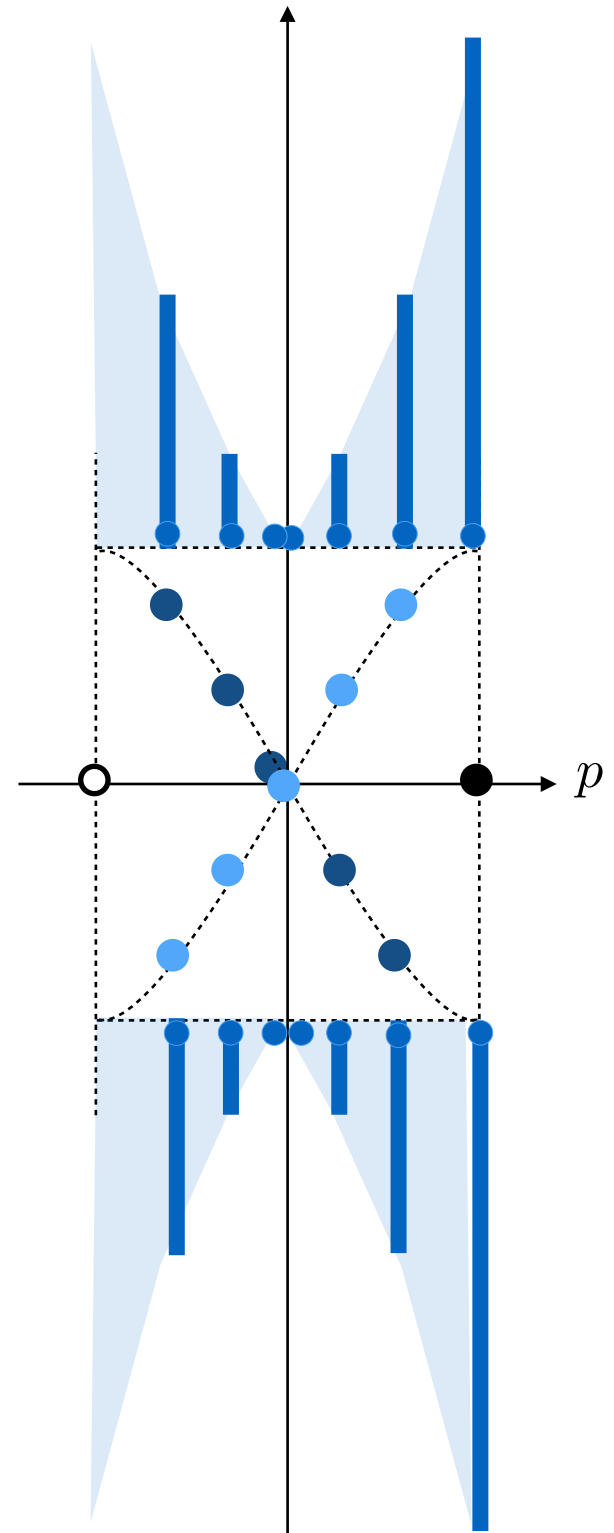
$$\text{Canonical (Unitary) tr. } u : \quad \begin{aligned} q &= Q + \frac{1}{D_c^{(N)} + 1} \psi \\ \chi &= -Q + \frac{D_c^{(N)}}{D_c^{(N)} + 1} \psi \end{aligned}$$

$$U = V^{-1} u V = u$$

# Eigenvalue spectrum of the DW fermion

$$\begin{aligned}
 H &= \sum_{x,t} \Psi(x,t)^\dagger \gamma_0 X_w \Psi(x,t)|_{\text{Dir.}} + \sum_x \chi(x)^\dagger \gamma_0 \chi(x) \\
 &= \sum_{x,t} \Psi'(x,t)^\dagger \gamma_0 X_w \Psi'(x,t)|_{\text{AP}} + \sum_x \psi(x)^\dagger \gamma_0 D_{\text{ov}} \psi(x)
 \end{aligned}$$

$(N \rightarrow \infty)$



## 't Hooft vertex operator in the Spin(10) CLGT

$$-\sum_x y E^a(x) \left\{ (Q + (1 - D)\psi)^T i\gamma_5 C_D P_+ T^a (Q + (1 - D)\psi) \right. \\ \left. + (\bar{Q} + \bar{\psi}(1 - D)) P_- i\gamma_5 C_D T^{a\dagger} (\bar{Q} + \bar{\psi}(1 - D))^T \right\}$$

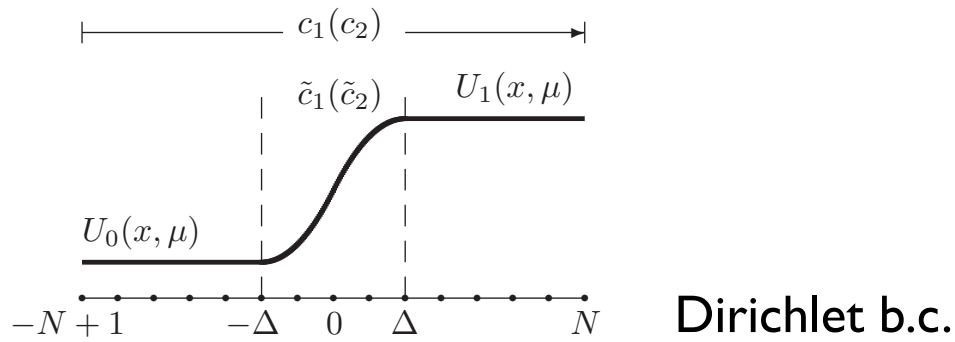
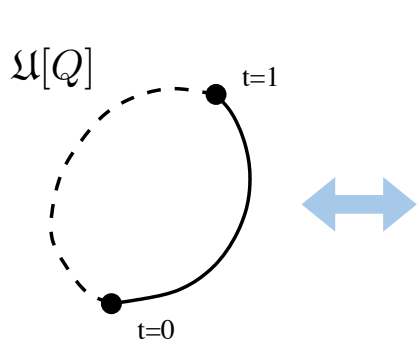
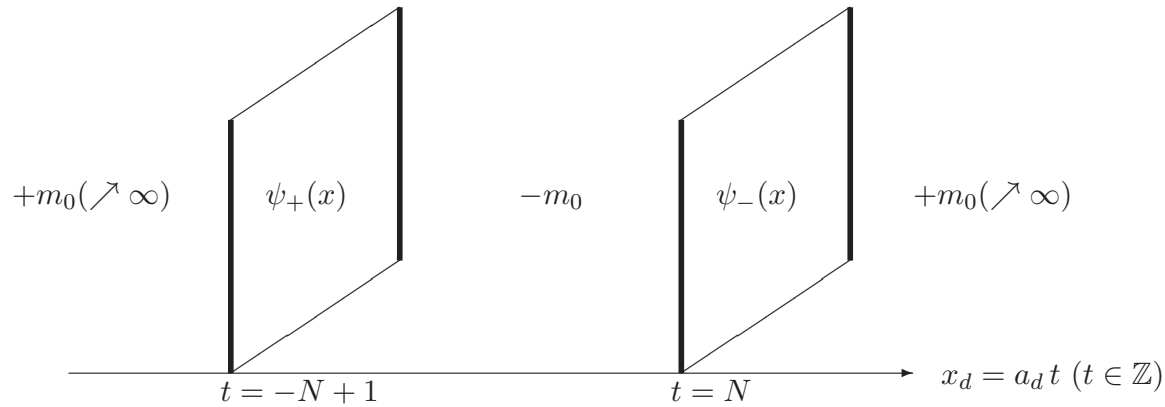
Creutz-Horvath-Neuberger fermion in Hamiltonian formulation is *not quite chiral* as Overlap fermion in Path-Integral formulation

It needs the mixing with the bulk massive Wilson fermion (AP, P) to reproduce the Weyl fermion d.o.f. and the associated anomaly inflow

It seems useful though, when one needs the separation of the bulk massive contributions from the boundary / low-lying modes

- **the Ground State of chiral DW fermions**  
**Integrability or Bulk independence?**

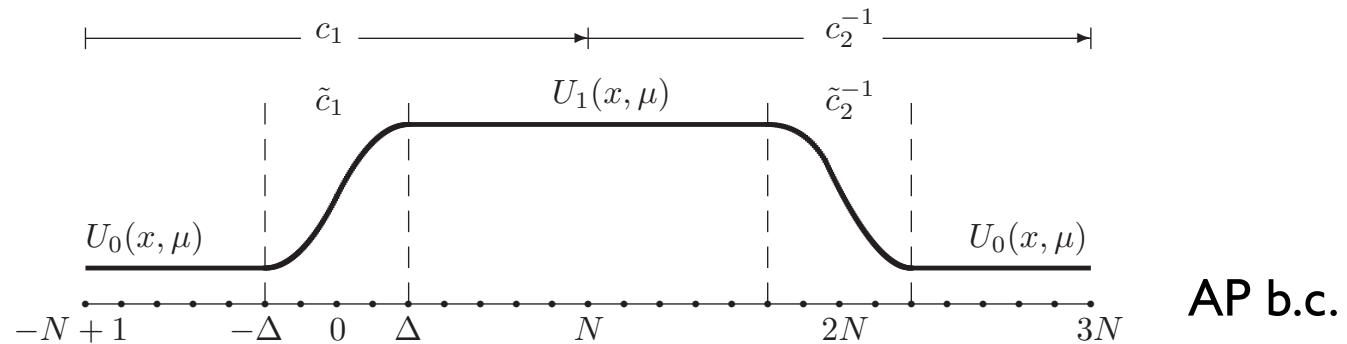
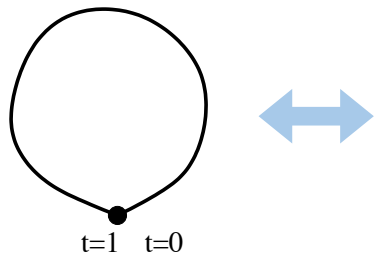
● Integrability condition for the G.S. of chiral DW fermions ?



Chiral set up

$$U(x, t, \mu)|_{t=-N+1} = U_0(x, \mu)$$

$$U(x, t, \mu)|_{t=N} = U_1(x, \mu)$$



- the G.S. of chiral DW fermions

$$H = \sum_{x,t} \Psi(x,t)^\dagger \gamma_0 X_w \Psi(x,t) |_{\text{Dir.}}$$

$$H|GS\rangle = E_{\text{min.}}|GS\rangle$$

- Bulk dependence

$$|GS\rangle \otimes |\overline{GS}\rangle \otimes |gs\rangle_\chi \otimes |gs\rangle_{\chi'} = |GS \oplus \overline{GS}\rangle \otimes |gs\rangle_{\psi(0)} \otimes |gs\rangle_{\psi(1)}$$

$$|GS \oplus \overline{GS}\rangle = \Lambda_- |GS \oplus \overline{GS}\rangle$$

$$\Lambda_- = \sum_{x,t} \Psi(x,t)^\dagger \left( \frac{1 + \hat{\gamma}_0}{2} \right) \Psi(x,t)$$

$$\hat{\gamma}_0 = -\gamma_0 X_w \frac{1}{\sqrt{X_w^\dagger X_w}} \Big|_{\text{AP,P}}$$

$$|GS \oplus \overline{GS}\rangle = \prod_I (\Psi^\dagger V_I) |0\rangle \quad \hat{\gamma}_0 V_I = +V_I$$

=> Integrability condition

$$cf. \quad \hat{\gamma}_5 v_i = -v_i$$

[Luscher (1998)]

## Summary

- **Reflection positivity in the Spin(10) CLGT with overlap Weyl fermions**

The Spin(10) CLGT with overlap Weyl fermions is R.P. in the weak gauge-coupling limit

At a finite gauge coupling, R.P. is not maintained even in the gauge-field-action, due to the admissibility condition *cf. improved gauge actions* [Luscher-Weisz (1984)]

- **Relation between Creutz-Horvath-Neuberger fermion and DW fermion**

Canonical (Unitary) transformation:  $q(x) = Q(x) + (1 - D_{\text{ov}})\psi(x)$   
 $\chi(x) = -Q(x) + D_{\text{ov}}\psi(x)$

CHN fermion needs the mixing with the bulk massive Wilson fermion to reproduce the Weyl fermion d.o.f. and the associated anomaly inflow

- **'t Hooft vertex in the Spin(10) CLGT with CHN fermion/DW fermion**

$$-\sum_x y E^a(x) \left\{ (Q + (1 - D)\psi)^T i\gamma_5 C_D P_+ T^a (Q + (1 - D)\psi) \right. \\ \left. + (\bar{Q} + \bar{\psi}(1 - D)) P_- i\gamma_5 C_D T^{a\dagger} (\bar{Q} + \bar{\psi}(1 - D))^T \right\}$$

