Higher-Form Anomalies on Lattices (for Bosons and Fermions)

Ryohei Kobayashi

University of Tokyo

Anomaly of global symmetry

‘t Hooft anomaly: Constrains low-energy dynamics of quantum systems

- Constrains dynamics of QFT, e.g., forbids confinement in gauge theory
- Obstruction to gauging the symmetry

Applications to condensed matter systems:

- Forbids trivial gapped phases, through **Lieb-Schultz-Mattis (LSM) theorem**
- Enforces nontrivial edge state of SPT phases, through **anomaly inflow**

[Lieb-Schultz-Mattis, Oshikawa]

Higher-form symmetry

p-form symmetry: topological operator w/ codimension (p+1) in spacetime [Gaiotto-Kapustin-Seiberg-Willett]

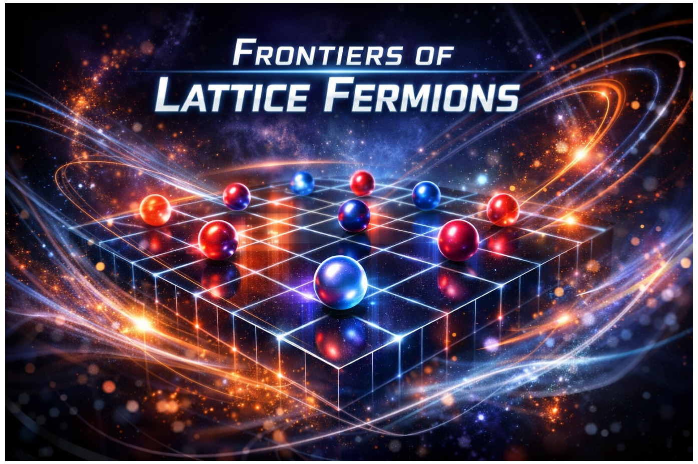
Ex. Anyons in (2+1)D TQFT generates 1-form symmetry

't Hooft anomaly: spin and braiding of anyons.

Applications to condensed matter systems:

- Line operators with nontrivial spins forbid short-range entangled (SRE) state
- Obstruction to gauging = obstruction to anyon condensation/proliferation. Only bosons can be condensed.

Aim of the talk



How to define anomaly on lattices?

We define anomaly of higher-form symmetry on **lattice models**.

[Feng-RK-Chen-Ryu, PRL]

Higher-form symmetry in fermionic systems?

We discuss anomaly of higher-form symmetry intrinsic in **fermionic systems**.

[RK-Prem-Yu, arXiv:2606.28682]

Higher-form anomaly of lattice models

Internal Symmetry on the Lattice

0-form internal symmetry on the lattice

Tensor product Hilbert space: $\mathcal{H}$

$$\mathcal{H} = \bigotimes_j \mathcal{H}_j$$

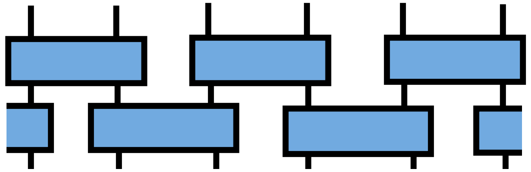
$$U : G \to \mathcal{U}(\mathcal{H})$$

satisfying

(1) $U(g_1)U(g_2) = U(g_1 g_2)$

(2) $U(g)$ preserves locality: **Quantum Cellular Autamata (QCA)**

Finite depth quantum circuit (FDQC) $\subset$ QCA. Not all QCAs are finite depth circuits! (e.g., lattice translation)



0-form Anomalies on the Lattice

Let's consider finite 0-form internal symmetry in (1+1)D.

It's generated by finite depth circuits. (Translations cannot generate a finite group)

't Hooft Anomaly on (1+1)D lattice: Else and Nayak (2014)

[Else-Nayak]

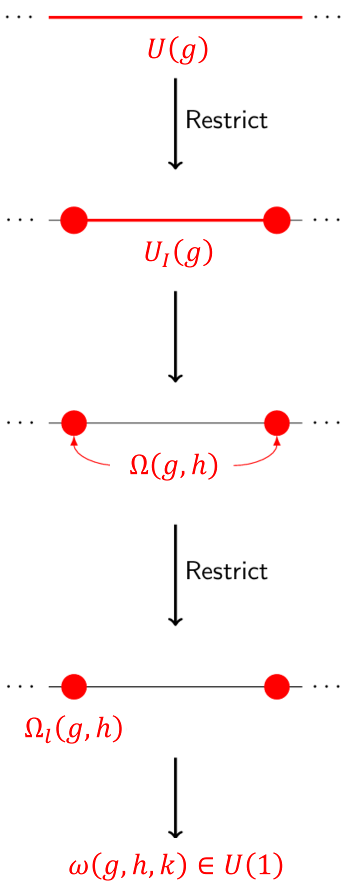
- Obstruction to **short-range entangled (SRE)** state

For finite internal symmetry in (1+1)D on lattice, it's equivalent to:

- Obstruction to **gauging** the symmetry on lattice
- Obstruction to **onsite realization** of internal symmetry

[Seifnashri-Shirley]

Else-Nayak Approach (Else and Nayak, 2014)



Reduction process:

Finite 0-form symmetry $U(g)$ in (1+1)D ⇨ $U(g)U(h) = U(gh)$

⇩

It's a circuit, so one can take a truncation: $U_I(g)$

⇩

$U_I(g)U_I(h) = \Omega(g,h)U_I(gh)$ ⇨ $\Omega(g,h)\Omega(gh,k) = {}^{g}\Omega(h,k)\Omega(g,hk)$

⇩

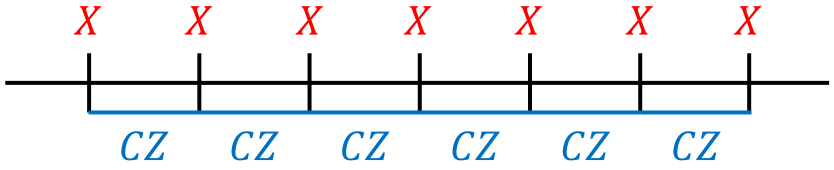
$\Omega = \Omega_l \sqcup \Omega_r$ ⇨ $\Omega_l(g,h)\Omega_l(gh,k) = \omega(g,h,k)\,{}^{g}\Omega_l(h,k)\Omega_l(g,hk)$

The Else-Nayak anomaly index: $[\omega] \in H^3\big(BG, U(1)\big)$

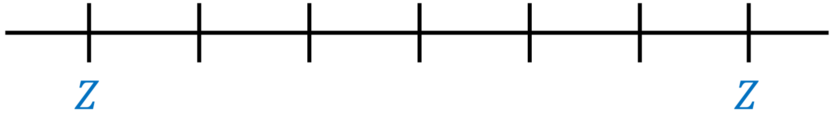
Else-Nayak Approach (Else and Nayak, 2014)

Example: Anomalous $\mathbb{Z}_2$ symmetry

$$U = \prod_i X_i \cdot \prod_i CZ_{i,i+1} =$$



$$\Omega(0,g) = \Omega(g,0) = 0, \ \ \Omega(1,1) = U^2 =$$



$$\Omega_l(g,h)\Omega_l(gh,k) = \omega(g,h,k)\, {}^{g}\Omega_l(h,k)\Omega_l(g,hk)$$

$$\omega(g,h,k) = (-1)^{ghk} = \begin{cases} -1, & g = h = k = 1 \\ 1, & \text{else} \end{cases}$$

$[\omega]$ is the generator of $H^3\big(B\mathbb{Z}_2, U(1)\big) = \mathbb{Z}_2$

Obstruction to SRE

When H3 index is nontrivial, the symmetric state cannot be SRE.

SRE state: $V|0\rangle$ V: Finite depth circuit. $|0\rangle$: Product state.

Show that when $U(g)V|0\rangle = V|0\rangle$, then $U(g)$ is anomaly free.

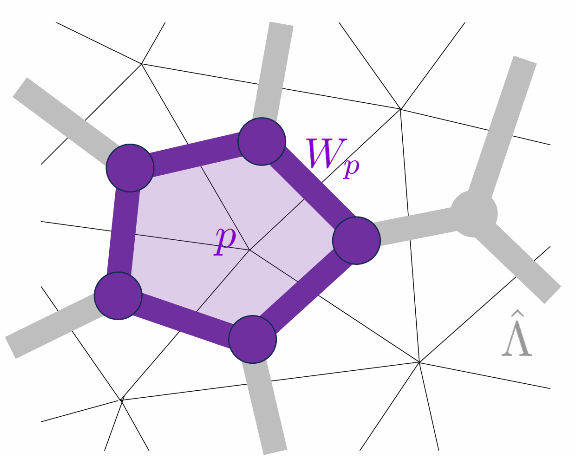
1. Redefine symmetry operator $U'(g) = V^\dagger U(g)V$. This preserves a product state.
2. A circuit preserving product state has a truncation s.t. $U'_I(g)|0\rangle = |0\rangle$
3. This implies that H3 index is trivial.

$$U_I(g)U_I(h) = \Omega(g,h)U_I(gh)$$

$$\Omega_l(g,h)\Omega_l(gh,k) = \omega(g,h,k)\,{}^{g}\Omega_l(h,k)\Omega_l(g,hk)$$

Higher-form symmetry on the Lattice

Let’s consider 1-form symmetry and its anomaly.



Finite 1-form symmetry in $(2+1)$D:

Generated by finite depth circuits on a line.

Higher-form symmetry on tensor product Hilbert space is “always emergent”:

- Small closed loops of symmetry operators (Gauss law operators) W_p
- Symmetry operators are topological within a superselection sector of W_p
- ‘t Hooft anomaly is independent of the choice of superselection sector, and constrains dynamics

Higher-form symmetry on the Lattice

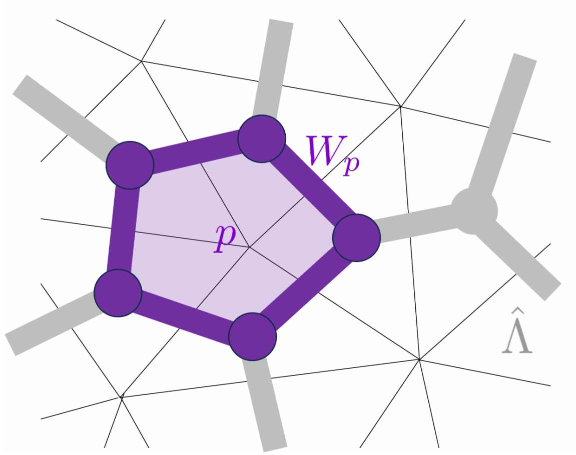
1-form symmetry on 2D tensor product Hilbert space:

Gauss law operators $W_p^{(g)}$ for each plaquette p and $g \in G$, satisfying

(P1) $W_p^{(g_1)} W_p^{(g_2)} = W_p^{(g_1+g_2)}, \qquad W_p^{(0)} = 1$ (group homomorphism)

(P2) $\left[W_p^{(g)}, W_{p'}^{(g')}\right] = 0$ (loops commute with each other)

(P3) $\prod_p W_p^{(g)} = 1$ (operators are topological within Gauss law)



mesoscopic lattice

Global symmetry operator (network of line operators):

$$U(\epsilon) = \prod_p W_p^{\epsilon(p)}$$

satisfying

$U(\epsilon_1)U(\epsilon_2) = U(\epsilon_1 + \epsilon_2 + g),$ $\forall$0-form ϵ_1, ϵ_2 and constant 0-form g

Generalized Else-Nayak Approach

Dimensional reduction process

$$U(\epsilon) = \prod_p W_p^{\epsilon(p)}$$

$$\Downarrow$$

Truncation at region R

$$U_R(\epsilon) = \prod_p W_{p;R}^{\epsilon(p)}$$

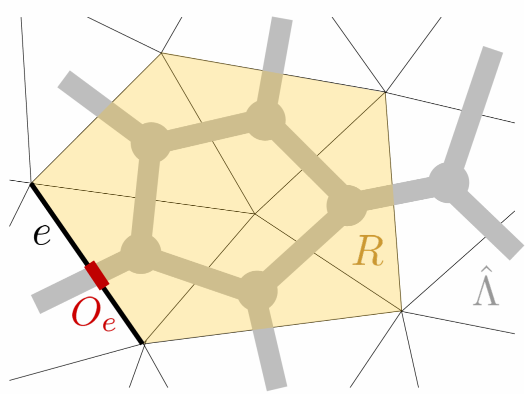
$$\Downarrow$$

$$\Omega(\epsilon_{01}, \epsilon_{12}, g_{012}) = U_R(\epsilon_{01}) U_R(\epsilon_{12}) U_R(\epsilon_{01} + \epsilon_{12} - g_{012})^{-1}$$

$$\Downarrow$$

$$\Omega(\epsilon_{01}, \epsilon_{12}, g_{012}) = \prod_e O_e(\epsilon_{01}, \epsilon_{12}, g_{012})$$

Reduction to point operator



Generalized Else-Nayak Approach

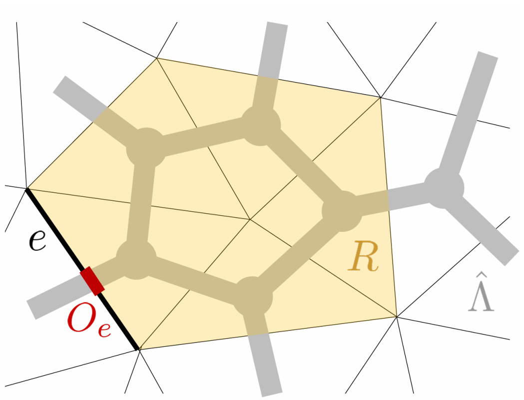
$$\Omega(\epsilon_{01},\epsilon_{12})\Omega(\epsilon_{02},\epsilon_{23}) = {}^{\epsilon_{01}}\Omega(\epsilon_{12},\epsilon_{23})\Omega(\epsilon_{01},\epsilon_{13})$$

$$\Omega(\epsilon_{01},\epsilon_{12},\ g_{012}) = \prod_e O_e(\epsilon_{01},\epsilon_{12},\ g_{012})$$

$$\Downarrow$$

$$e^{2\pi i F_e(\epsilon_{01},\epsilon_{12},\epsilon_{23},\{g\})} = O_e(\epsilon_{01},\epsilon_{12})O_e(\epsilon_{02},\epsilon_{23})\left({}^{\epsilon_{01}}O_e(\epsilon_{12},\epsilon_{23})O_e(\epsilon_{01},\epsilon_{13})\right)^{-1}$$

$$\sum_e F_e(\epsilon_{01},\epsilon_{12},\epsilon_{23},\{g\}) = 0 \quad \Longrightarrow \quad \int_I F = A_l - A_r$$



Finally,

$\omega(\epsilon_{01},\epsilon_{12},\epsilon_{23},\epsilon_{34},\{g\})$

$= A_l(\epsilon_{01},\epsilon_{12},\epsilon_{23}) + A_l(\epsilon_{01},\epsilon_{13},\epsilon_{34}) + A_l(\epsilon_{12},\epsilon_{23},\epsilon_{34}) - A_l(\epsilon_{02},\epsilon_{23},\epsilon_{34}) - A_l(\epsilon_{01},\epsilon_{12},\epsilon_{24})$

Because

$\delta F_e(\epsilon_{01},\epsilon_{12},\epsilon_{23},\epsilon_{34})$

$= F_e(\epsilon_{01},\epsilon_{12},\epsilon_{23}) + F_e(\epsilon_{01},\epsilon_{13},\epsilon_{34}) + F_e(\epsilon_{12},\epsilon_{23},\epsilon_{34}) - F_e(\epsilon_{02},\epsilon_{23},\epsilon_{34}) - F_e(\epsilon_{01},\epsilon_{12},\epsilon_{24})$

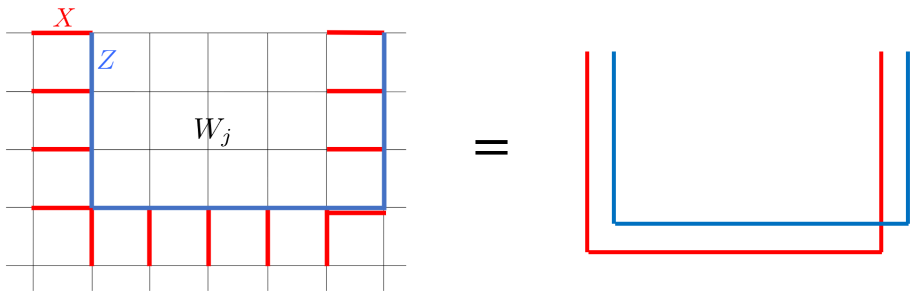
$= 0,$

we obtain that $\omega(\epsilon_{01},\epsilon_{12},\epsilon_{23},\epsilon_{34},\{g\})$ does not depend on ϵ_{ij}.

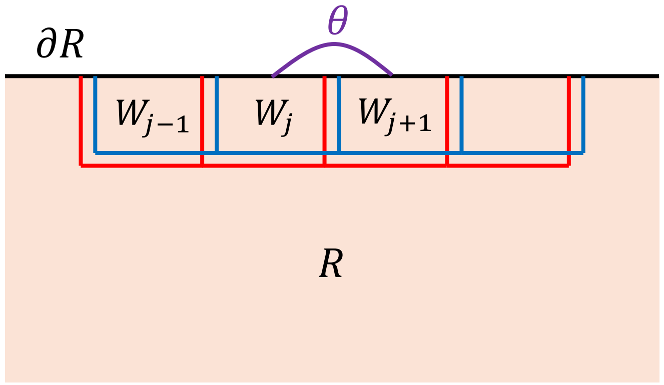
Anomaly index: $[\omega] \in H^4\left(B^2G, U(1)\right)$

[Feng-RK-Chen-Ryu]

Generalized Else-Nayak Approach



Example: Fermionic *em* line in TC



Truncated symmetry operator:

$$W_j W_{j+1} = -W_{j+1} W_j \qquad W_j^2 = -1$$

Reduction to boundary point operator:

$$O_{j,j+1} = (-1)^{\theta_{j,j+1}}, \qquad \theta = \epsilon_{12} d\epsilon_{01} + g_{012} d\epsilon_{02}$$

$$F_{j,j+1} = g_{012} \cdot d\epsilon_{23} = g_{012} \cdot \big(\epsilon_{23}(j) - \epsilon_{23}(j+1)\big)$$

$$A_l = g_{012}\epsilon_{23}$$

Anomaly index: $\omega = g_{012} g_{234} \leftrightarrow B_2 \cup B_2 = w_2 \cup B_2$

$$[\omega] \in H^4\big(B^2 Z_2, U(1)\big)$$

$H^4\big(B^2 Z_2, U(1)\big) = Z_4$. Z2 anyon can be boson, semion, fermion, or anti-semion. (for bosonic models)

What we can show:

Higher-form anomaly on lattice: Classified by $H^4\left(B^2G, U(1)\right)$.

Dynamical consequence: When H4 index is nontrivial, short-range-entangled (SRE) state is forbidden

[Feng-RK-Chen-Ryu]

Onsite realization: Gives a criteria for possibility of onsite realization (related to possibility of higher gauging)

[Feng-Chen-Hsin-RK]

Higher-form anomaly of fermionic systems

Symmetry and anomaly of fermionic systems

In general, fermionic systems have much richer anomalies than bosonic systems

Ex. Z2 symmetry in (1+1)D.

Bosons: Z2 anomaly

Fermions: Z8 anomaly (chiral Z2 symmetry of Majorana fermion)

Does higher-form symmetry have intrinsically fermionic anomalies?

Spin/braiding of anyons intrinsic in fermionic systems?

Constructing fermionic theory: Fermion condensation

Take $U(1)_4$. Abelian anyon theory with $\{1, v, \psi, v\psi\}$.

A fermion ψ generates anomalous Z2 1-form symmetry. Let's write background gauge field B. Anomaly is:

$$\pi \int B \cup B$$

We can get fermionic theory by performing "fermion condensation":

[Bhardwaj-Gaiotto-Kapustin,
Kapustin-Thorngren, Thorngren,
Aasen-Lake-Walker, …]

Gauge B by coupling the theory to spin structure: $d\xi = w_2$

$\pi \int B \cup B = \pi \int w_2 \cup B = 0.$ (spin str: $w_2 = 0$). B can be gauged as a fermionic theory.

It yields a fermionic invertible phase (Chern insulator).

New Fermionic topological order

Take U(1)$_4$. Abelian anyon theory with $\{1, v, \psi, v\psi\}$.

The theory has Z4 1-form symmetry. Let's write background gauge field as a pair of Z2 gauge fields B, C:

ψ line $dB = \frac{d\hat{C}}{2}$ v line

It's 't Hooft anomaly is: $\pi \int B \cup B + B \cup C + \frac{2\pi}{8} \int C \cup C + \ \ldots$

Fermion
Self statistics

v, ψ
Mutual braiding

v particle
1/8 spin statistics

We gauge B by coupling to twisted spin structure: $d\xi = w_2 + C$

Resulting theory is topological order.

New Fermionic topological order

Why the resulting theory is topological order?

the 't Hooft anomaly: $\pi\int B\cup B + B\cup C + \frac{2\pi}{8}\int C\cup C + \ \ldots$

Fermion
Self statistics

v,ψ
Mutual braiding

v particle
1/8 spin statistics

Twisted spin structure trivializes mixed anomaly: $\pi\int B\cup B + B\cup C = \pi\int d(\xi\cup B) = 0$

After gauging B, the theory has a Z2 1-form symmetry C. It is anomalous:

$$\frac{2\pi}{8}\int C\cup C + \ \ldots$$

Spin 1/8: this anomaly is illegally quantized as bosonic 't Hooft anomaly ⇨ Fermionic anomaly.

New Fermionic topological order

What we have done: coupling twisted spin structure to U(1)$_4$, and condensing/proliferating a fermion

Properties of the theory:

- Z2 Abelian anyon v. Fusion rule $v\times v=f$. f is a local fermion.
- v has spin 1/8. Implies mod 8 fermionic ’t Hooft anomaly
- $v\times v=f$ implies that Z2 symmetry exhibits fermionic anomaly: Junction of anyons violates fermion parity
- Two states on a torus. Distinct from bosonic/spin Chern-Simons theory.

Fermionic anomaly of Z2 1-form symmetry

The anomaly of Z2 1-form symmetry w/ twisted spin structure in (2+1)D: Z16

It's classified through three layers of group cohomologies:

- $H^4\left(B^2Z_2, U(1)\right) = Z_4$ (bosonic layer)
- $H^3(B^2Z_2, Z_2) = Z_2$ (Complex fermion layer)
- $H^2(B^2Z_2, Z_2) = Z_2$ (Kitaev's Majorana layer)

Central extensions of these groups yield Z16.

Our theory from $U(1)_4$ is within complex fermion layer; $\nu = 2 \in Z_{16}$.

Bulk topological response

Bulk SPT phase in (3+1)D with twisted spin structure : Z16

Topological response at $\nu = 4 \in Z_{16}$ (bosonic layer): $\frac{2\pi}{4}\int P(C) = \frac{2\pi}{4}\int C \cup C + C \cup_1 dC$

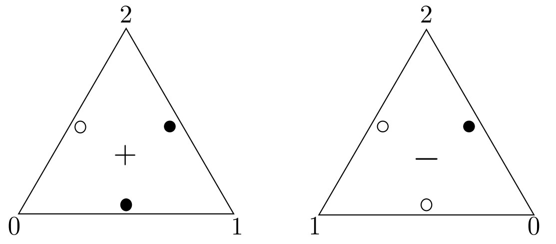
Bosonic SPT w/ Z2 symmetry. Z4 classification

Topological response at $\nu = 2 \in Z_{16}$ (complex fermion layer). It looks like:

$$z_\xi\left(\frac{dC}{2}\right) \cdot \exp\left(\frac{2\pi i}{8}\int C \cup C + C \cup_1 dC\right)$$

Twisted spin theory (Grassmann integral)

Fractional response for Z2 1-form symmetry, spin 1/8



Grassmann variables on (d-1) simplices, and path integrate them

This becomes a gauge invariant response coupled to twisted spin structure $d\xi = w_2 + C$.

Describes anomaly for the spin 1/8 particle w/ fusion rule v x v = f .

Bulk topological response

What’s the bulk topological response for $\nu = 1 \in Z_{16}$ (Kitaev’s Majorana layer)?

With twisted spin structure $d\xi = w_2 + C$ on oriented 4-manifold, 2-form background C has integral lift $\hat{C}$.

The response is then given in the form of

$$\mathrm{Arf}(PD(\hat{C})) \cdot \exp\left(\frac{2\pi i}{16} \int \hat{C} \cup \hat{C}\right)$$

Kitaev’s Majorana chain

Fractional response for Z2 1-form symmetry, spin 1/16

Arf = partition function of Kitaev’s Majorana chain (Kitaev’s Majorana layer)

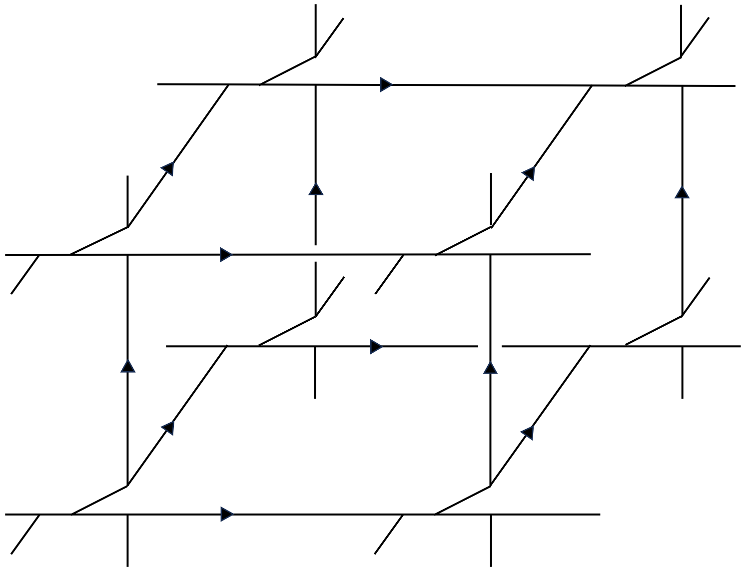
This gives signature of 4-manifold mod 16, hence gauge invariant.

[Guillou-Marin, Kirby-Taylor]

Lattice model for $\nu = 2 \in Z_{16}$ theory

Our lattice model is an analogue of Walker-Wang Hamiltonian model.

Walker-Wang: anyon diagrams on edges of cubic lattice [Walker-Wang]



$$|\Psi\rangle = \sum |\text{ anyon diagram }\rangle$$

Modular tensor category (2+1D bosonic TQFT data)

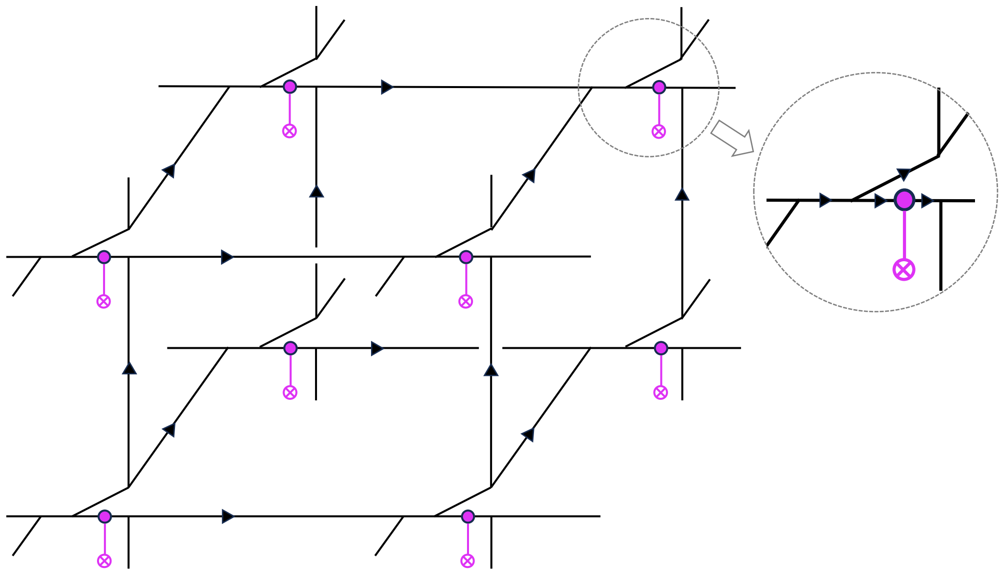
⇨ (3+1)D bosonic invertible phase,

with gapped boundary realized by 2+1D TQFT

Lattice model for $\nu = 2 \in Z_{16}$ theory

Our lattice model is an analogue of Walker-Wang Hamiltonian model.

Our model: $\mathrm{U}(1)_4 = \{1, v, \psi, v\psi\}$ Walker-Wang type model decorated with local fermions



$$|\Psi\rangle = \sum |\text{ anyon diagram }\rangle \otimes |\text{ local fermions }\rangle$$

- We put { 1, v } anyons on cubic lattice edges
- Diagram may not be closed, and could fuse into pink ψ
- When ψ is present, we put a complex fermion

Lattice model for $\nu = 2 \in Z_{16}$ theory

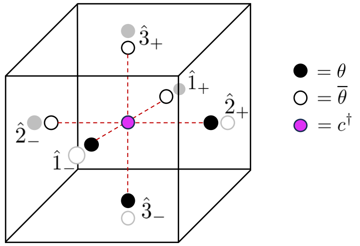
$$|\Psi\rangle = \sum |\,\text{anyon diagram}\,\rangle \otimes |\,\text{local fermions}\,\rangle$$

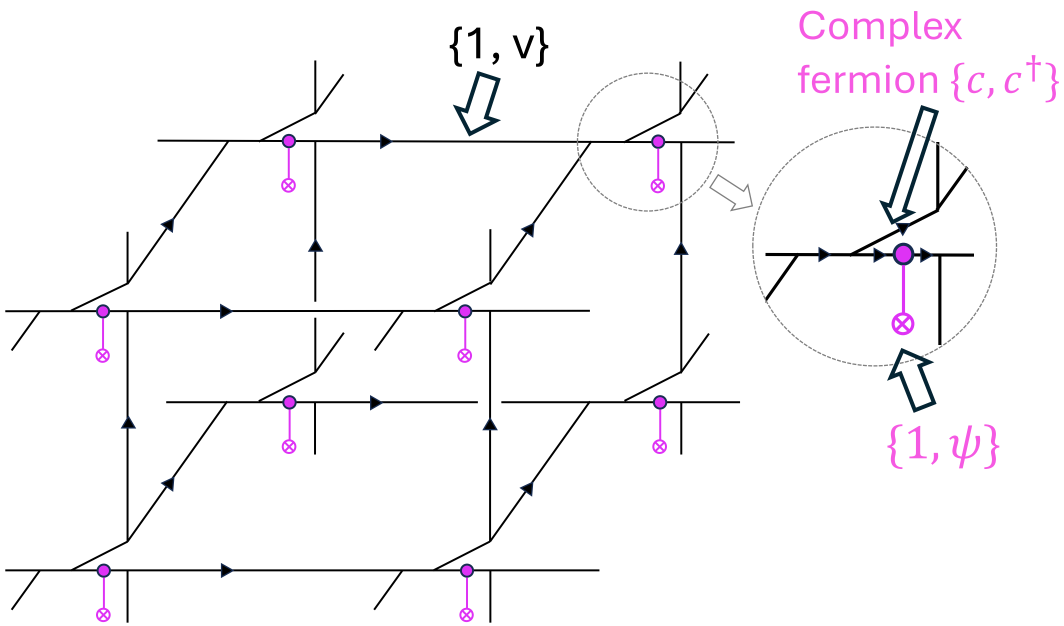
“Walker-Wang type”

[Walker-Wang]

“Grassmann integral”

[Gu-Wen, Gaiotto-Kapustin, RK ...]





∃ commuting projector Hamiltonian in the form of:

$$H = -\sum_v P_v - \sum_p P_p$$

Vertex term

Fusion constraints

Plaquette term

Fusing anyon loop along plaquette

Gapped boundary: Gapped boundary has Z2 anyon {1, v}. Satisfies Z2 fusion rule v x v = f .

$\boldsymbol{\nu = 1 \in Z_{16}}$ theory: Cousin of Ising

What's the TQFT that saturates $\nu = 1 \in Z_{16}$ anomaly?

Let's consider Ising TQFT $\{1, \sigma, \psi\}$: non-invertible fusion rule $\sigma \times \sigma = 1 + \psi$

One can turn it into fermionic, twisted spin TQFT: non-invertible fusion rule $\tau \times \tau = 1 + f$

f is a local fermion. Then, the above fusion rule is interpreted as anomalous invertible Z2 1-form symmetry.

- done by condensing a fermion ψ , with additional insertion of σ line along Poincare dual of w2.
- With this insertion, σ turns into an anyon in untwisted sector. Denoted it by τ.

Sixteen-fold way for fermionic topological order

Kitaev's sixteen-fold way: Family of 16 topological orders, by gauging fermion parity in (2+1)D invertible phase

Spin(ν)$_1$ ($\nu = 1$ mod 16 is Ising, $\nu = 2$ mod 16 is U(1)$_4$ …)

Correspondingly, there is a sixteen-fold family of fermionic topological orders:

- Consists of single Z2 anyon, with the fusion rule:

$\tau \times \tau = 1 + f$ ($\nu = 1$ mod 2, Kitaev's Majorana layer)

$v \times v = f$ ($\nu = 2$ mod 4, complex fermion layer)

$v \times v = 1$ ($\nu = 0$ mod 4, bosonic layer)

- Generates anomalous Z2 1-form symmetry with $\nu \in Z_{16}$ anomaly. Carries spin $\nu/16$.

Summary

- Higher-form anomaly of lattice models (on tensor product Hilbert space)
- Mod 16 fermionic anomaly of Z2 1-form symmetry in twisted spin theories
- New sixteen-fold family of fermionic topological order. Lattice Hamiltonian model for $\nu = 2 \in Z_{16}$.

Future directions

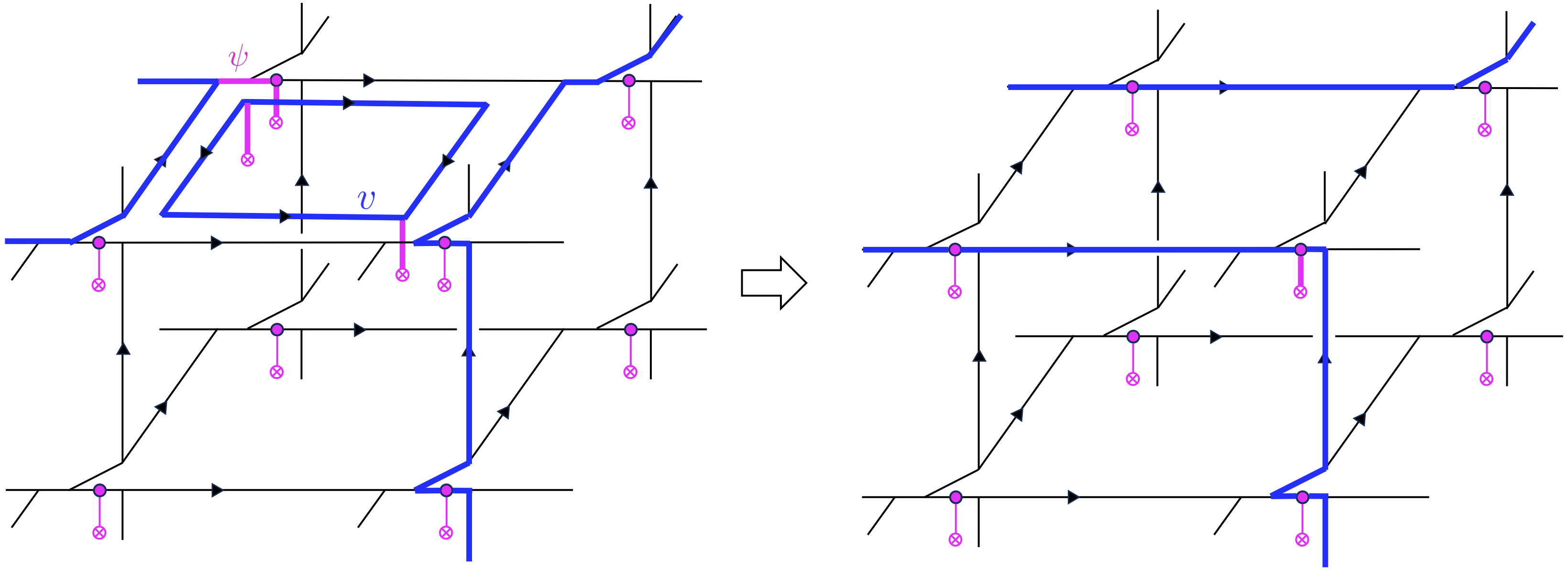
- Lattice model for $\nu = 1 \in Z_{16}$. Would involve Majorana chain decorations.
- Exploring classification of twisted spin TQFT and new fermionic topological orders.

Backup slides

Lattice model for $\nu = 2 \in Z_{16}$ theory

$$H = -\sum_{v} P_v - \sum_{p} P_p$$

Plaquette terms fluctuate anyon diagram by fusing v loop:



together with fermion fluctuation.