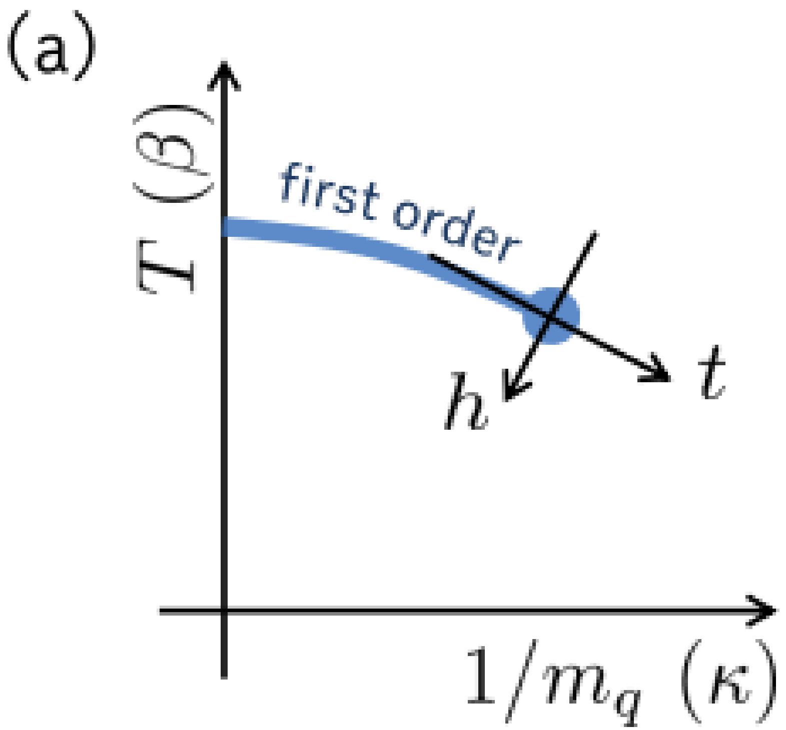
cumulant expansion of thermodynamic potential in heavy-quark QCD

Takahiro Doi (Kyoto Univ.)

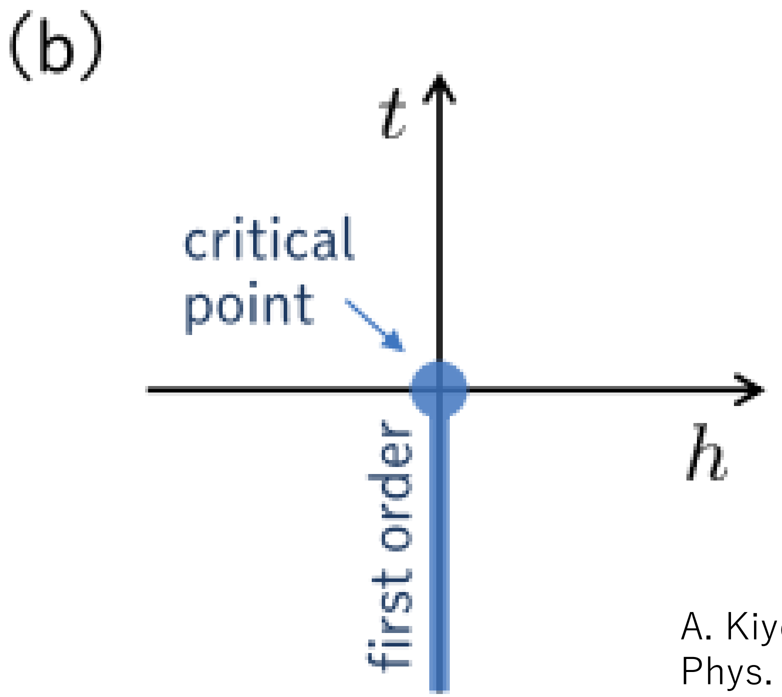
・Kei Tohme, Takahiro M. Doi, Masakiyo Kitazawa, Krzysztof Redlich, and Chihiro Sasaki
Phys. Rev. D 112, 094515 (2025), arXiv:2508.09927
・Takahiro M. Doi and Masakiyo Kitazawa in preparation

2026/06/03 QCD Critical Point and Hydrodynamic Evolution, YITP

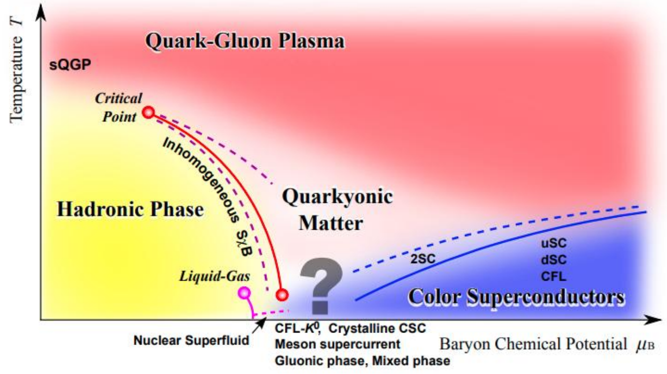
Heavy quark QCD



Heavy Quark QCD



Ising model



K. Fukushima and T. Hatsuda, *Rep. Prog. Phys.* **74** 014001, 2011

Important but not focused in this talk

A. Kiyohara, M. Kitazawa, S. Ejiri, and K. Kanaya, Phys. Rev. D 104, 114509, 2021

In this talk, we consider heavy quark QCD and its phase diagram using hopping parameter expansion on a lattice.

≃Heavy quark mass expansion

Hopping parameter expansion in lattice effective action

Setup
- Wilson(plaquette) gauge action for pure gluons(SU(3) Yang Mills)
- Degenerated Wilson fermion with Nf flavor with chemical potential(we set μ =0 later)
- Finite temperature system

Partition function

$$Z = \int \mathcal{D}U\mathcal{D}\bar{q}\mathcal{D}q \exp\left[-S_g - S_Q\right] = \int \mathcal{D}U e^{-S_g} \exp\left[\ln\mathrm{Det}(1-\kappa H)\right] = \int \mathcal{D}U e^{-S_g} \exp\left[-N_\mathrm{f}\sum_{l=1}^{\infty}\frac{\kappa^l}{l}\mathrm{Tr}[H^l]\right]$$

hopping parameter expansion

Wilson fermion action

$$S_Q = N_f \sum_{s,s'} \bar{q}(s)\left[1-\kappa \underline{H(s,s')}\right] q(s)$$

hopping matrix

$$\kappa \simeq 1/m_q \ll 1$$

heavy quark

Hopping matrix

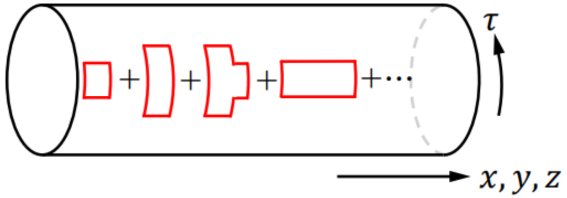
$$H(s,s') \equiv \sum_{k=1}^{3}\left[(1-\gamma_k)\,U_k(s)\delta_{s+\hat{k},s'} + (1+\gamma_k)\,U_k^\dagger(s-\hat{k})\delta_{s-\hat{k},s'}\right]$$
$$+\left[(1-\gamma_4)\,e^{\mu_q a}U_4(s)\delta_{s+\hat{4},s'} + (1+\gamma_4)\,e^{-\mu_q a}U_4^\dagger(s-\hat{4})\delta_{s-\hat{4},s'}\right],$$

Hopping parameter expansion in lattice effective action

Fermion action

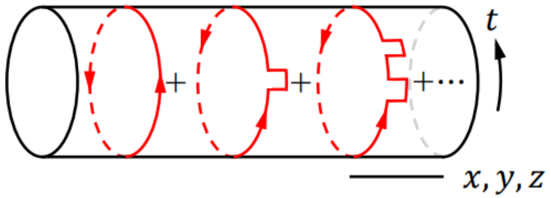
$$\ln \mathrm{Det}(1-\kappa H) = -N_{\mathrm{f}} \sum_{l=1}^{\infty} \frac{\kappa^l}{l} \mathrm{Tr}\, H^l = \sum_{k=-\infty}^{\infty} e^{k\hat{\mu}_{\mathrm{q}}} \Lambda_k \qquad (\hat{\mu}_{\mathrm{q}} \equiv \mu_{\mathrm{q}}/T)$$

Spatially closed loops $(k=0)$

$\Lambda_0 =$ 

$$\Lambda_0 \equiv (2\kappa)^4 N_{\mathrm{f}} \sum_{s} \sum_{\mu<\nu} \mathrm{Re\,tr_c}\, P_{\mu\nu}(s) + \mathcal{O}(\kappa^6)$$

Temporally closed loops $(k \neq 0)$

$\Lambda_k =$ 

$$\Lambda_k \equiv (2\kappa)^{kN_t} 2 N_{\mathrm{f}} N_{\mathrm{c}} \frac{(-1)^{k+1}}{k} L_k + \mathcal{O}\left(\kappa^{kN_t+2}\right)$$

$$L_k \equiv \frac{1}{N_{\mathrm{c}}} \sum_{\boldsymbol{s}} \mathrm{tr_c} \left[\prod_{j=0}^{kN_t-1} U_t(\boldsymbol{s}+j\hat{4}) \right].$$

k-winding Polyakov loop

Higher k means higher order(small contribution to effective action) in hopping parameter expansion.
Hopping parameter expansion in action level is considered to be converged within the physical κ .

N. Wakabayashi , S. Ejiri , K. Kanaya , M. Kitazawa, Progress of Theoretical and Experimental Physics, 2022-3, 033B05, 2022

Our motivation: How about thermodynamic potential? $\Omega = -T \ln Z$

Hopping parameter expansion of thermodynamic potential

The partition function

$$\boldsymbol{Z = Z_{\mathrm{g}} \left\langle \exp\left[\sum_{k=-\infty}^{\infty} e^{k\hat{\mu}_{\mathrm{q}}} \Lambda_k\right]\right\rangle}$$

$$Z_{\mathrm{g}} = \int \mathcal{D}U \exp[-S_{\mathrm{g}}],$$

$$\langle \mathcal{O} \rangle = \lim_{\alpha \to 0} \frac{1}{Z_{\mathrm{g}}} \int \mathcal{D}U \exp[-S_{\mathrm{g}} - \alpha S_{\mathrm{Q}}]\mathcal{O}.$$

$$(\hat{\mu}_{\mathrm{q}} \equiv \mu_{\mathrm{q}}/T)$$

The grand potential

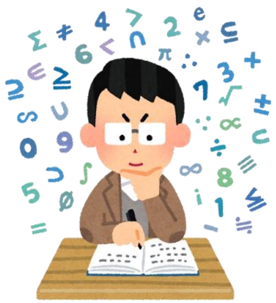
$$\Omega(T, \mu_{\mathrm{q}}) = -T \ln Z_{\mathrm{g}} - T \ln \underline{\left\langle \exp\left[\sum_{k=-\infty}^{\infty} e^{k\hat{\mu}_{\mathrm{q}}} \Lambda_k\right]\right\rangle}$$

→ cumulant expansion

Hopping parameter expansion of thermodynamic potential

$$\Omega(T,\mu_{\mathrm{q}}) = -T\ln Z_{\mathrm{g}} - T\sum_{\{m_k\}} \frac{e^{\sum_k k m_k \hat{\mu}_{\mathrm{q}}}}{\prod_k m_k!}\left\langle \prod_{k=-\infty}^{\infty}\Lambda_k^{m_k}\right\rangle_{\mathrm{c}}$$

Ω_{q} : quark contributions



calculate

$$\frac{\Omega_{\mathrm{q}}(T,\mu_{\mathrm{q}})}{-T} = X_0 + \sum_{w=1}^{\infty}\left(e^{w\hat{\mu}_{\mathrm{q}}}+e^{-w\hat{\mu}_{\mathrm{q}}}\right)X_w$$

X_w : written by Λ_k

$$\Lambda_0 \equiv (2\kappa)^4 N_{\mathrm{f}} \sum\sum \mathrm{Retr_c}\, P_{\mu\nu}(s) + \mathcal{O}(\kappa^6)$$

$$\Lambda_k \equiv (2\kappa)^{kN_t} 2N_{\mathrm{f}}N_{\mathrm{c}}\frac{(-1)^{k+1}}{k}L_k + \mathcal{O}\left(\kappa^{kN_t+2}\right)$$

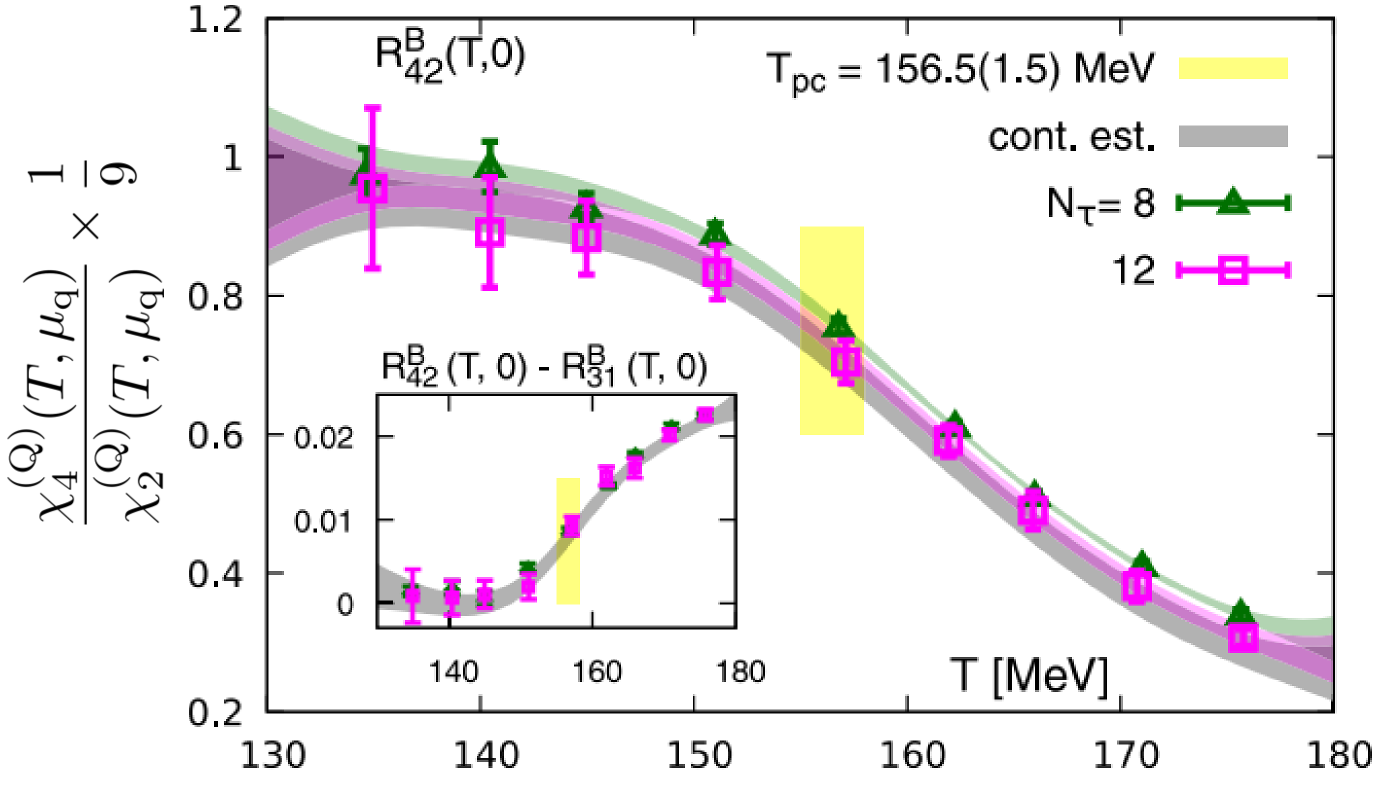
I show the 2 topics from this formula.

1. **Quark number susceptibility**
2. **Convergence radius of cumulant exp.**

Quark number susceptibilities

Tohme, et al., PRD112, 094515 (2025)

lattice QCD result(physical quark mass)



A. Bazavov, D. Bollweg, H.-T. Ding, P. Enns, J. Goswami, P. Hegde, O. Kaczmarek, F. Karsch, R. Larsen et al. (HotQCD Collaboration), Phys. Rev. D 101, 074502 (2020)

Heavy quark limit

$$\frac{\chi_4^{(Q)}(T,\mu_q)}{\chi_2^{(Q)}(T,\mu_q)} \times \frac{1}{9} = \begin{cases} 1/9 & (T > T_c), \\ 1 & (T < T_c), \end{cases}$$

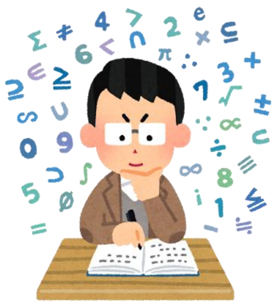
$$\chi_n^{(Q)}(T,\mu_1) \equiv -\frac{\partial^n}{\partial \hat{\mu}_q^n} \frac{\Omega(T,\mu_q)}{TV} \qquad (\hat{\mu}_q \equiv \mu_q/T)$$

9:1 (low T: high T) is approximately reproduced even in the physical quark mass lattice QCD!

The (part of) properties of real QCD can be investigated from the heavy quark QCD.

Convergence radius of cumulant expansion

$$\Omega(T,\mu_{\mathrm{q}}) = -T\ln Z_{\mathrm{g}} - T\sum_{\{m_k\}} \frac{e^{\sum_k k m_k \hat{\mu}_{\mathrm{q}}}}{\prod_k m_k!}\left\langle \prod_{k=-\infty}^{\infty}\Lambda_k^{m_k}\right\rangle_{\mathrm{c}}$$



calculate

Ω_{q} : quark contributions

$$\frac{\Omega_{\mathrm{q}}(T,\mu_{\mathrm{q}})}{-T} = X_0 + \sum_{w=1}^{\infty}\left(e^{w\hat{\mu}_{\mathrm{q}}} + e^{-w\hat{\mu}_{\mathrm{q}}}\right)X_w$$

X_w : written by Λ_k

Truly converged? finite convergence radius? asymptotic expansion?

We check the convergence radius of the cumulant expansion in the simplest case.

Convergence radius of cumulant expansion

We check the convergence radius of the cumulant expansion in the simplest case.

- Nt=4
- We take Leading order(LO) in hopping parameter expansion for the effective action.
- zero chemical potential μ=0.

$$Z = \int \mathcal{D}U e^{-S_g(\beta)+\Lambda_0+2\mathrm{Re}\Lambda_1}$$
$$= Z_{\mathrm{YM}}\langle \exp(\Lambda_0 + 2\mathrm{Re}\Lambda_1)\rangle_{\mathrm{YM}}$$
$$= Z_{\mathrm{YM}}\langle \exp(\lambda a)\rangle_{\mathrm{YM}}$$

plaquette

$$\Lambda_0 = (2\kappa)^4 N_f \sum_s \sum_{\mu<\nu} \mathrm{Retr}P_{\mu\nu}(s)$$

Polyakov loop

$$\mathrm{Re}\Lambda_1 = (2\kappa)^4 2 N_f N_c \mathrm{Re}L_1$$

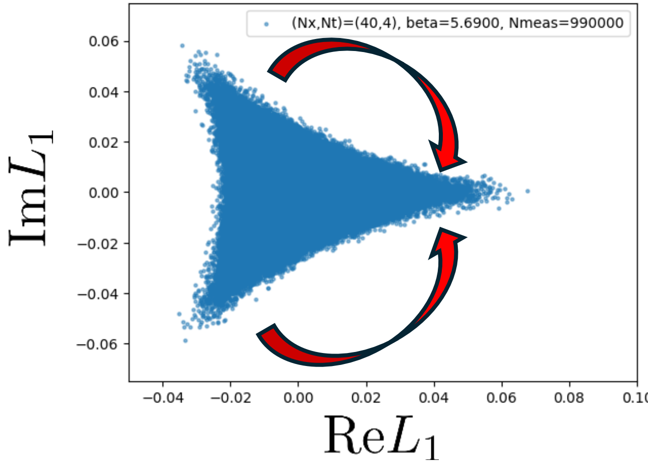
κ -dependent part $\lambda \equiv 2^{N_t+2} N_f N_c \kappa^{N_t} = 2^6 N_f N_c \kappa^4 (N_t = 4)$

κ -independent part $a \equiv \frac{1}{4N_c}\sum_s \sum_{\mu<\nu} \mathrm{Retr}P_{\mu\nu}(s) + \mathrm{Re}L_1$

$$\Omega = -T \ln Z$$

Our target

$$-\Omega_Q/T \equiv \ln\langle \exp(\lambda a)\rangle_{\mathrm{YM}}$$



We take Z3 rotations to real L sector in the average $\langle\cdot\rangle_{\mathrm{YM}}$ to consider the situation of the thermodynamic limit.

Simulation: Nconf=100000

Convergence radius of cumulant expansion

Cumulant expansion of the thermodynamic potential

$$\ln\langle\exp(\lambda a)\rangle_{\mathrm{YM}} = \sum_{k=1}^{\infty} c_k \lambda^k$$

: hopping parameter expansion of thermodynamic potential

We estimate the convergence radius λ_c by 2 methods.

$$c_1 = \langle a\rangle_{\mathrm{YM}}$$
$$c_2 = \langle a^2\rangle_{\mathrm{YM}} - \langle a\rangle_{\mathrm{YM}}^2$$
$$c_3 = \langle a^3\rangle_{\mathrm{YM}} - 3\langle a^2\rangle_{\mathrm{YM}}\langle a\rangle_{\mathrm{YM}}^2 + 2\langle a\rangle_{\mathrm{YM}}^2$$
$$\ldots$$

method 1. Cauchy-Hadamard formula

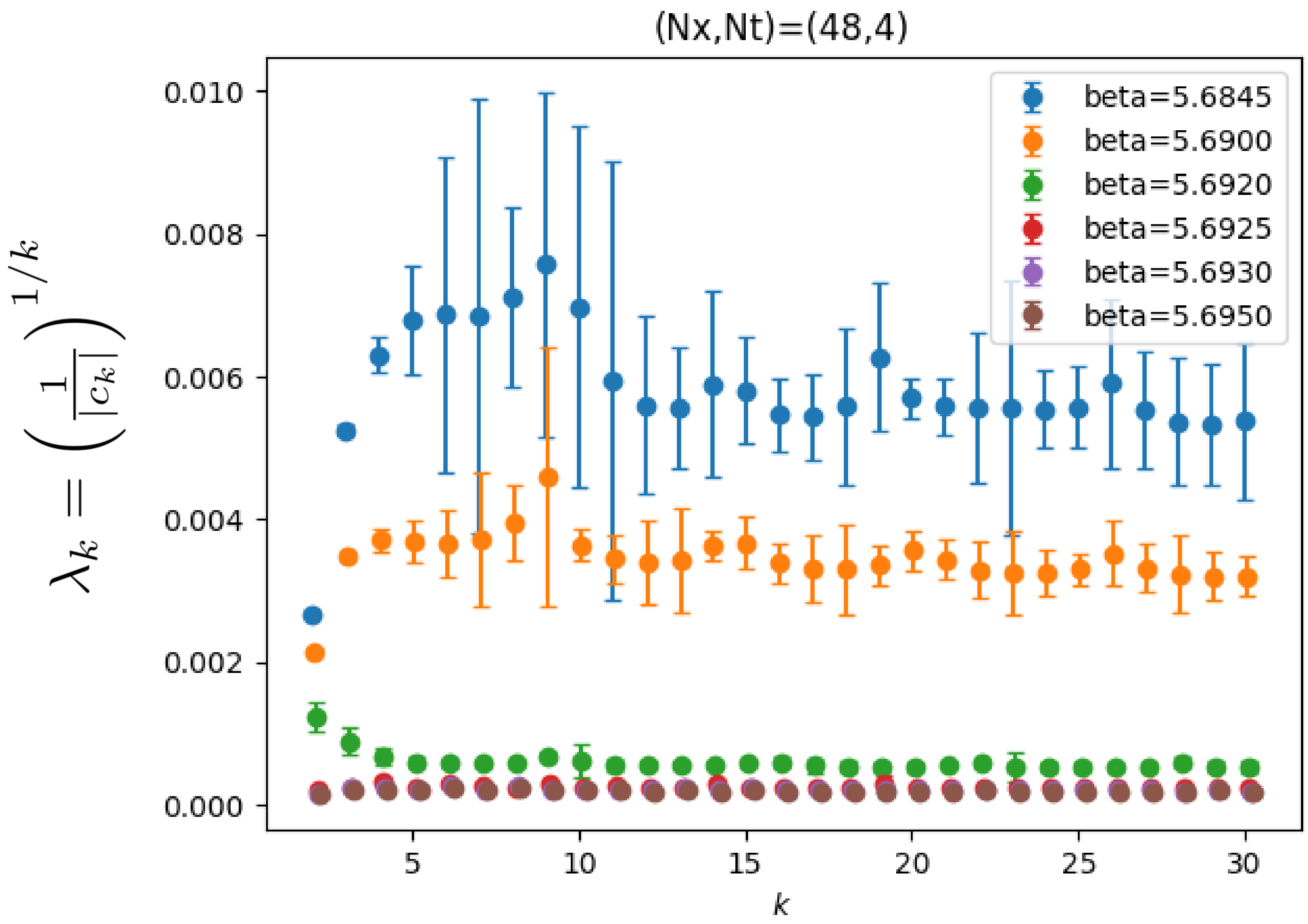
$$\lambda_c = \lim_{k\to\infty} \lambda_k \qquad \lambda_k = \left(\frac{1}{|c_k|}\right)^{1/k}$$

method 2. Lee-Yang zero

$$\lambda_c = \min_n |\lambda_{\mathrm{LYZ}}^{(n)}| \qquad \lambda_{\mathrm{LYZ}}^{(n)} \in \mathbb{C} \text{ such that } \langle\exp(\lambda_{\mathrm{LYZ}}^{(n)} a)\rangle_{\mathrm{YM}} = 0$$

1. Convergence radius from Cauchy-Hadamard formula

$$\ln\langle\exp(\lambda a)\rangle_{\mathrm{YM}} = \sum_{k=1}^{\infty} c_k \lambda^k \qquad \lambda_c = \lim_{k\to\infty} \lambda_k \qquad \lambda_k = \left(\frac{1}{|c_k|}\right)^{1/k}$$



lattice parameter | Temperature

$$\beta = \frac{6}{g^2} \longleftrightarrow T$$

corresponds

- We estimate the convergence radius

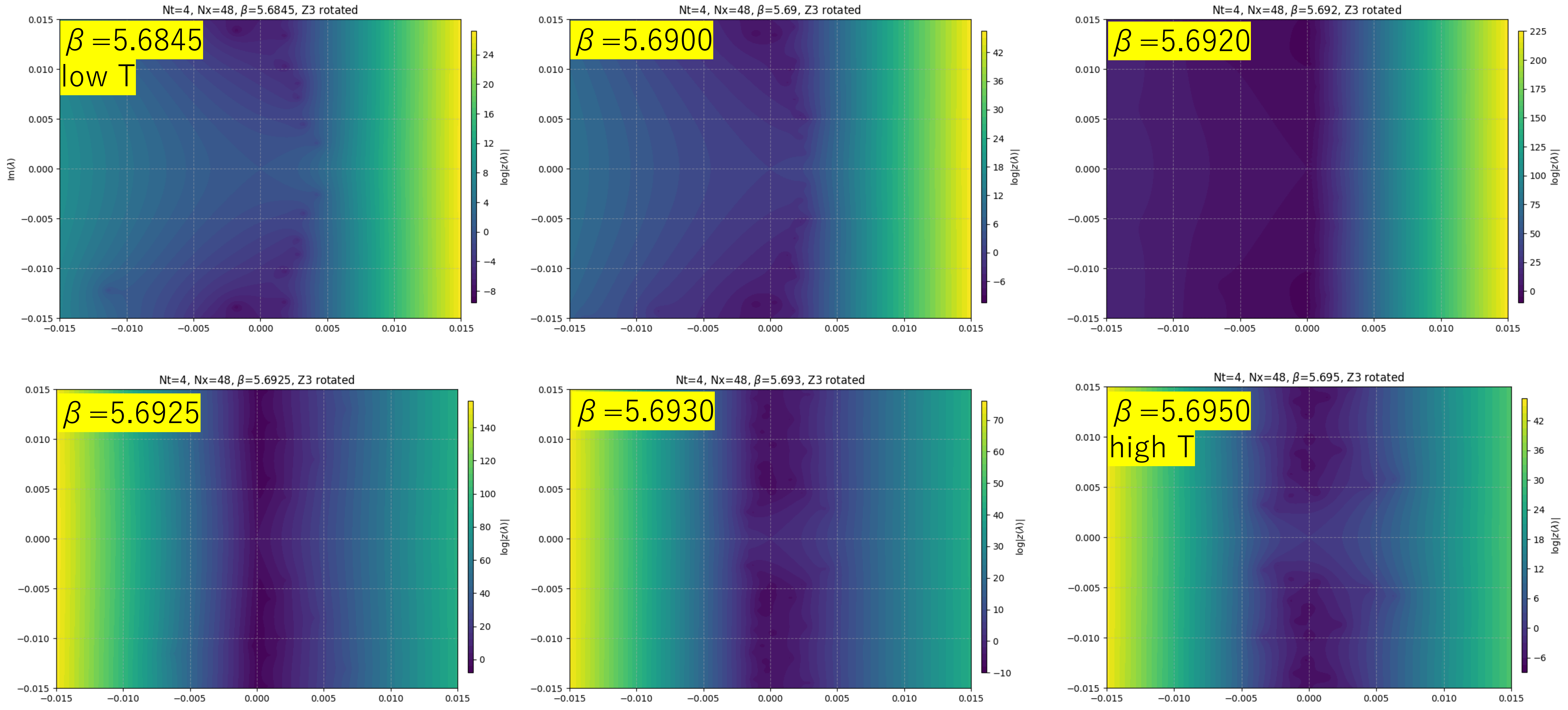
$$\lambda_c^{(\mathrm{CH})} \simeq \lambda_{30}$$

- Error bar is statistical error of Monte Carlo calc.
- We find the convergence radii depend on β .

2. Convergence radius from Lee-Yang zeros(LYZ)

$$\lambda_c = \min_n |\lambda^{(n)}_{\mathrm{LYZ}}| \qquad \lambda^{(n)}_{\mathrm{LYZ}} \in \mathbb{C} \text{ such that } \langle \exp(\lambda^{(n)}_{\mathrm{LYZ}} a) \rangle_{\mathrm{YM}} = 0$$

Horizontal axis: Re λ
Vertical axis: Im λ
color: $|\langle \exp(\lambda a) \rangle_{\mathrm{YM}}|$



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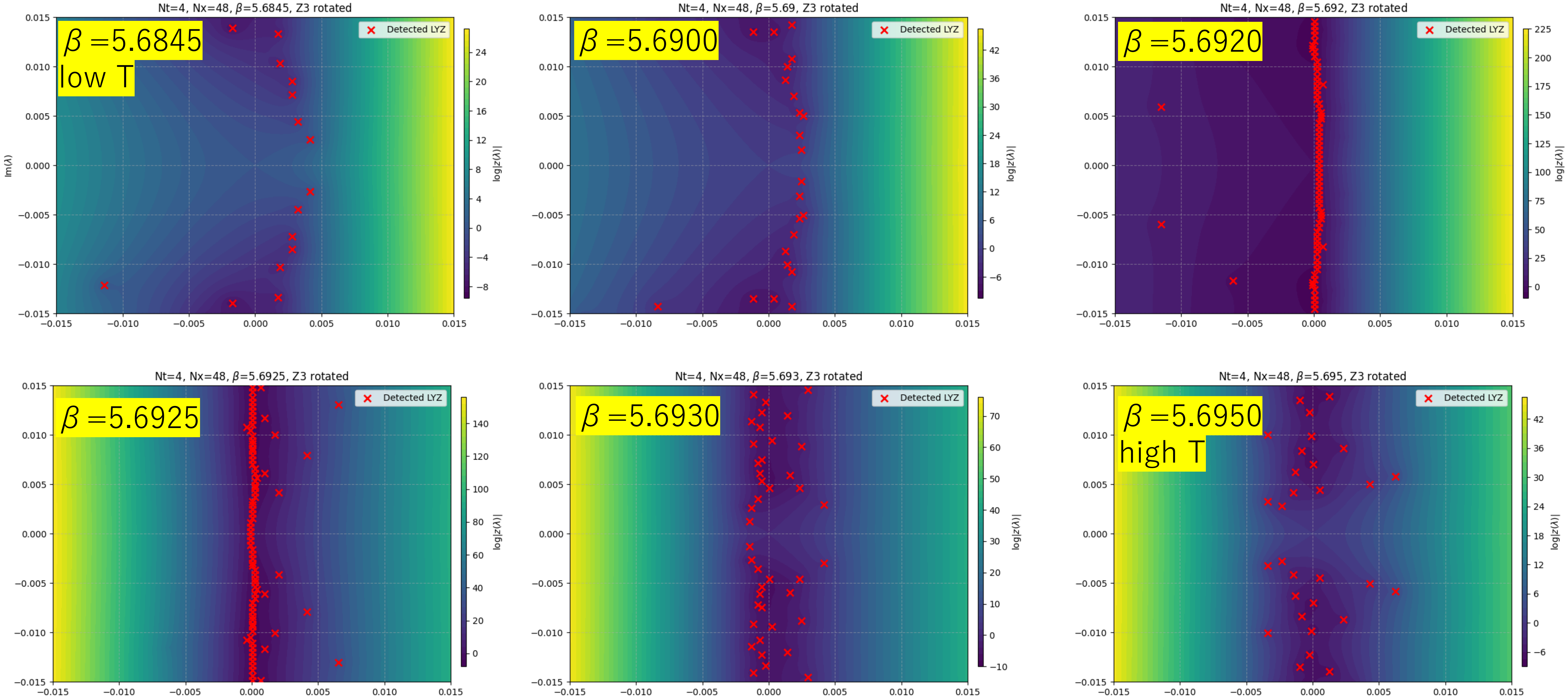

2. Convergence radius from Lee-Yang zeros(LYZ)

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Horizontal axis: Re λ
Vertical axis: Im λ
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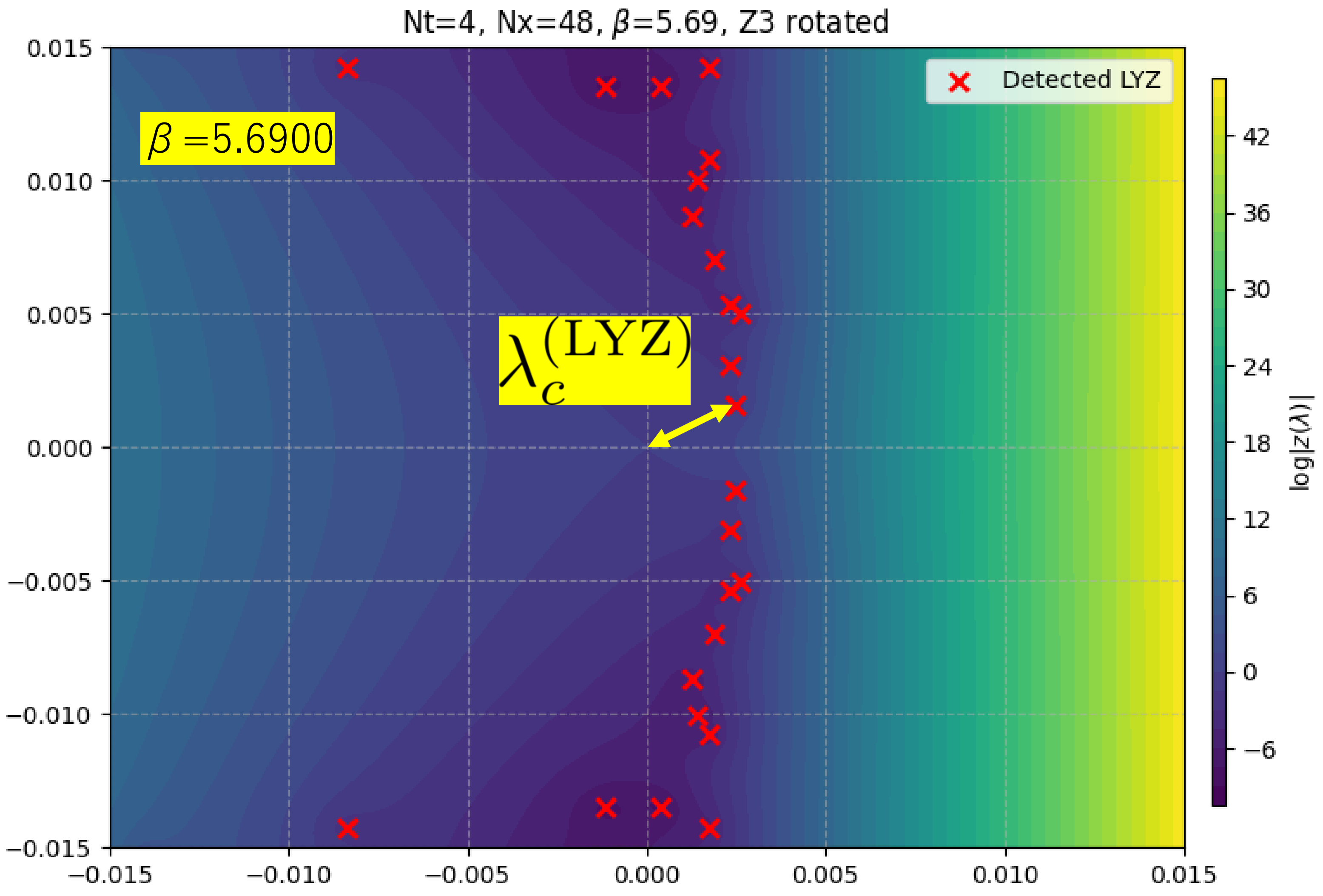
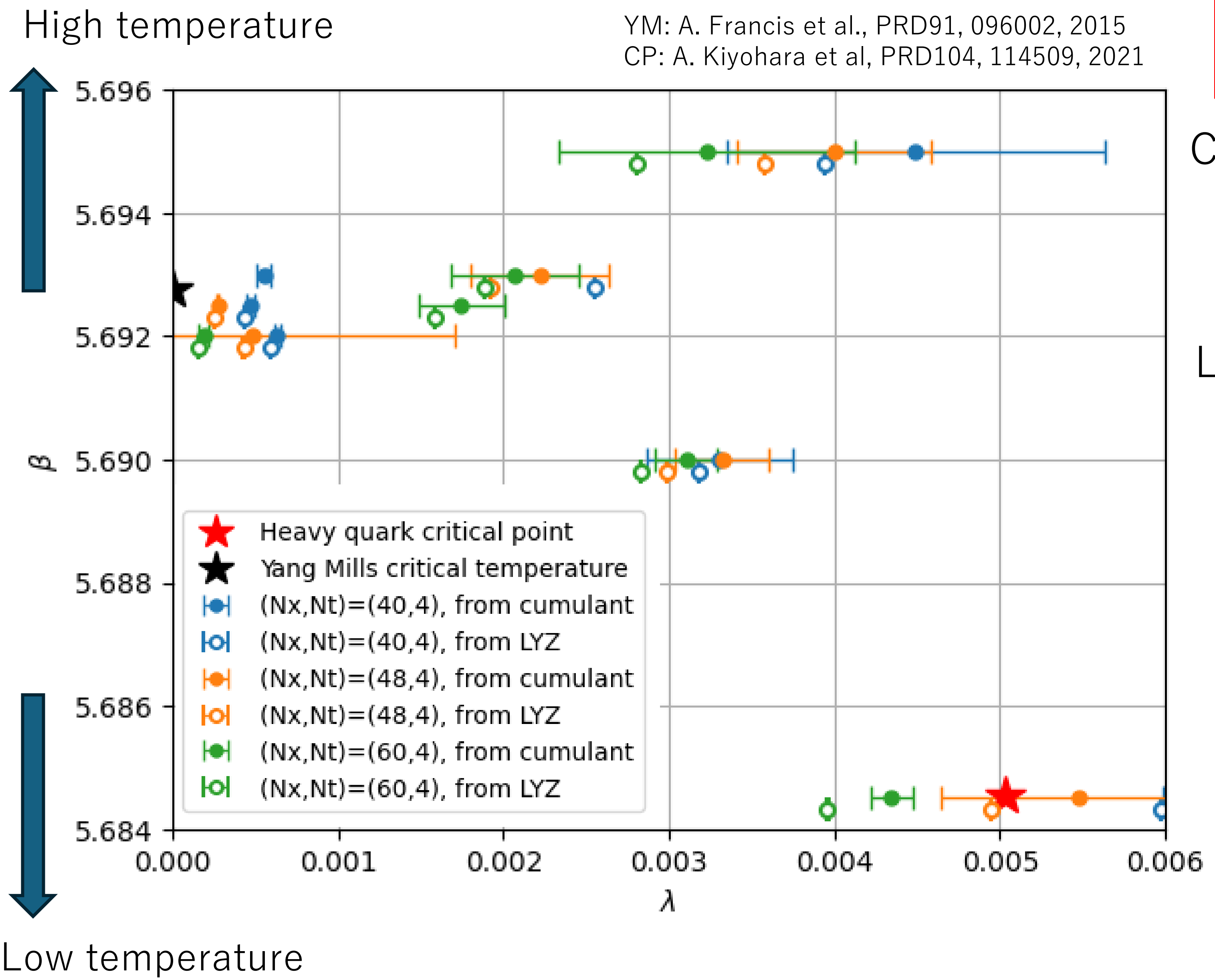


$$\lambda_c^{(\mathrm{LYZ})} = \min_n |\lambda_{\mathrm{LYZ}}^{(n)}|$$

Convergence radius of cumulant expansion



Data points indicate the convergence radii at β

Cauchy-Hadamard formula

$$\lambda_c^{(\mathrm{CH})} \simeq \lambda_{k=30}$$

Lee-Yang zeros

$$\lambda_c^{(\mathrm{LYZ})} = \min_n |\lambda_{\mathrm{LYZ}}^{(n)}|$$

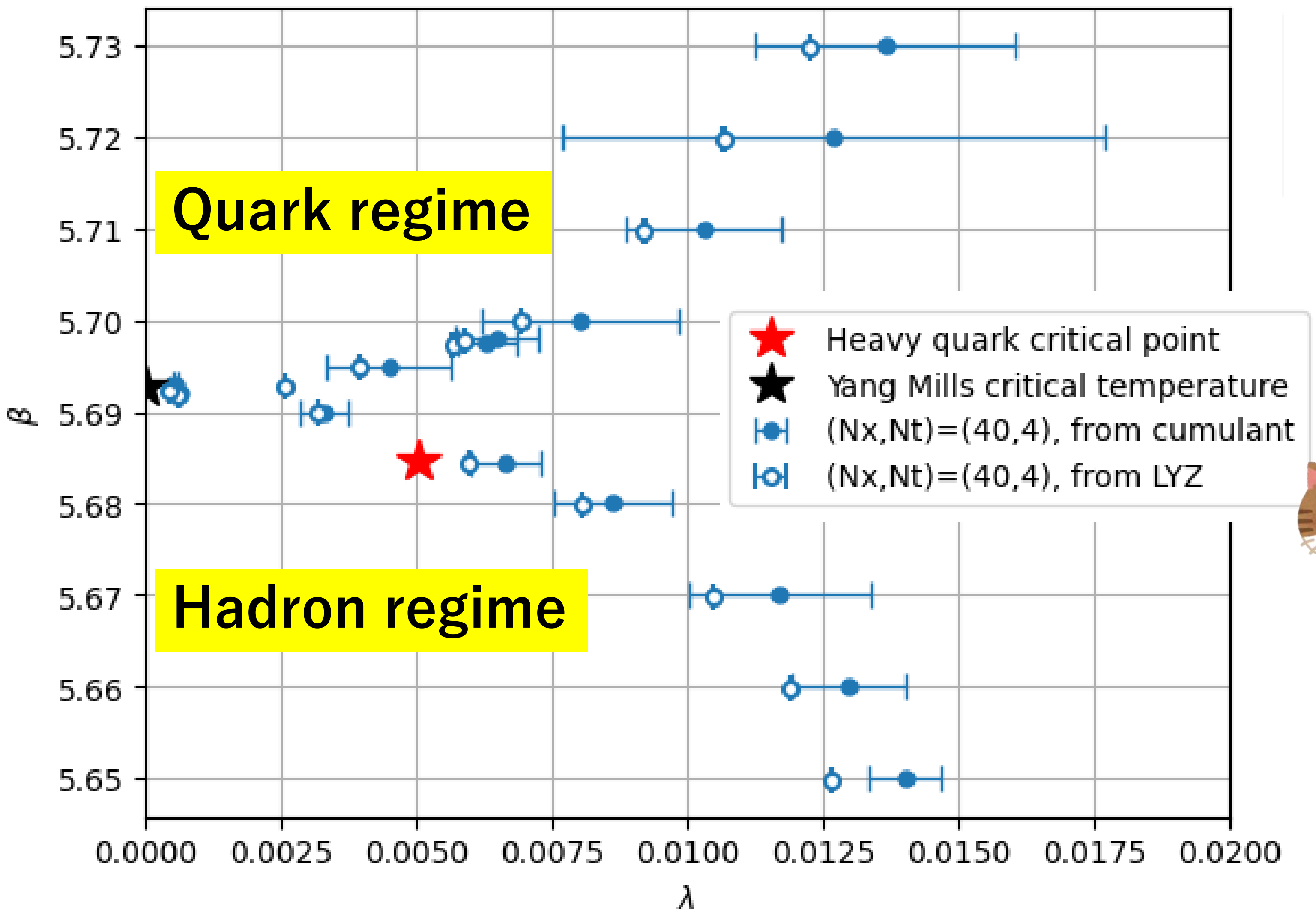
$V_3 = N_x^3$: 3-dim (lattice) volume

We found that

- two methods are consistent.
- The convergence radii are small near $\beta \sim \beta_c^{\mathrm{YM}}$
- The convergence radii are larger as larger $|\beta - \beta_c^{\mathrm{YM}}|$

Convergence radius of cumulant expansion

Nx=40 (not so large volume)but large range of beta.



It seems that
the convergence radius will be larger
as larger $|\beta - \beta_c^{\rm YM}|$.
The hadron/quark regimes
are disconnected
by the convergence radii.

How about the two limits:

- $\beta \to \infty$ (free gluons):
  convergence radius will be ∞.
- $\beta \to 0$ (strong coupling limit):
  convergence radius is ∞ ? finite?

We will answer the question
in the future.

Summary

Future works: Landau potential analysis, larger Nt, finite μ, higher order in hopping parameter expansion, observables, ⋯

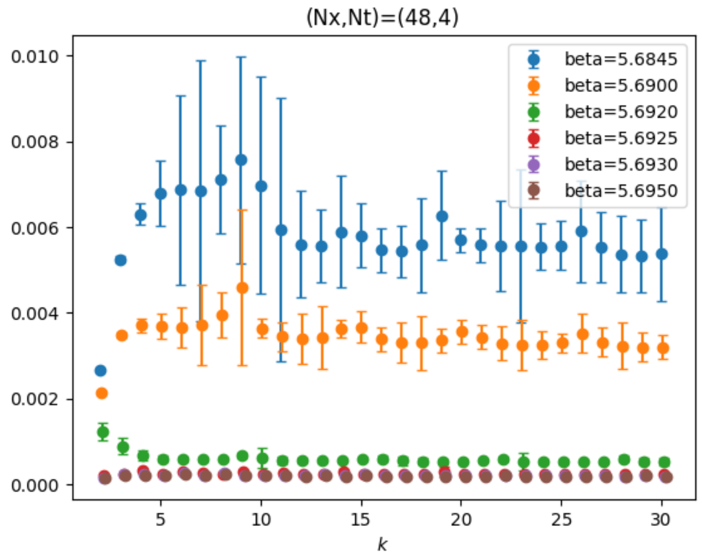
- We consider heavy quark QCD.
- We consider hopping parameter expansion of the thermodynamic potential in addition to the lattice action using the technique of the cumulant expansion.
- Heavy quark limit approximately reproduce the quark number fluctuation.

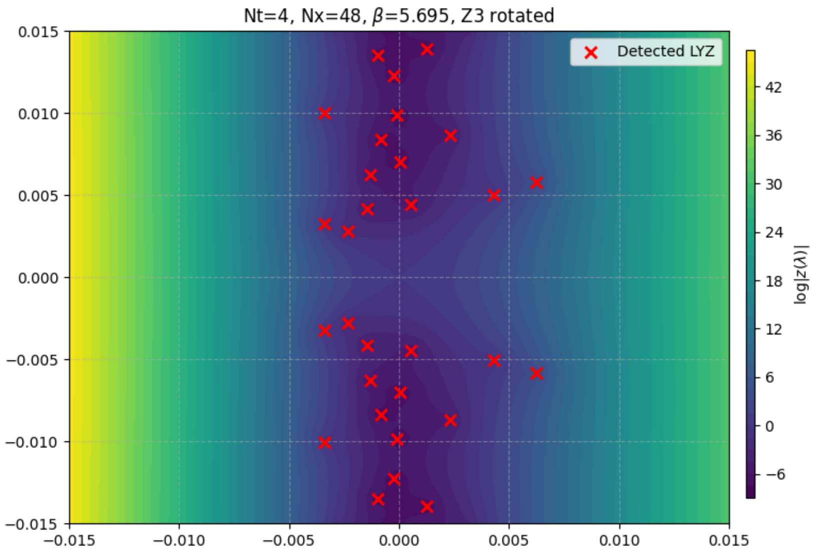
$$\frac{\chi_4^{(\mathrm{Q})}(T,\mu_{\mathrm{q}})}{\chi_2^{(\mathrm{Q})}(T,\mu_{\mathrm{q}})} = \begin{cases} 1 & (T > T_{\mathrm{c}}) \\ 9 & (T < T_{\mathrm{c}}). \end{cases}$$

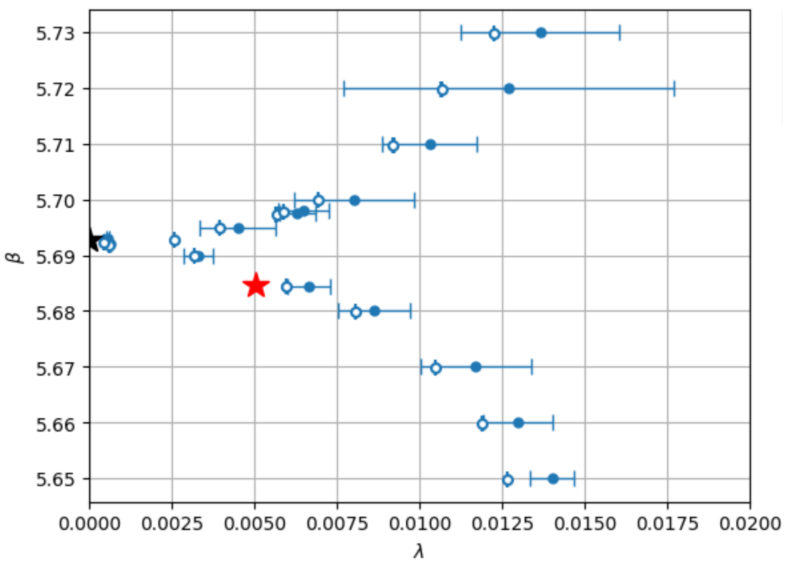
($\kappa \to 0$ limit)

- Using the Leading order of hopping parameter expansion for the effective action, we estimate the convergence radius of the cumulant expansion by two methods, Cauchy-Hadamard approach and Lee-Yang zero approach.
- The two methods are consistent.
- quark regime at high temperature and hadron regime at low temperature are disconnected by the finite convergence radii.

$$\lambda_k = \left(\frac{1}{|c_k|}\right)^{1/k}$$







Backup

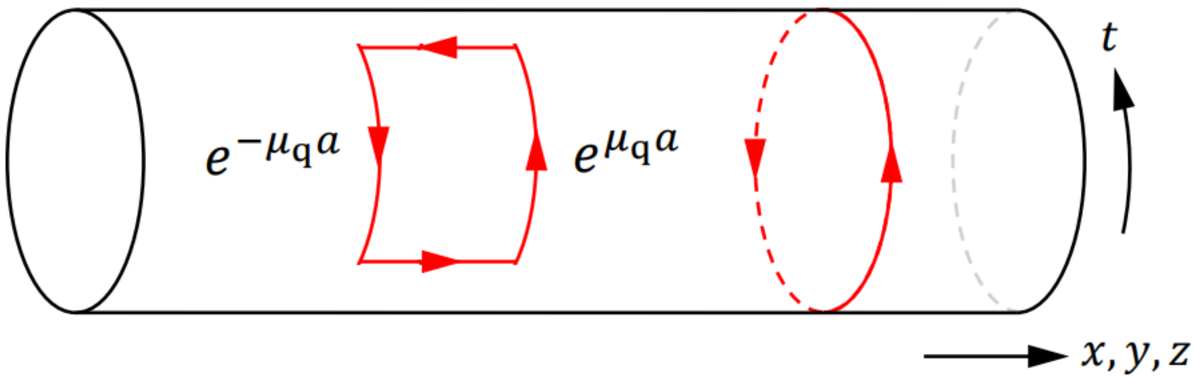
Hopping parameter expansion in lattice effective action

Partition function

$$Z = \int \mathcal{D}U\mathcal{D}\bar{q}\mathcal{D}q \exp[-S_g - S_Q] = \int \mathcal{D}U\, e^{-S_g} \exp\left[-N_f \sum_{l=1}^{\infty} \frac{\kappa^l}{l} \mathrm{Tr}\, H^l\right]$$

Closed trajectories of length l.

Spatially closed loops   Temporally closed loops



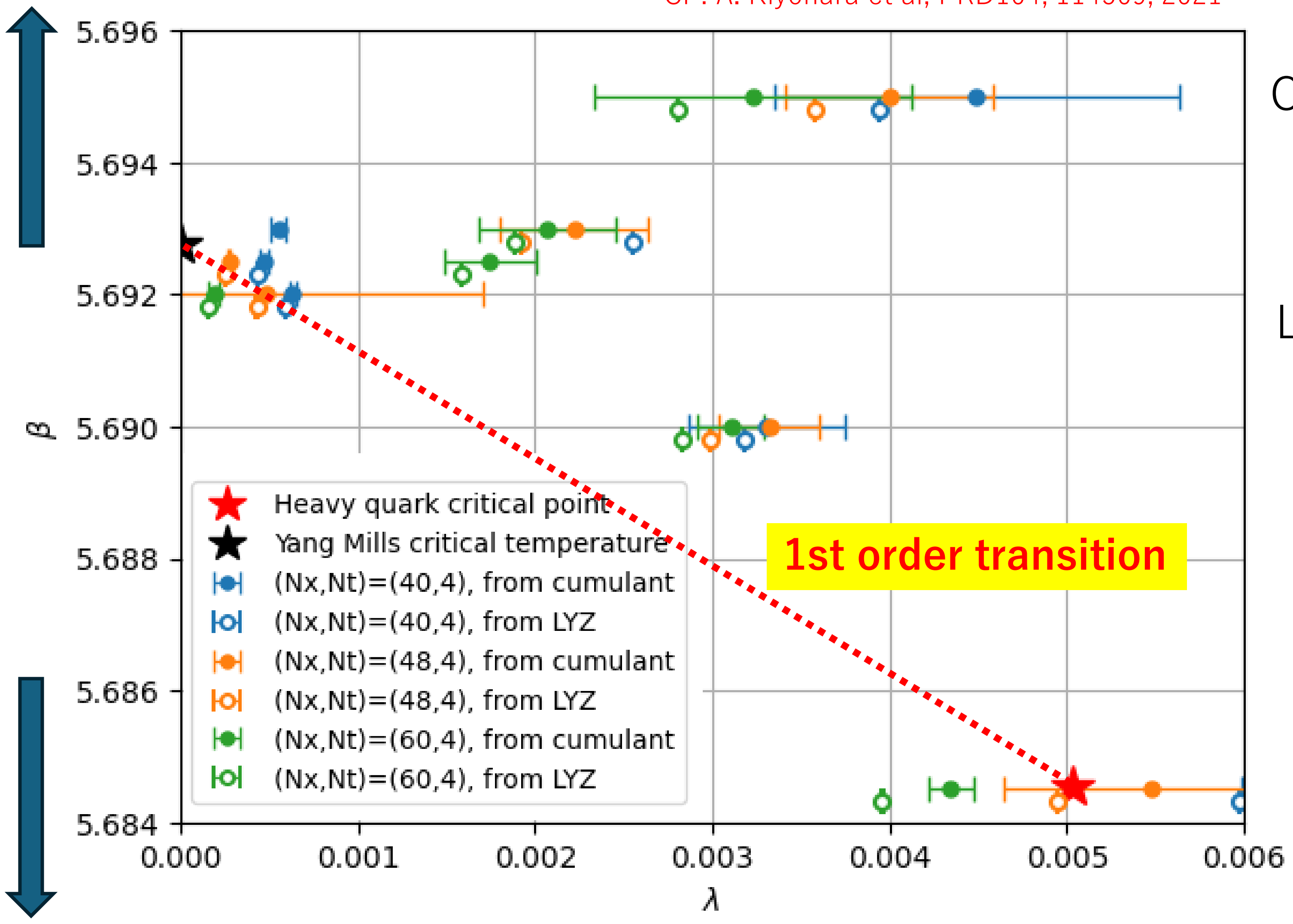
μ_q-dependences

- Spatially closed loops: μ_q-independent
- Temporally closed loops: μ_q-dependent

Convergence radius of cumulant expansion

High temperature

YM: A. Francis et al., PRD91, 096002, 2015
CP: A. Kiyohara et al, PRD104, 114509, 2021



Low temperature

Data points indicate the convergence radii at β

Cauchy-Hadamard formula

$$\lambda_c^{(\mathrm{CH})} \simeq \lambda_{k=30}$$

Lee-Yang zeros

$$\lambda_c^{(\mathrm{LYZ})} = \min_n |\lambda_{\mathrm{LYZ}}^{(n)}|$$

$V_3 = N_x^3$: 3-dim (lattice) volume

We found that

- two methods are consistent.
- The convergence radii are small near $\beta \sim \beta_c^{\mathrm{YM}}$
- The convergence radii are larger as larger $|\beta - \beta_c^{\mathrm{YM}}|$

Quark number susceptibilities

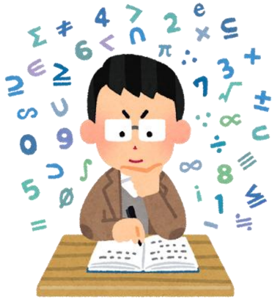
$$\chi_n^{(\mathrm{Q})}(T,\mu_\mathrm{q}) \equiv -\frac{\partial^n}{\partial \hat{\mu}_\mathrm{q}{}^n}\frac{\Omega(T,\mu_\mathrm{q})}{TV} = -\frac{\partial^n}{\partial \hat{\mu}_\mathrm{q}{}^n}\frac{\Omega_\mathrm{q}(T,\mu_\mathrm{q})}{TV}\ .$$

$$\Omega_\mathrm{q}(T,\mu_\mathrm{q}) = X_0 + \sum_{w=1}^{\infty}\left(e^{w\hat{\mu}_\mathrm{q}} + e^{-w\hat{\mu}_\mathrm{q}}\right)X_w$$

$$= \frac{1}{V}\sum_{w=1}^{\infty} w^n\left[e^{w\hat{\mu}_\mathrm{q}} + (-1)^n e^{-w\hat{\mu}_\mathrm{q}}\right]X_w\ .$$

The ratio of 4th to 2nd :

$$\frac{\chi_4^{(\mathrm{Q})}(T,\mu_\mathrm{q})}{\chi_2^{(\mathrm{Q})}(T,\mu_\mathrm{q})} = \frac{\sum_{w=1}^{\infty} w^4 C_w X_w}{\sum_{w'=1}^{\infty} w'^2 C_{w'} X_{w'}}.\quad \left(C_w = e^{w\hat{\mu}_\mathrm{q}} + e^{-w\hat{\mu}_\mathrm{q}}\right)$$



We can analytically calculate the ratio!

Quark number susceptibilities

Tohme, et al., PRD112, 094515 (2025)

@ deconfined phase ($T > T_c$)

$\mathbb{Z}_3$-symmetry is broken and all X_w can have nonzero value.

→ The leading contribution: X_1.

$$\frac{\chi_4^{(\mathrm{Q})}(T,\mu_\mathrm{q})}{\chi_2^{(\mathrm{Q})}(T,\mu_\mathrm{q})} = \frac{C_1X_1 + 2^4 C_2X_2 + \cdots}{C_1X_1 + 2^2 C_2X_2 + \cdots}$$

$$= 1 + 12\frac{C_2}{C_1}\underbrace{\frac{X_2}{X_1}}_{\mathcal{O}(\kappa^{N_t})}.$$

@ confined phase ($T < T_c$)

$\mathbb{Z}_3$-symmetry is preserved and X_{3n} ($n \in \mathbb{Z}$) can have nonzero value.

→ The leading contribution: X_3.

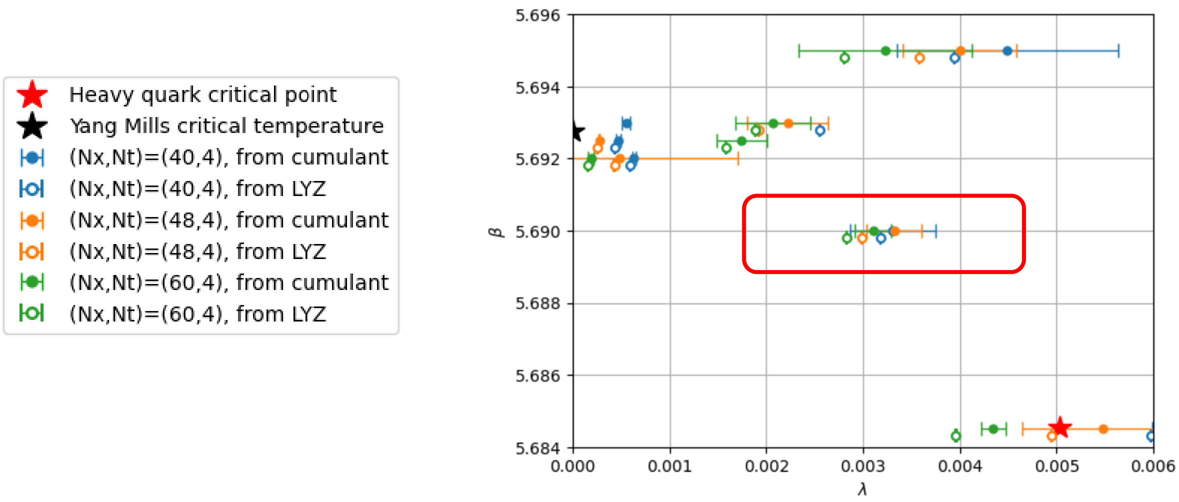
$$\frac{\chi_4^{(\mathrm{Q})}(T,\mu_\mathrm{q})}{\chi_2^{(\mathrm{Q})}(T,\mu_\mathrm{q})} = \frac{3^4 C_3X_3 + 6^4 C_6X_6 + \cdots}{3^2 C_3X_3 + 6^2 C_6X_6 + \cdots}$$

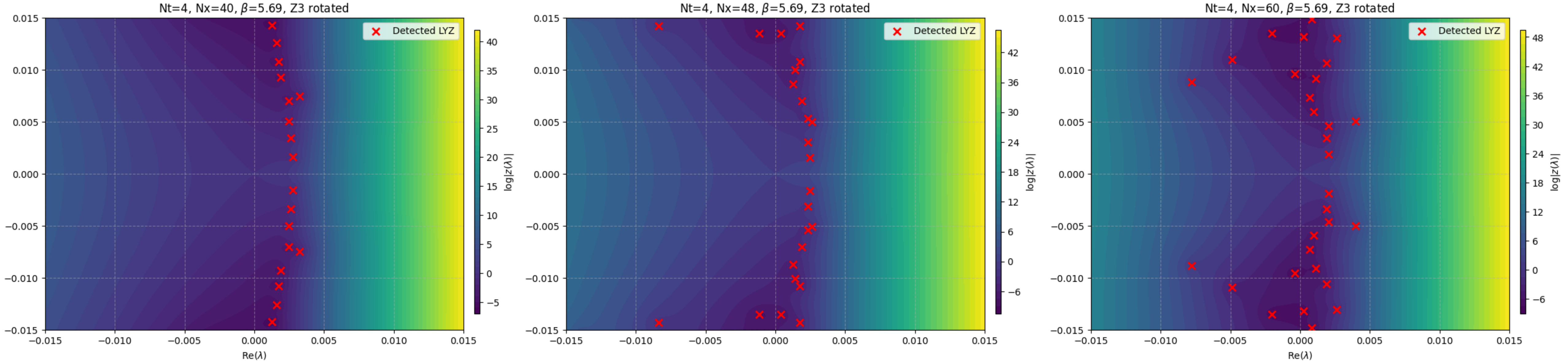
$$= 9 + 140\frac{C_6}{C_3}\underbrace{\frac{X_6}{X_3}}_{\mathcal{O}(\kappa^{3N_t})}.$$

We get the ratios in low/high temperature at heavy quark limit:

$$\frac{\chi_4^{(\mathrm{Q})}(T,\mu_\mathrm{q})}{\chi_2^{(\mathrm{Q})}(T,\mu_\mathrm{q})} = \begin{cases} 1 & (T > T_\mathrm{c}) \\ 9 & (T < T_\mathrm{c}) \end{cases}$$

Volume dependence of LYZs



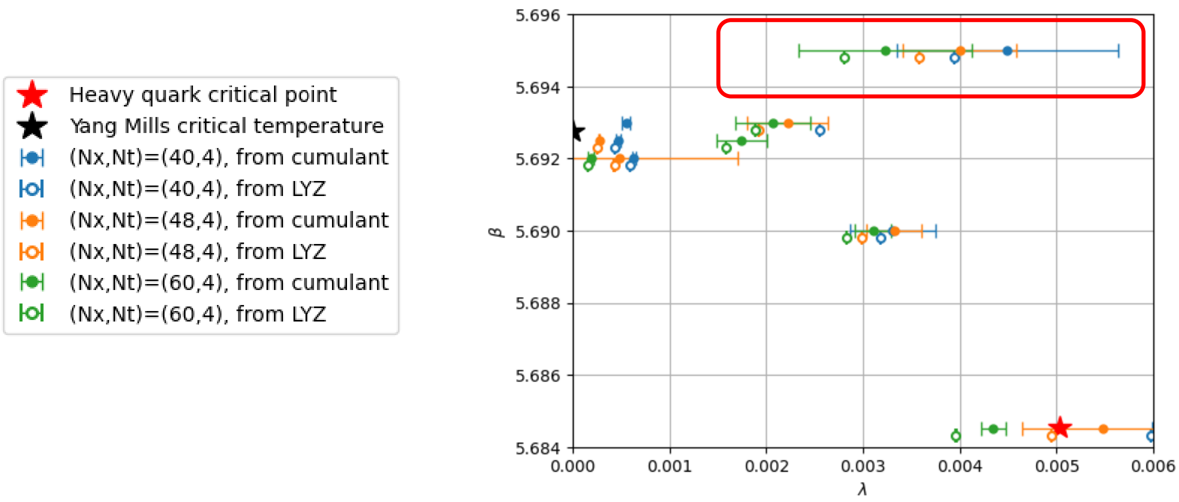


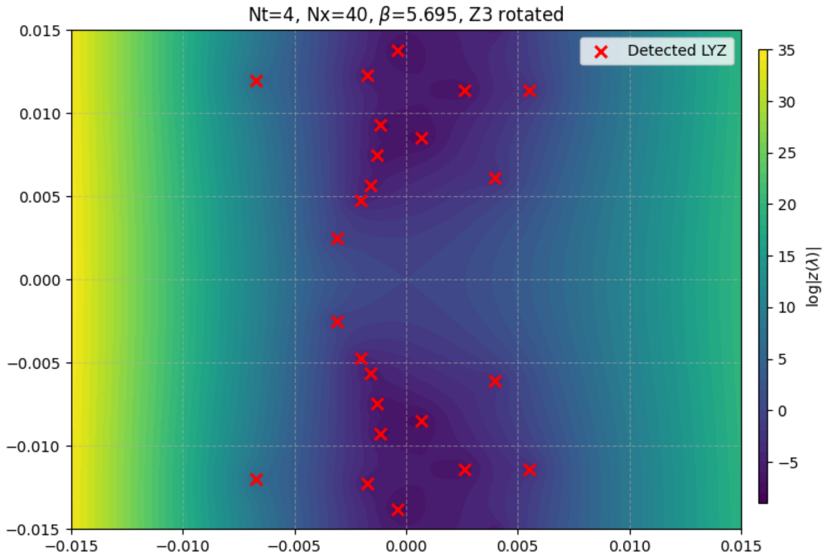
Nx=40 Nx=48 Nx=60

We know that the LYZs will touch the real axis in V→∞ limit in the case of beta=5.6900 corresponding to the 1st order transition.

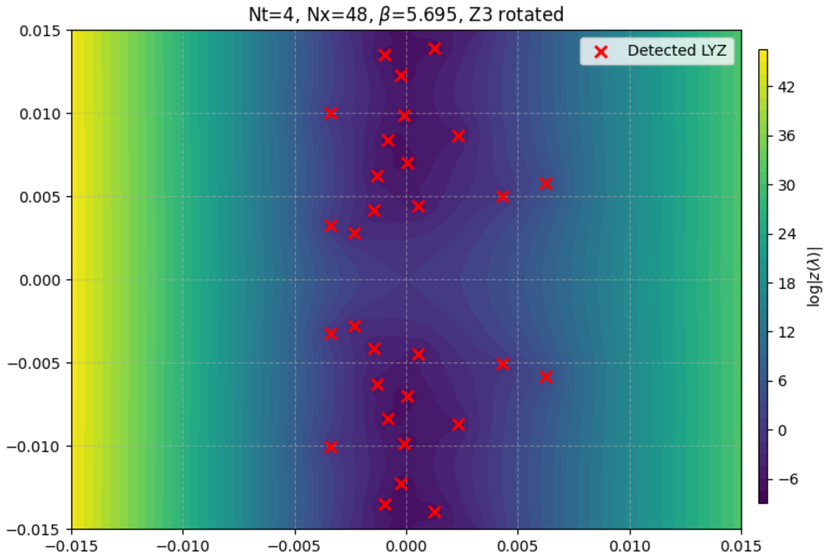
A. Kiyohara, M. Kitazawa, S. Ejiri, and K. Kanaya, Phys. Rev. D 104, 114509, 2021

Volume dependence of LYZs

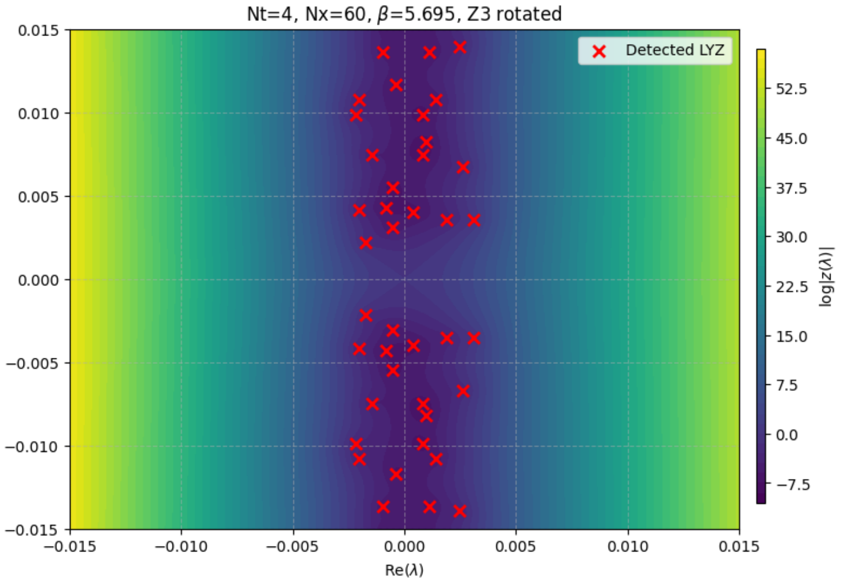




Nx=40



Nx=48



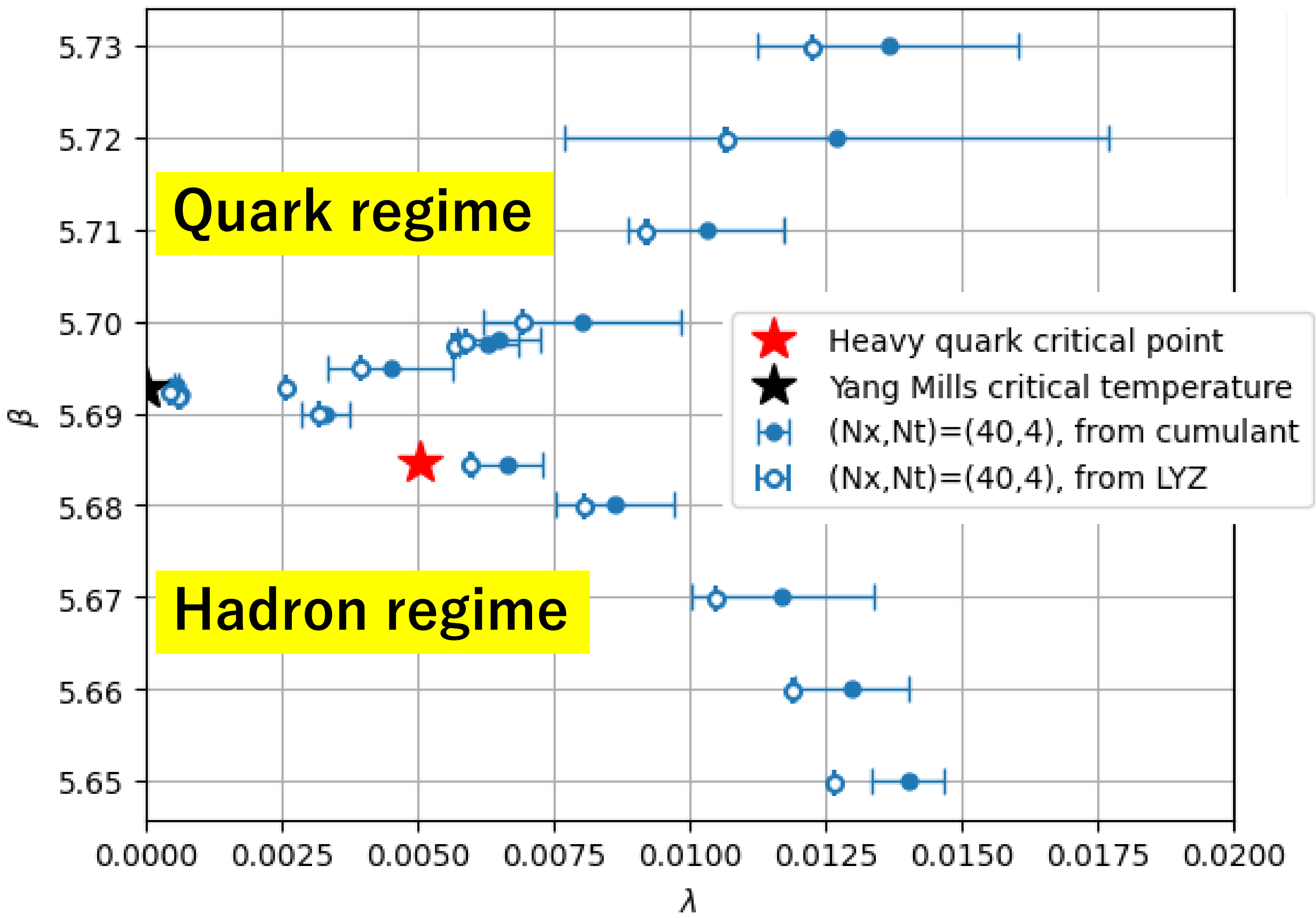
Nx=60

We know that the LYZs in $\lambda>0$ region does not touch the real axis in $V\to\infty$ limit in high T>Tc.
But LYZs are located near real axis $\lambda<0$ region. Then, the convergence radius is finite.

A. Kiyohara, M. Kitazawa, S. Ejiri, and K. Kanaya, Phys. Rev. D 104, 114509, 2021

Convergence radius of cumulant expansion

Nx=40 (not so large volume)but large range of beta.



It seems that
the convergence radius will be larger
as larger $|\beta - \beta_c^{\rm YM}|$.
The hadron/quark regimes
are disconnected
by the convergence radii.
This is good news for me because
my concern below is resolved.

The two phases should be
smoothly connected.

⇕ **? how consistent?**

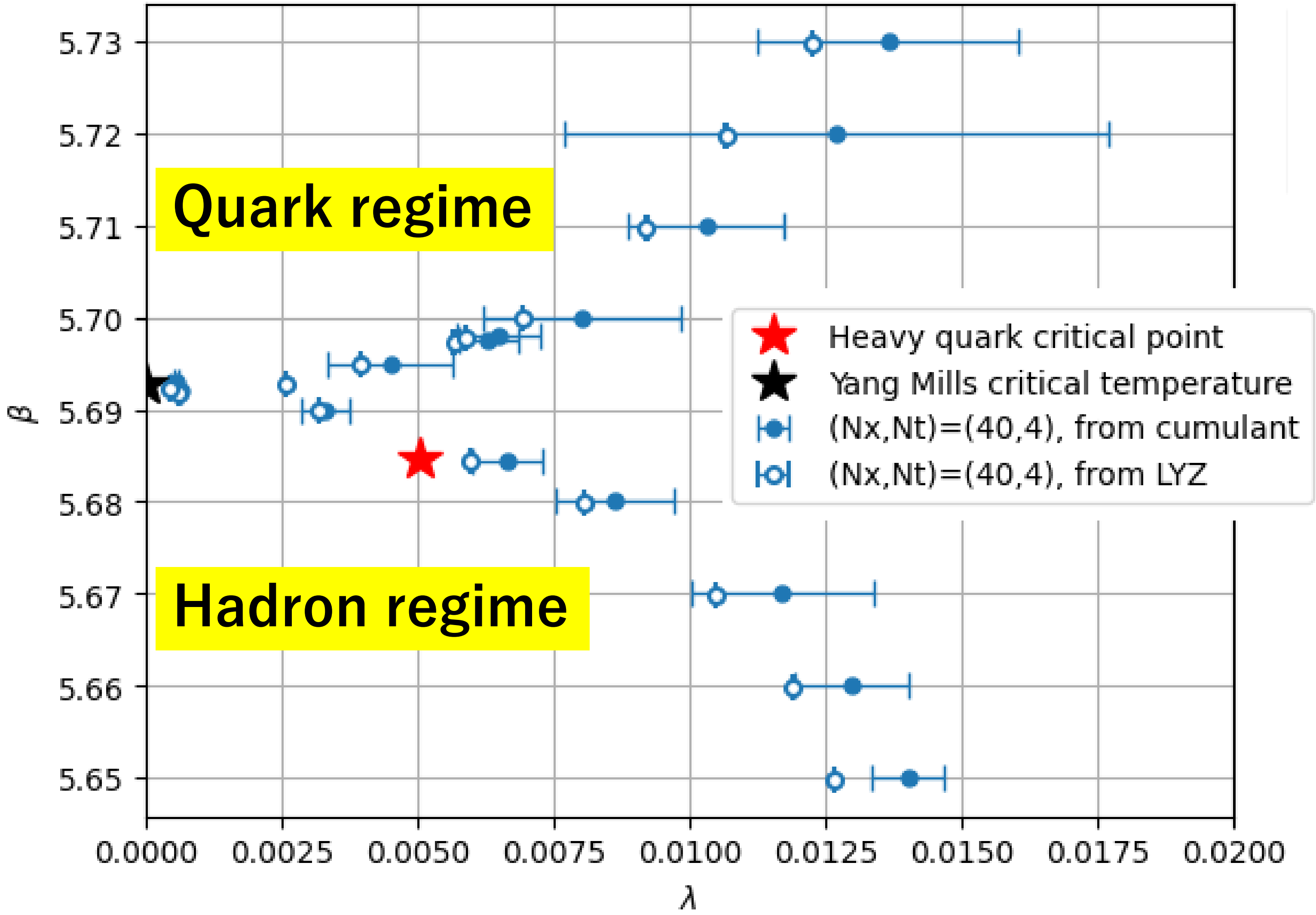
it is difficult to smoothly connect
the two regimes
in the viewpoint of our result

$$\frac{\chi_4^{(\rm Q)}(T,\mu_{\rm q})}{\chi_2^{(\rm Q)}(T,\mu_{\rm q})} = \begin{cases} 1 & (T > T_{\rm c}) \\ 9 & (T < T_{\rm c}). \end{cases}$$

($\lambda \to 0$ limit)

Convergence radius of cumulant expansion

Nx=40 (not so large volume)but large range of beta.



How about the two limits:

- $\beta \to \infty$ (free gluons): convergence radius will be ∞.
- $\beta \to 0$ (strong coupling limit): convergence radius is ∞ ? finite?

We will answer the question in the future.