Dense $\mathrm{QC_2D_2}$ **with uniform matrix product states**

QCD Critical Point and Hydrodynamic Evolution
(2026/06/02)

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Critical point and Hydrodynamics

My question is how soft modes (near a QCD critical point) can be characterized from microscopic dense QCD.

This question is difficult to address directly in real-world QCD, but QCD in (1+1)-dimension may be solvable in the Hamiltonian formulation.

In this talk, I will report our recent results from this perspective.

$\mathrm{QC_2D_2}$ at finite density

We numerically construct the translational and gauge invariant ground states of (1+1)-dimensional $SU(2)$ gauge theory with a single flavor at finite baryon density using tensor network method.

$$S=\int \mathrm{d}^2x\left[-\frac{1}{2}\mathrm{Tr}\left(F^{\mu\nu}F_{\mu\nu}\right)+i\bar{q}\gamma^{\mu}D_{\mu}q+\mu_q q^{\dagger}q-m_q\bar{q}q\right]$$

μ_q : quark chemical potential m_q : quark mass

*The choice $N_c=2$ and $N_f=1$ is made for purely technical reasons.

Previous studies on $\mathrm{QC_2D_2}$

- Two-color QCD at finite baryon density can be simulated by lattice Monte Carlo for even N_f in (1+3)-dimensions.

  [Review by V. V. Braguta (2023), E. Itou (2025).]

- The Hamiltonian formulation with a strong coupling expansion on the lattice

  [H.J. Hamer (1982)]

- Matrix product state with an open boundary condition on the lattice

  [M. C. Banuls, K. Cichy, J. I. Cirac, K. Jansen, S. Khun (2017), T. Hayata, Y. Hidaka, K. Nishimura (2023)]

- Grassmann tensor renormalization group approach

  [K. H. Pai, S. Akiyama, S. Todo (2025)]

- Bosonization with a certain regularization at strong coupling limit

  [V. Baluni (1980), P. J. Steinhardt (1980), T. Kojo (2011)]

Cold and dense $\mathrm{QCD_2}$ is *Tomonaga-Luttinger liquid!(Bosonization)*

[M. Lajer, R. M. Konik, R. D. Pisarski, A. M. Tsvelik (2021)]

Tomonaga—Luttinger Liquid (1)

[Tomonaga (1950), Luttinger (1960)]

Consider a (1+1)-dimensional many-particle system:

$\phi_B(t, x)$: Labeling field (particle number) $\langle \phi_B \rangle = N_B(x)$

When the density-density interaction is local, the Hamiltonian in the long-distance limit is described by *[Haldane (1981)]*

$$H_{\mathrm{TLL}} = \int \mathrm{d}x \left[2\pi c_s K \Pi_B^2 + \frac{c_s}{8\pi K} (\partial_1 \phi_B)^2 \right]$$

$[\Pi_B(t, x_1), \phi_B(t, x_2)] = -i\delta(x_1 - x_2)$ c_s : Sound speed, K : Luttinger parameter

Two parameters capture physics of a cold and dense quantum fluid.

Tomonaga—Luttinger Liquid (2)

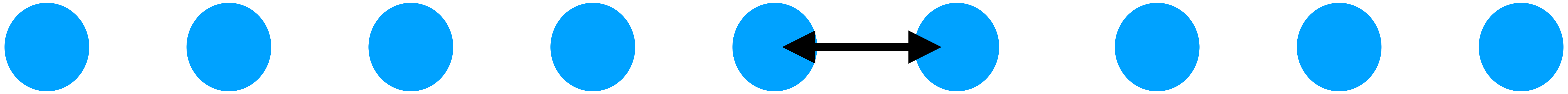
The density-density correlation function:

[Haldane (1981)]

$$\langle \delta n_B(t,x_1)\delta n_B(t,x_2)\rangle \sim \sum_{\ell=1}^{\infty} B_\ell \frac{\cos(2\pi\ell n_B|x_1-x_2|)}{|x_1-x_2|^{2\ell^2 K}}$$

A spatial modulation appears in two-point function:

$$d_B \sim 1/n_B$$



$$p_f = \pi\langle n_B\rangle$$

Our Results

We confirm the following using Hamiltonian lattice calculations.

- A gapless behavior is observed at any finite baryon number density, $\langle n_B \rangle \neq 0$.
- A central charge is $c = 1$ (A single baryon number density mode).
- There are spatial modulations in the baryon number density-density correlation with wavenumber, $p_f = 2\pi\ell\langle n_B \rangle$, $\ell = 1,2,\cdots$.
- We compute the EFT parameters using first-principles methods.

Cold and dense QC_2D_2 is indeed Tomonaga—Luttinger liquid!

Hamiltonian formulation

Hamiltonian formalism

In the Hamiltonian formalism, physical states can be constructed directly.

$$\langle \Psi | \hat{O} | \Psi \rangle$$

$|\Psi\rangle$: Physical state

Huge vector
(Hilbert space)

Huge and usually sparse matrix
(Operator)

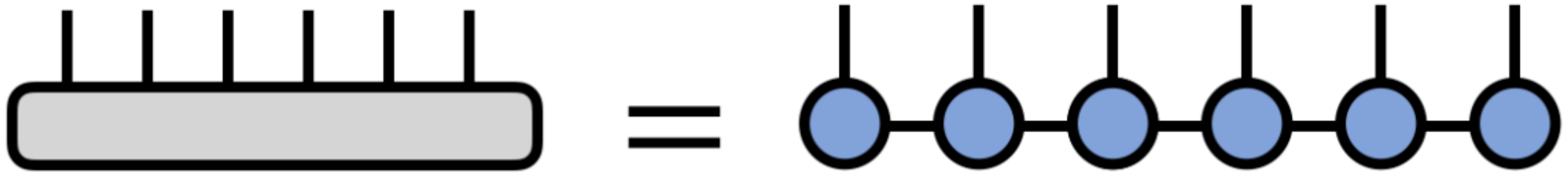
No sign problem, but dimension of $|\Psi\rangle$ is quite large…

Ex) $\dim |\Psi\rangle = 2^N$ even for N-site Ising model.

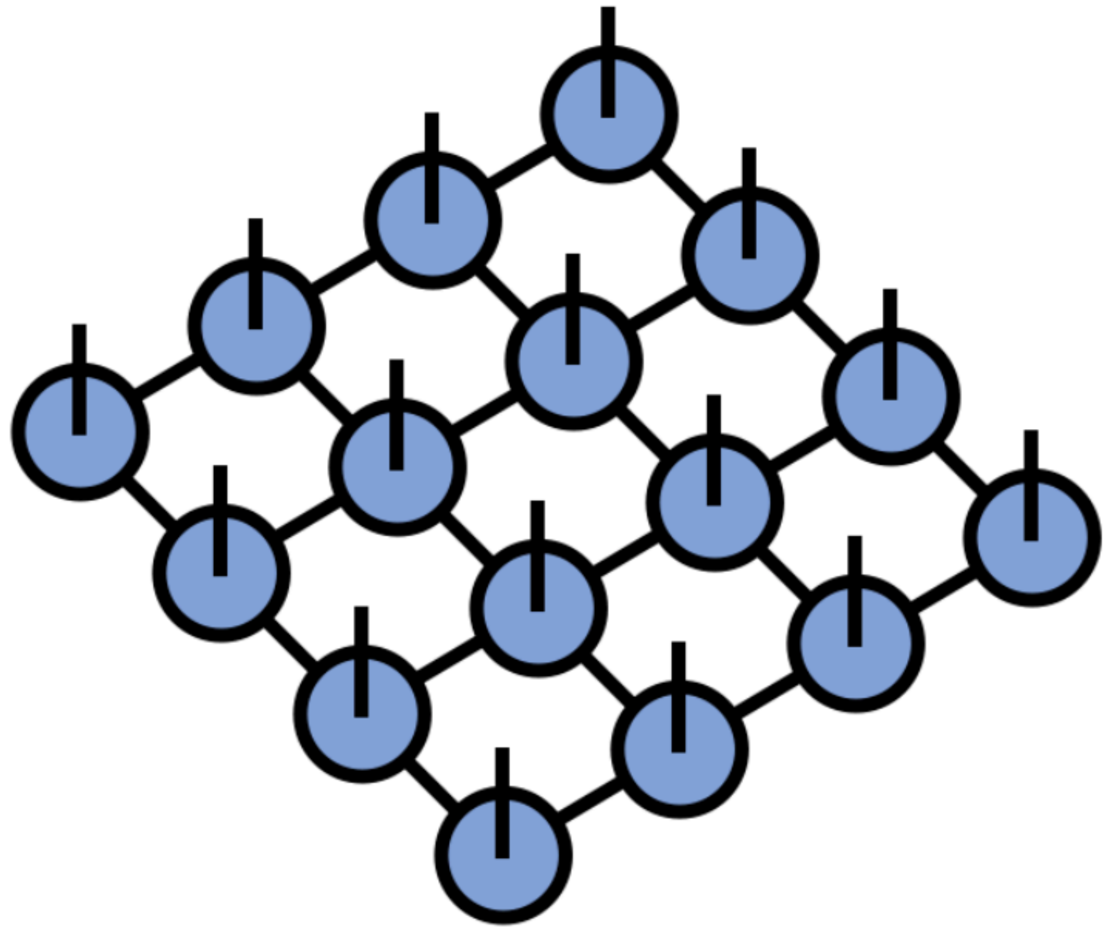
Tensor Network

Tensor network provides a good basis to construct the physical state by classical computing.

MPS



PEPS



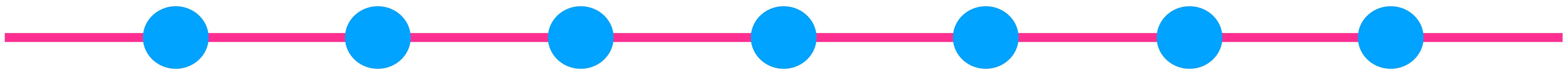
[TensorNetwork.org]

Kogut-Susskind Hamiltonian formulation

A spatial direction is discretized, while the temporal direction is continuous.

[Kogut, Susskind (1974)]

$$|\psi\rangle =$$



A gauge field lives on the link.

$$[R^a, U] = UT^a,\ \ [L^a, U] = T^a U$$

$$[R^a, R^b] = i f_c^{ab} R^c,\ \ [L^a, L^b] = -i f_c^{ab} L^c$$

L^a, R^a: Left and right electric fields

U: Wilson line in fund. rep. T^a: Generator of $SU(2)$

A matter field lives on the site.

$$\{\phi^\dagger_{c_1}(n), \phi^{c_2}(m)\} = \delta_{n,m}\delta^{c_2}_{c_1}$$

ϕ_c : (Staggered) fermion in the fundamental rep.

The calculation is mostly performed using group theory.

Hamiltonian in 1+1 dimension

The total Hamiltonian with a single flavor

$$H_{\mathrm{tot}} = H_E + H_{\mathrm{hop}} + H_{\mathrm{mass}}$$

$$H_E = \sum_n \frac{ag^2}{2} E^2(n),$$ $E^2 = (R^a)^2 = (L^a)^2$: square of electric field

$$H_{\mathrm{hop}} = \sum_n \frac{1}{2a} \left(\phi^\dagger_{c_1}(n+1) U^{c_1}_{c_2}(n) \phi^{c_2}(n) + \mathrm{h.c.} \right),$$

Hopping term

$$H_{\mathrm{mass}} = m \sum_n (-1)^n \phi^\dagger_c(n) \phi^c(n).$$

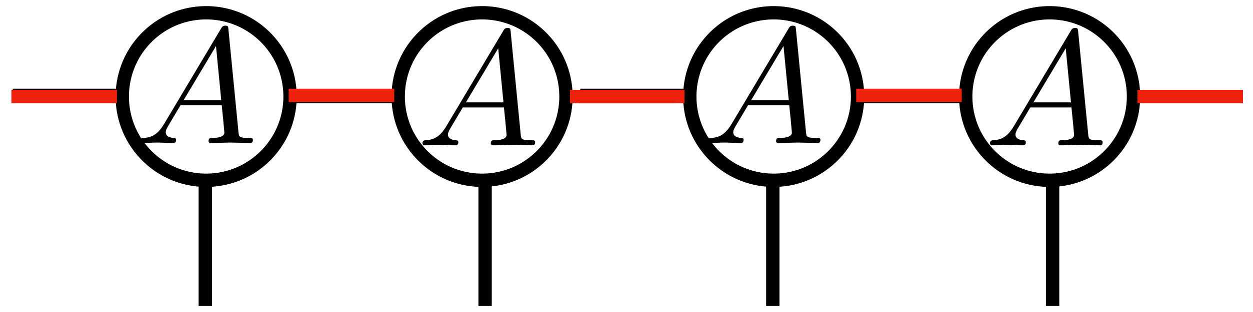
Quark mass term

a: lattice spacing, *g*: gauge coupling, *m*: quark mass

uniform Matrix Product States (uMPS)

- Assuming the translational invariance and using the truncation, uniform matrix product states reduce the number of variational parameters!

$$|\Psi\rangle = v_L^\dagger \cdots A^{s_n} A^{s_{n+1}} \cdots v_R | \cdots, s_n, s_{n+1}, \cdots \rangle = \cdots$$



$\cdots$

Number of parameters ~ D^2×(dim. of local Hilbert space)

D: bond dimension

[M. Fannes, B. Nachtergaele, R. F. Werner (1992)]

- An exact physical state is realized for $D \to \infty$.

[Verstraete, Cirac (2006)]

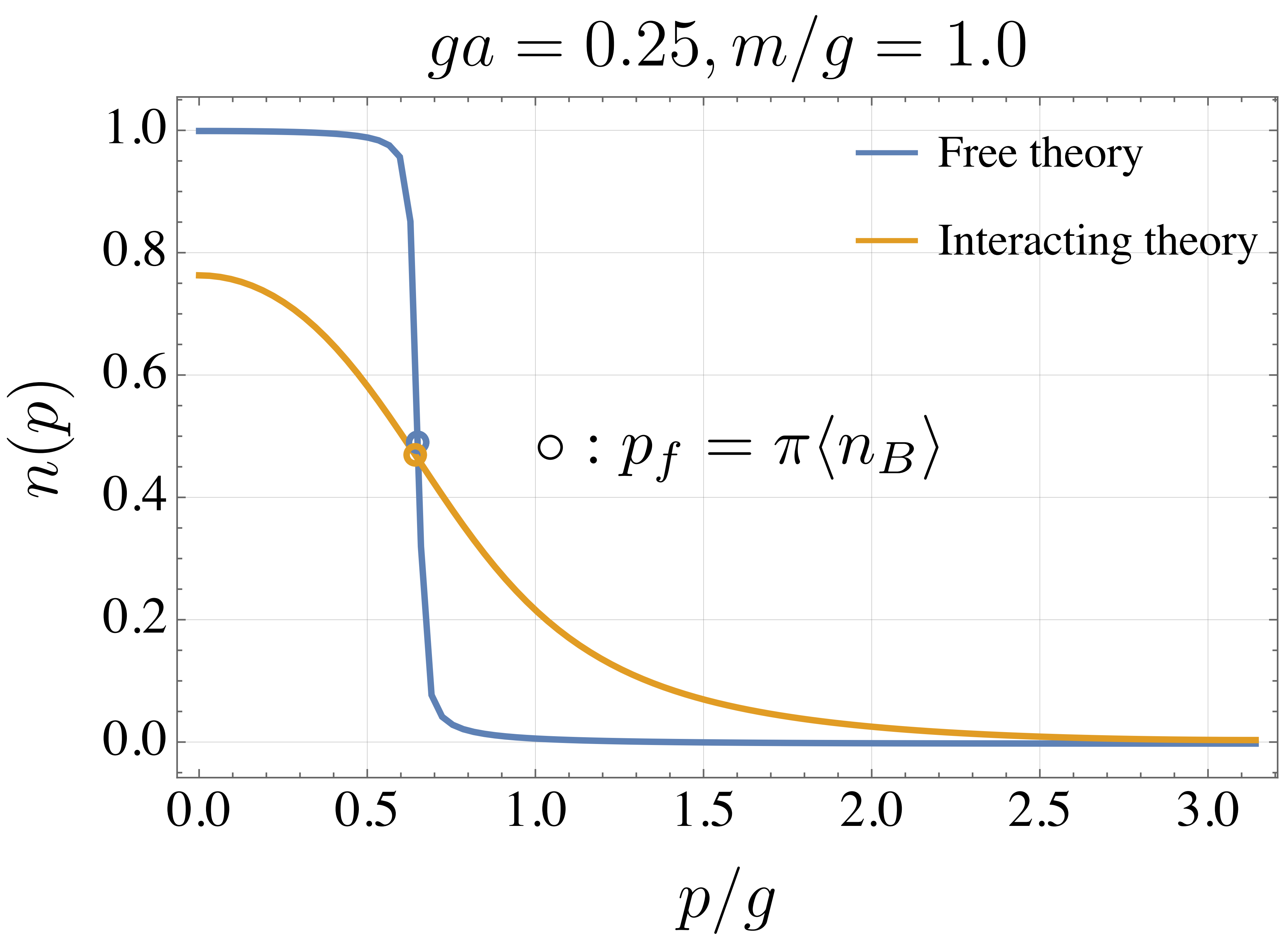
- We use gauge and translational invariant uMPS ansatz and construct the ground state by minimizing the grand potential by variational method.

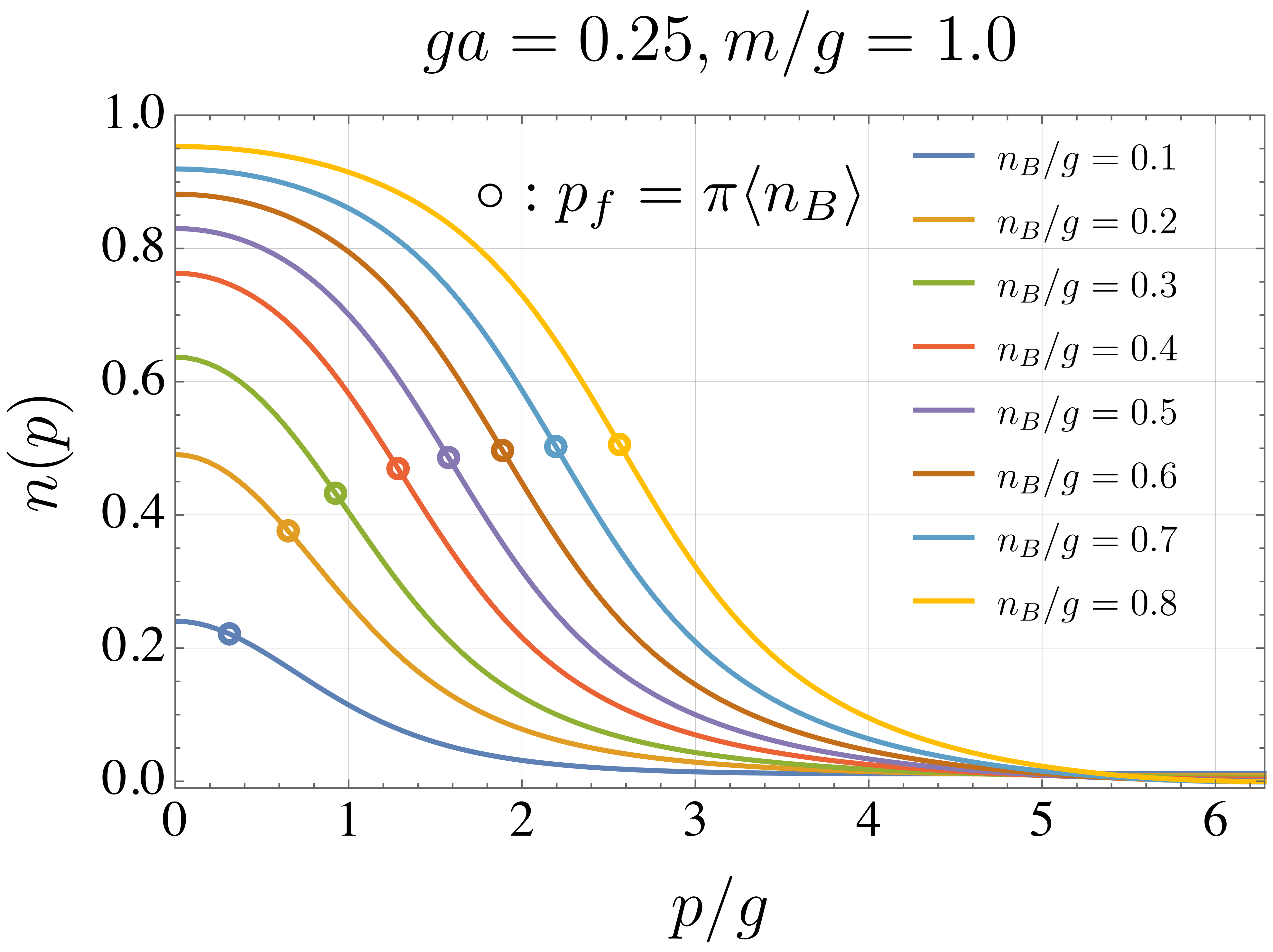
[Zauner-Stauber, Vanderstraeten, Fishman, Verstraete, Haegeman (2017)]

Result

Quark distribution function

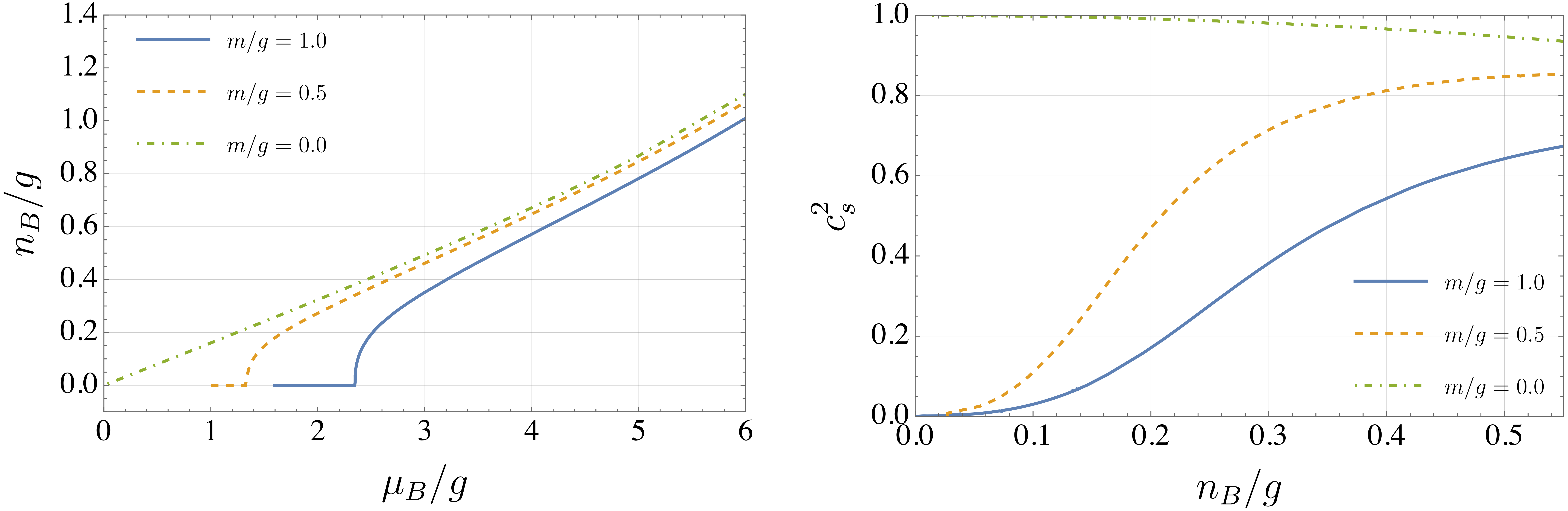
Remark: Fermi surface is destabilized by the gauge interaction.





Cold, Dense and Uniform

$\langle n_B \rangle \neq 0$ for $\mu_B \geq M_B$. (M_B: the lightest baryon mass)



Caution: No continuous global symmetry is spontaneously broken $\langle B \rangle \simeq 0$.

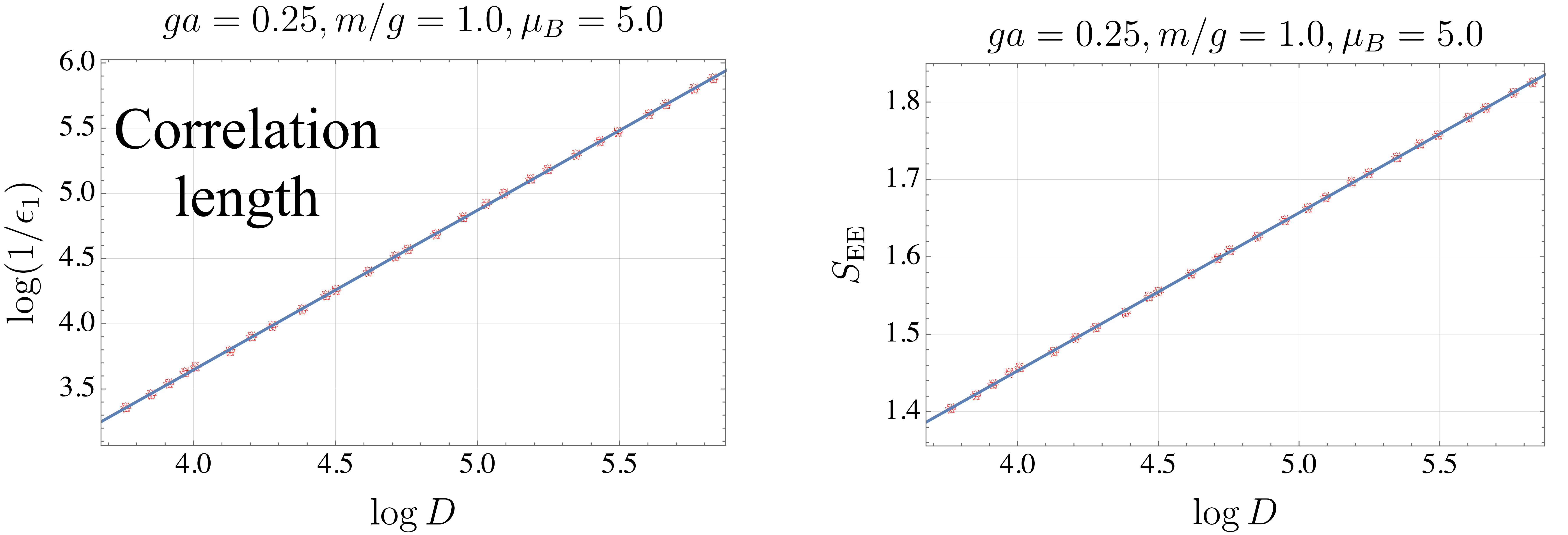
$B \sim \epsilon_{c_1 c_2} q^{c_1} q^{c_2}$ carries $U(1)_B$ quantum number.

Cold, Dense, Uniform and Gapless

The system behaves as the gapless phase for $\langle n_B \rangle \neq 0$!

$$S_{\mathrm{EE}} \sim \frac{c}{6} \log(\xi / a)$$

[P. Calabrese, J. Cardy (2009)]



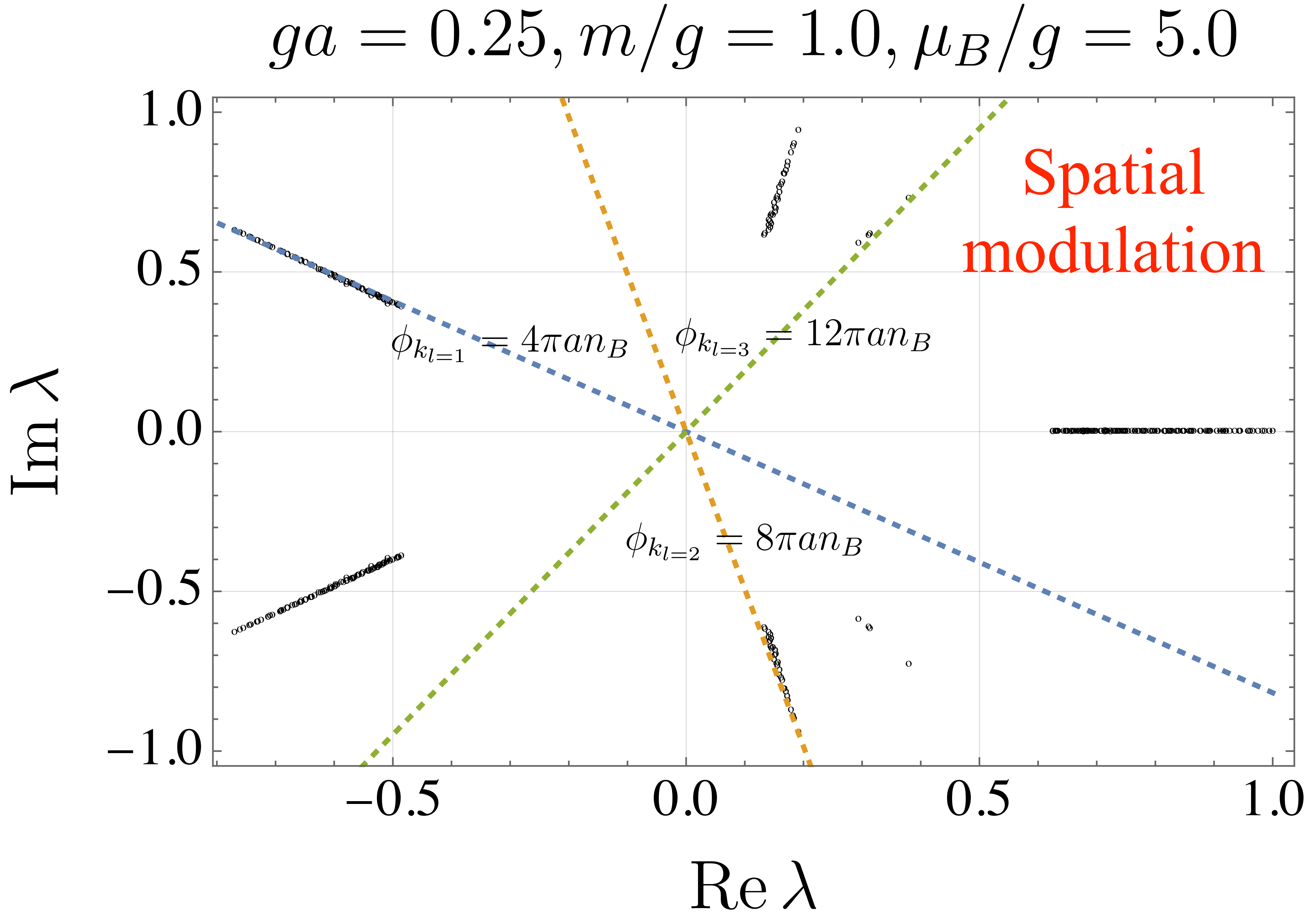
The best fitted-value of the central charge is $c = 1.00083 \pm 0.005$

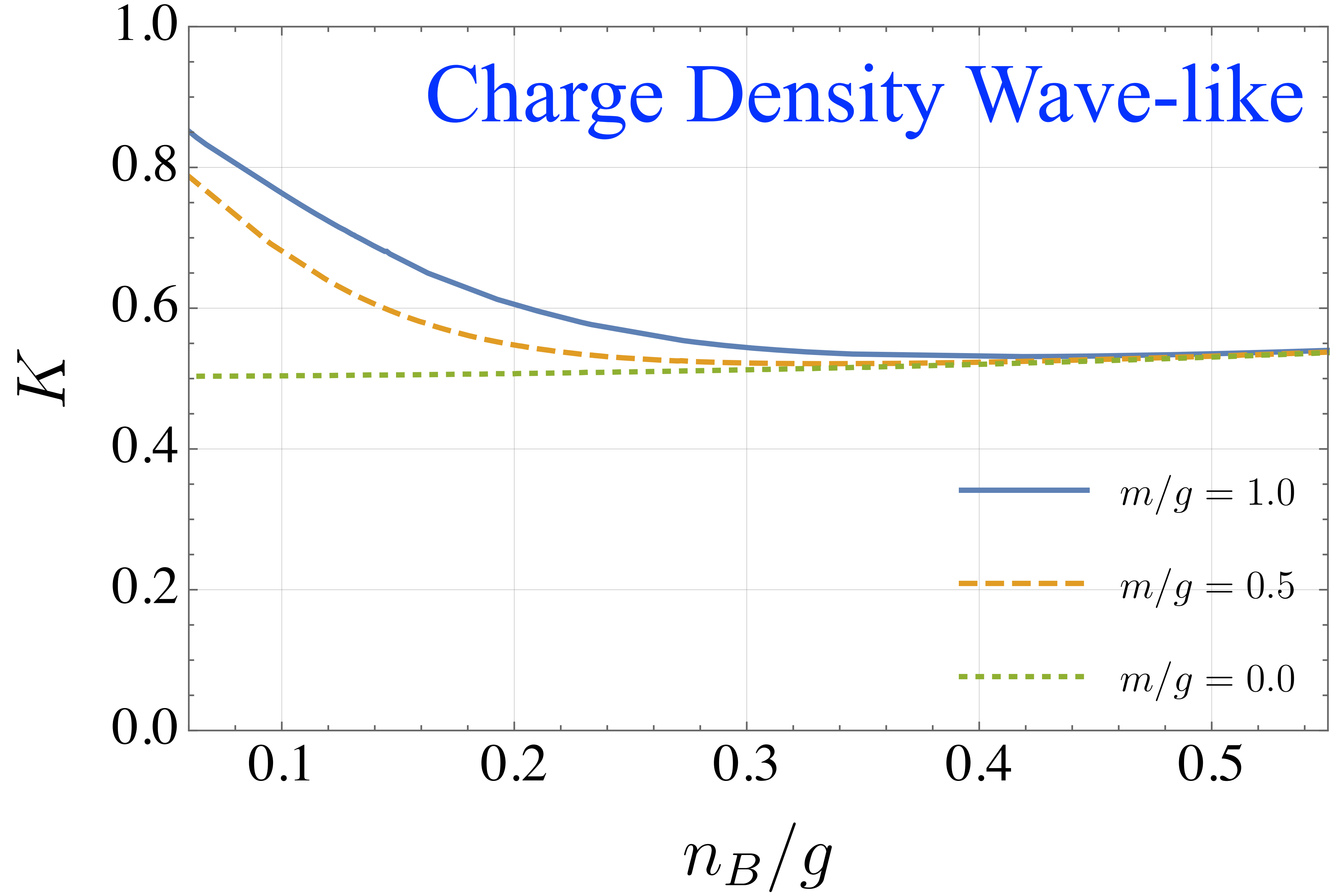
Tomonaga—Luttinger liquid theory

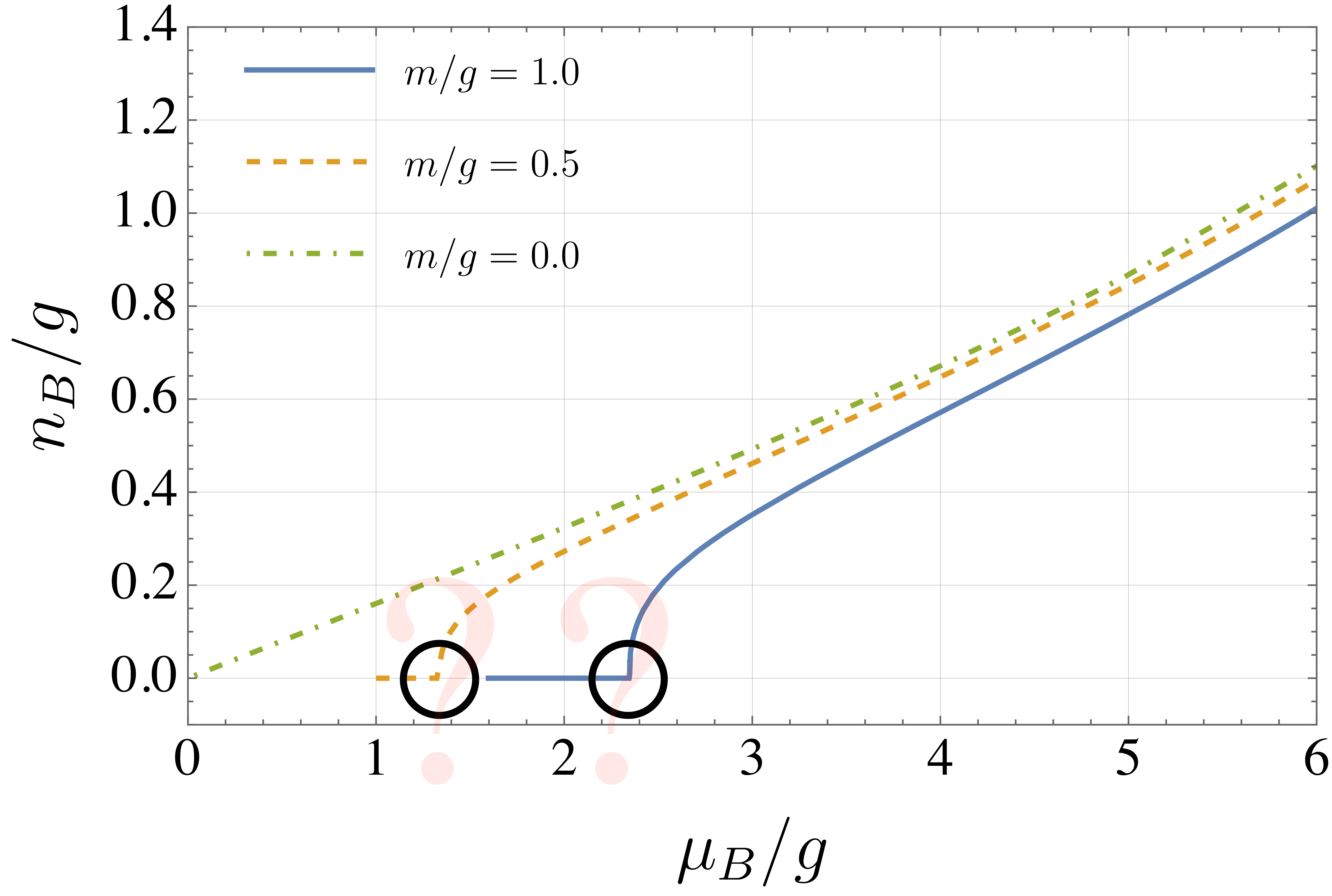
A low-energy behavior is described by

$$H_{\mathrm{TLL}}(\mu_B > M_B) = \int \mathrm{d}x \left[2\pi c_s K \Pi_B^2 + \frac{c_s}{8\pi K} (\partial_1 \phi_B)^2 \right]$$

Density-density correlation: $\langle \delta n_B(t, x_1) \delta n_B(t, x_2) \rangle \sim \sum_{\ell=1}^{\infty} B_\ell \frac{\cos(2\pi \ell n_B |x_1 - x_2|)}{|x_1 - x_2|^{2\ell^2 K}}$

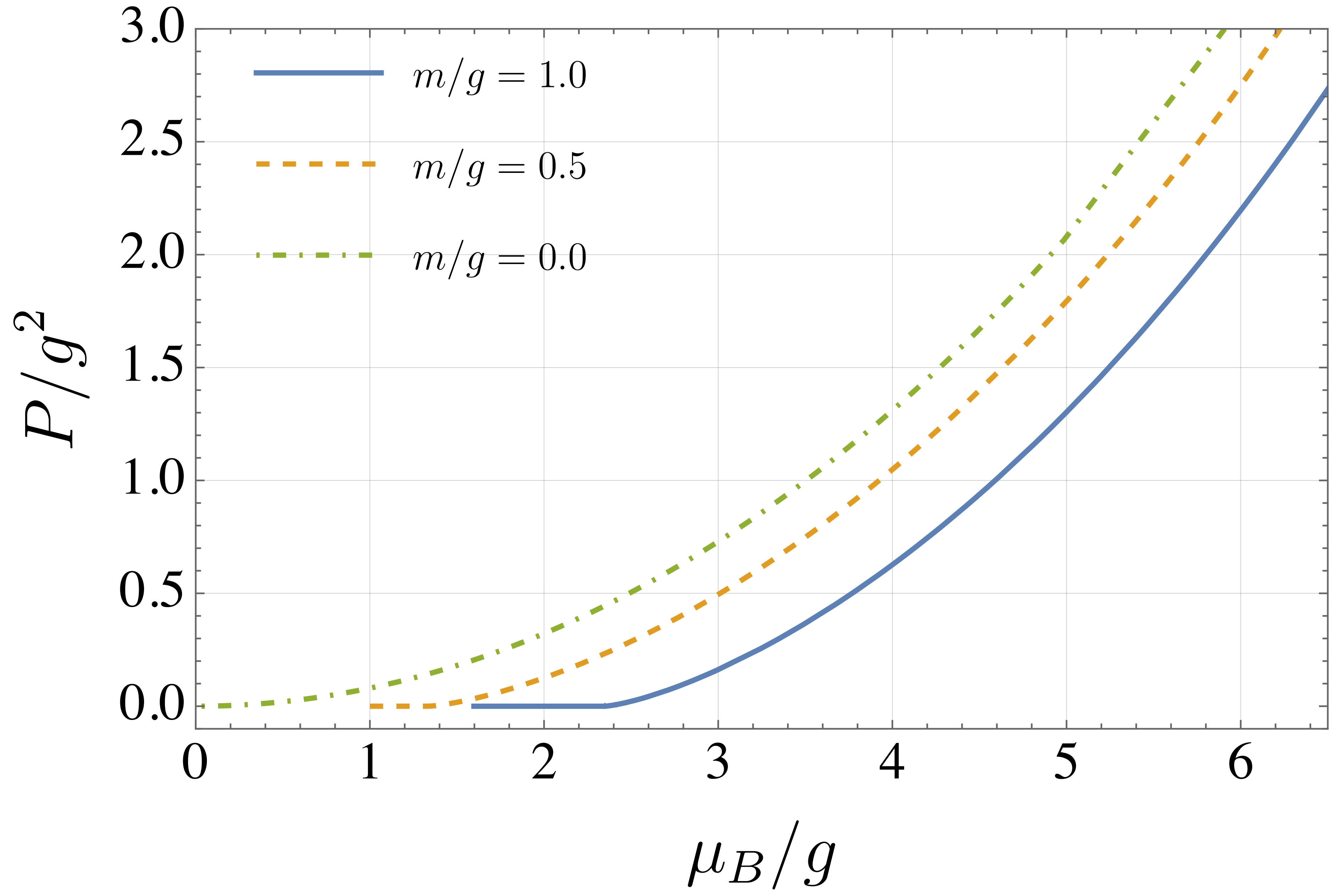


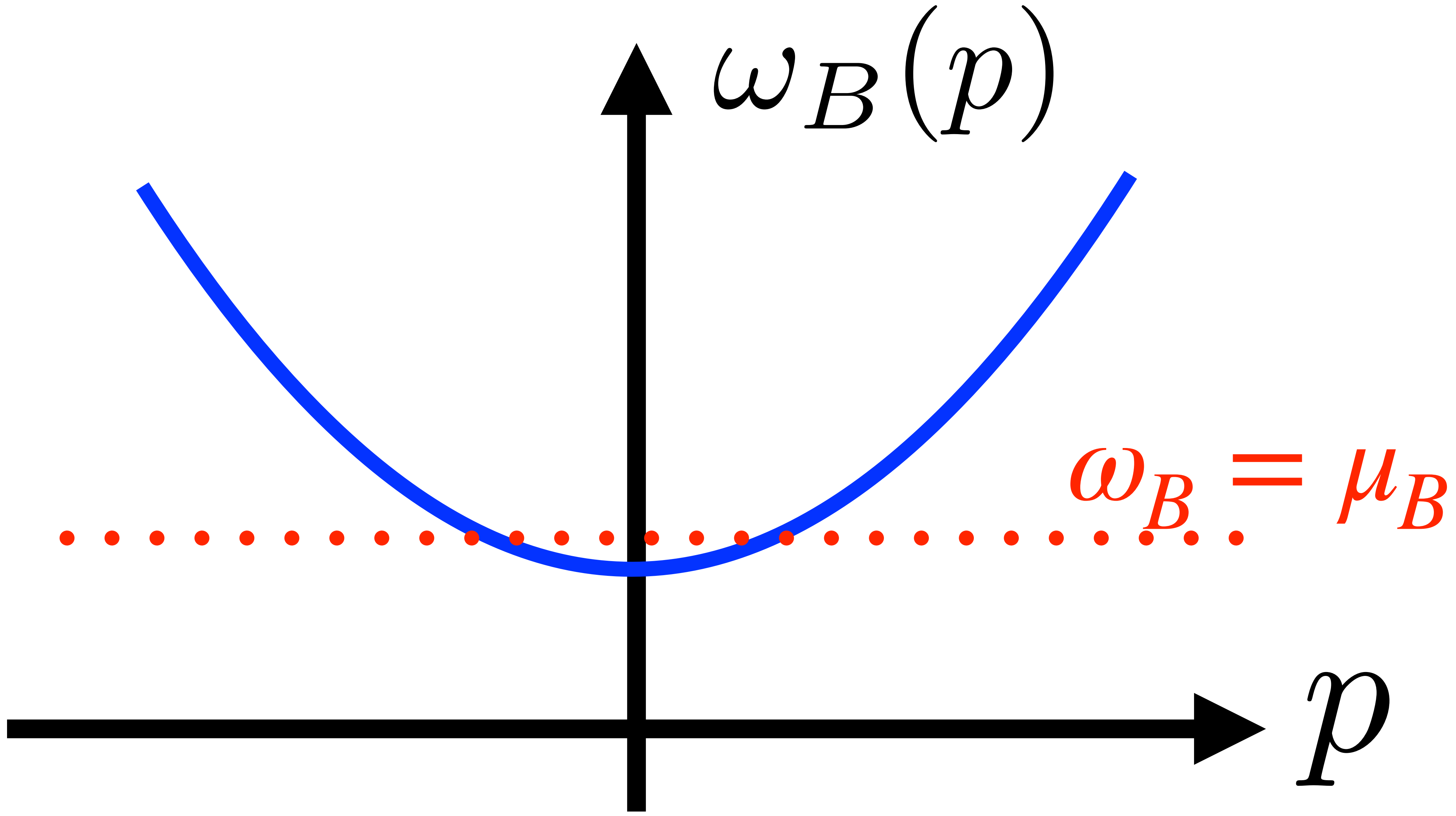



$m/g = 1.0$
$m/g = 0.5$
$m/g = 0.0$
n_B/g
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
1
2
3
4
5
6
μ_B/g
?
?

Dilute Baryon Liquid

Unfortunately, I couldn't construct the ground state at very dilute baryon density region, $\langle n_B \rangle / g \leq 0.04$…





As far as I understand, the most plausible scenario is a Pokrovsky—Talapov transition, which is described by a non-relativistic free-fermion gas.

(*However, baryon is boson for $N_c = 2$!)

[Pokrovsky, Talapov (1980)]

Conclusion

- We explore the two-color single flavor dense QCD in (1+1)-dimension with uniform matrix product states.

- A gapless behavior is observed at any finite baryon number density $\langle n_B \rangle \neq 0$, and the central charge is $c = 1$.

- There are spatial modulations in the baryon number density-density correlation with wavenumber, $p_f = 2\pi\ell\langle n_B \rangle$, $\ell = 1,2,\cdots$.

- We compute EFT parameters using first-principles methods for cold and dense $\mathrm{QC_2D_2}$.

VUMPS

Variational Uniform Matrix Product State (VUMPS) algorithm

[Zauner-Stauber, Vanderstraeten, Fishman, Verstraete, Haegeman (2017)]

Energy is the target function: $E \equiv \dfrac{\langle \Psi_{\mathrm{uMPS}} | \hat{H} | \Psi_{\mathrm{uMPS}} \rangle}{\langle \Psi_{\mathrm{uMPS}} | \Psi_{\mathrm{uMPS}} \rangle}$

uMPS: $|\Psi\rangle = \mathrm{tr}(\cdots \underline{A^{s_n}}\, A^{s_{n+1}} \cdots)|\cdots, s_n, s_{n+1}, \ldots\rangle$

Variational parameters

VUMPS is based on the gradient descent method:

Find the MPS tensor minimizing energy: $\dfrac{\delta E}{\delta (\mathbf{A}^{s_m})^\dagger} = 0$

Cold and Dense Ground states

The ground state property is investigated by minimizing,

$$\langle\Psi_{\mathrm{uMPS}}|H_{\mathrm{tot}} - \mu_B N_B|\Psi_{\mathrm{uMPS}}\rangle$$

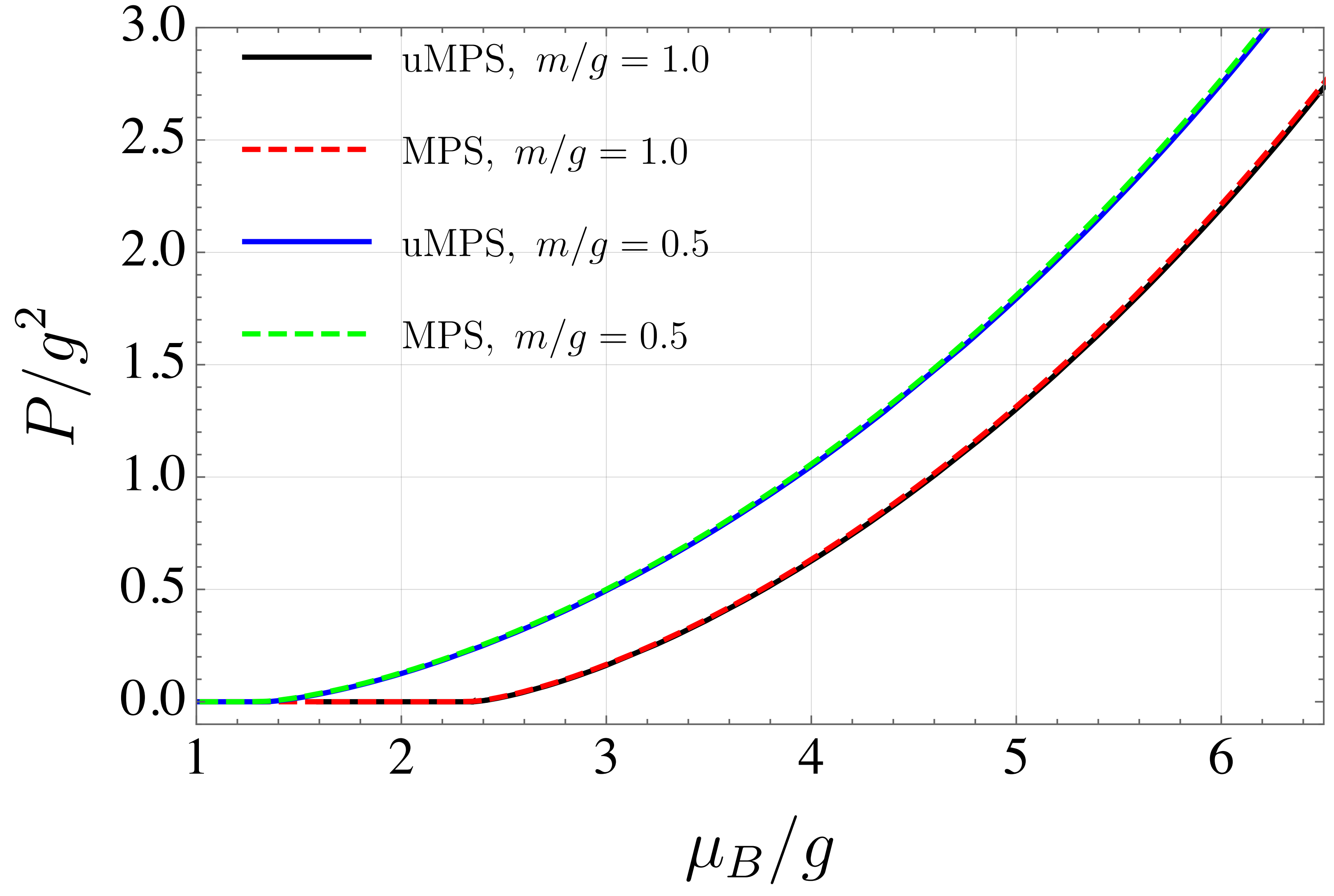
N_B: baryon number operator

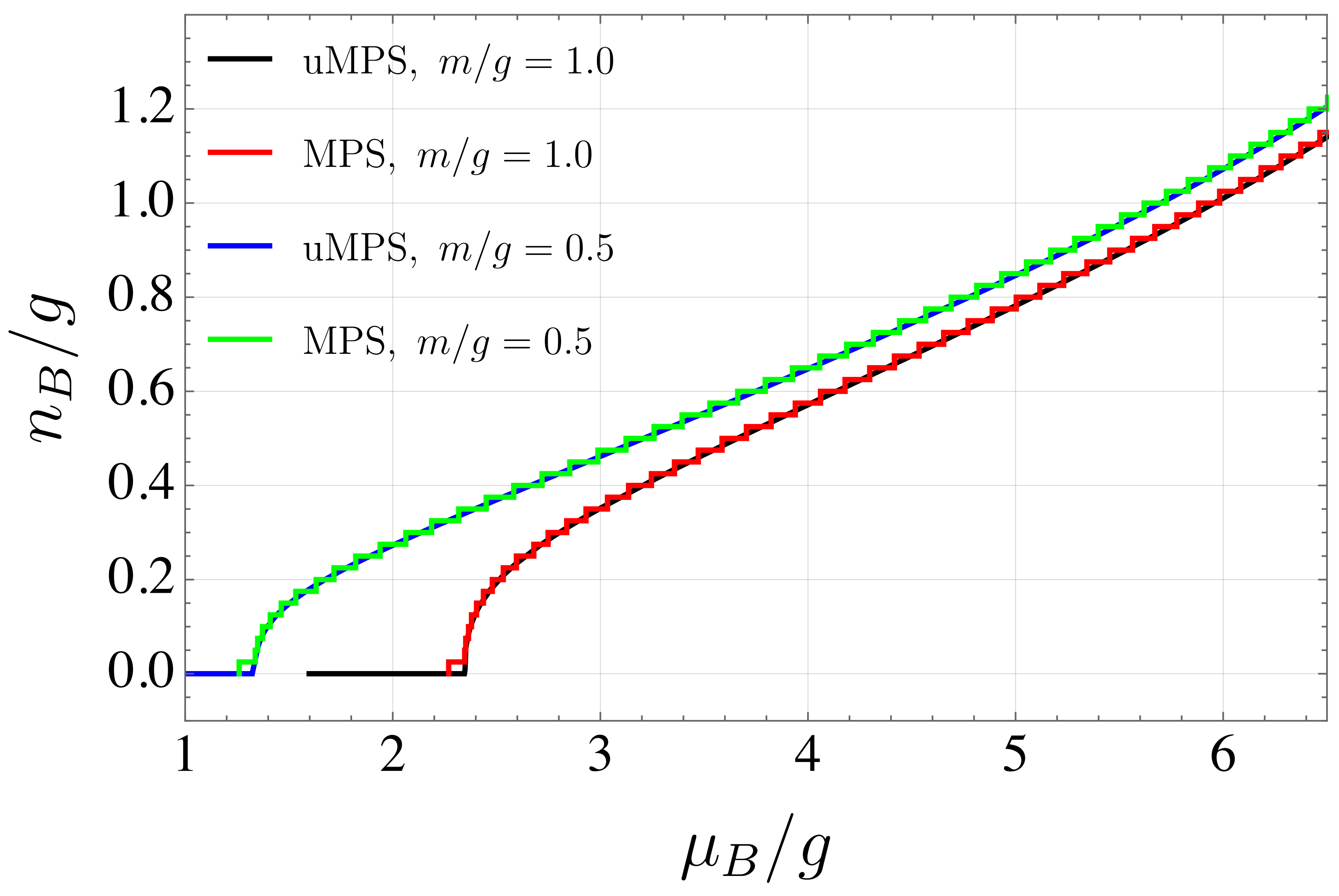
under fixed lattice input parameters: m/g, μ_B/g, D, ga,
(The gauge coupling has mass dimension 1 in (1+1)-dim.)

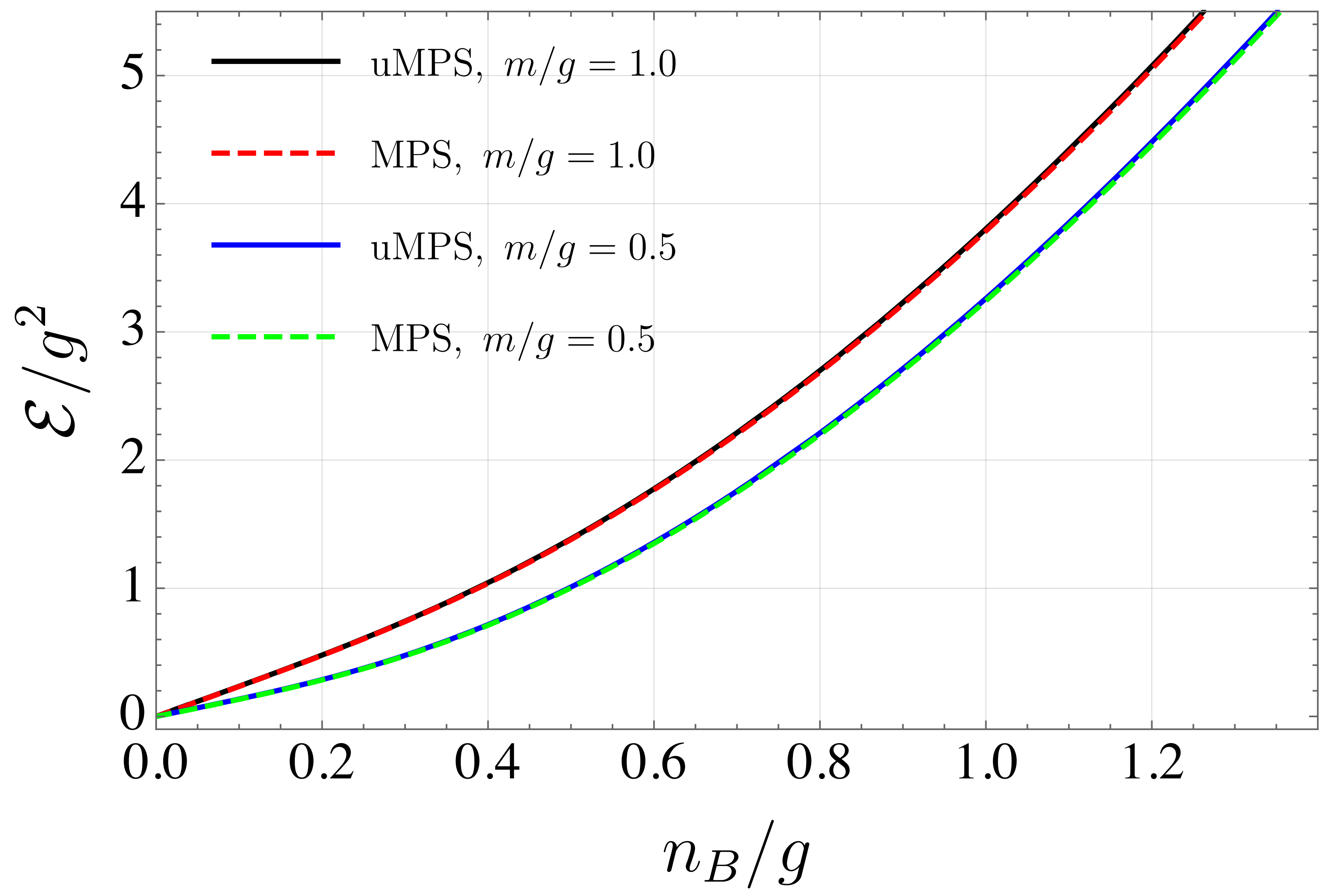
with a translational and gauge invariant MPS ansatz $|\Psi_{\mathrm{uMPS}}\rangle$.

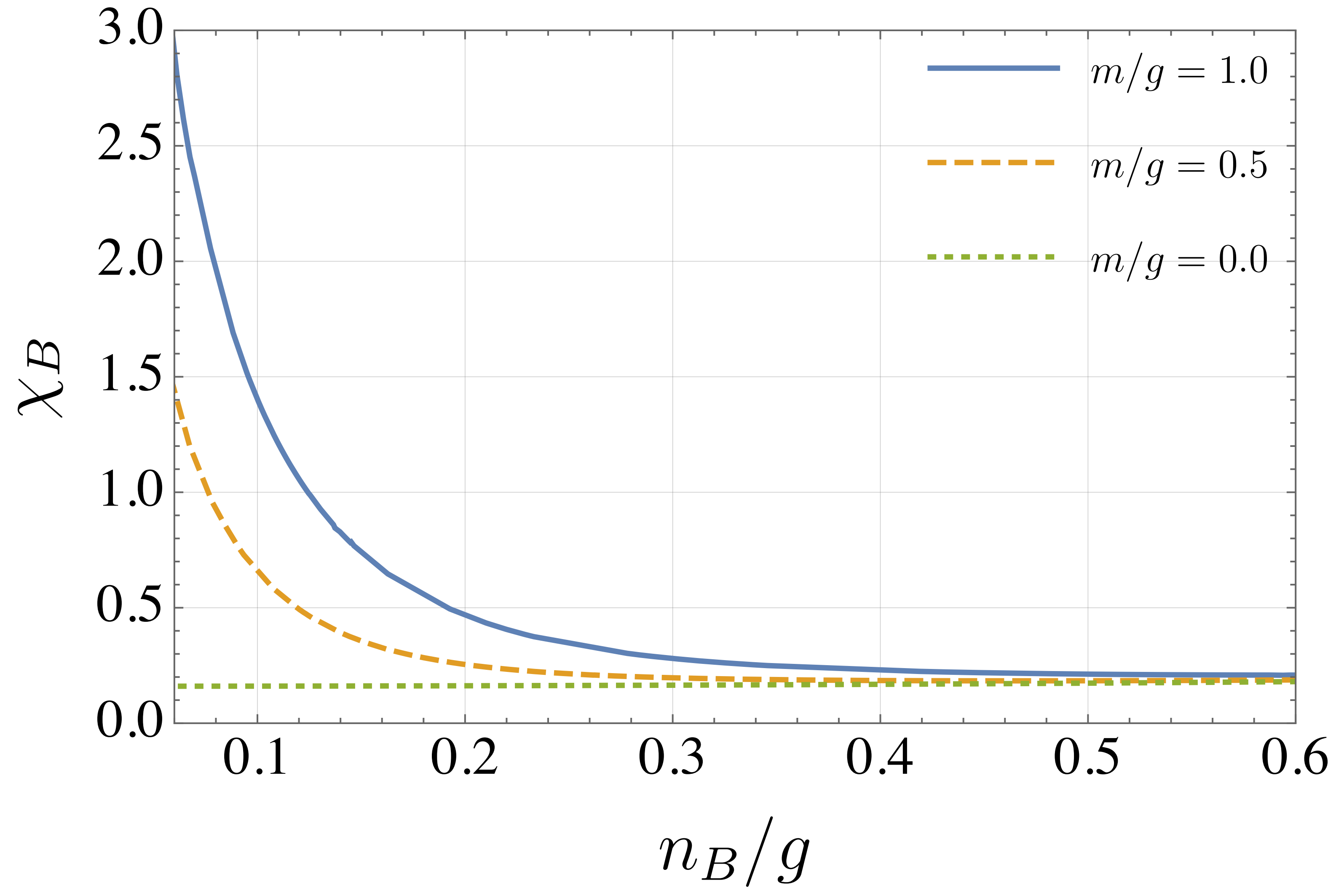
Remark: The translational invariance plays a crucial role to obtain new results.

Thermodynamic quantities









Gauss law constraint

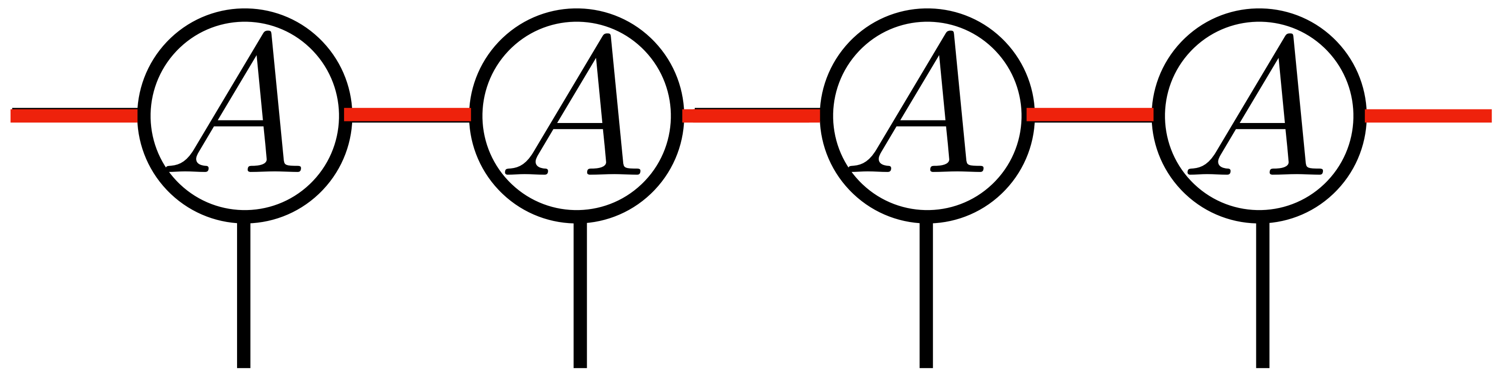
Gauss law constraint (gauge invariance) is imposed on the physical state:

$$G^a(n) = R^a(n) - L^a(n-1) + Q^a(n) \quad Q^a(n) = \phi^\dagger(n) T^a \phi(n)$$

$$[H_{\text{tot}}, G^a(n)] = 0, \quad G^a(n)|\Psi\rangle = 0 \ (\forall n)$$

Gauge invariant variational ansatz:

$$|\Psi\rangle = \quad \cdots$$



$$\cdots$$

$$[A_n^{j_{L_n}, m_{L_n}, M_n, j_{R_n}, m_{R_n}}]_{J_L \alpha_{J_L}; J_R \beta_{J_R}} = \underline{b^{j_{L_n}, j_{M_n}, q_n, j_{R_n}}_{\alpha_{J_L} \beta_{J_R}}} \frac{1}{\sqrt{d_{j_{L_n}}}} C^{j_{L_n}, m_{L_n}}_{j_{R_n}, m_{R_n}, j_{M_n}, m_{M_n}} \delta^{j_{L_n}}_{J_L} \delta^{j_{R_n}}_{J_R}$$

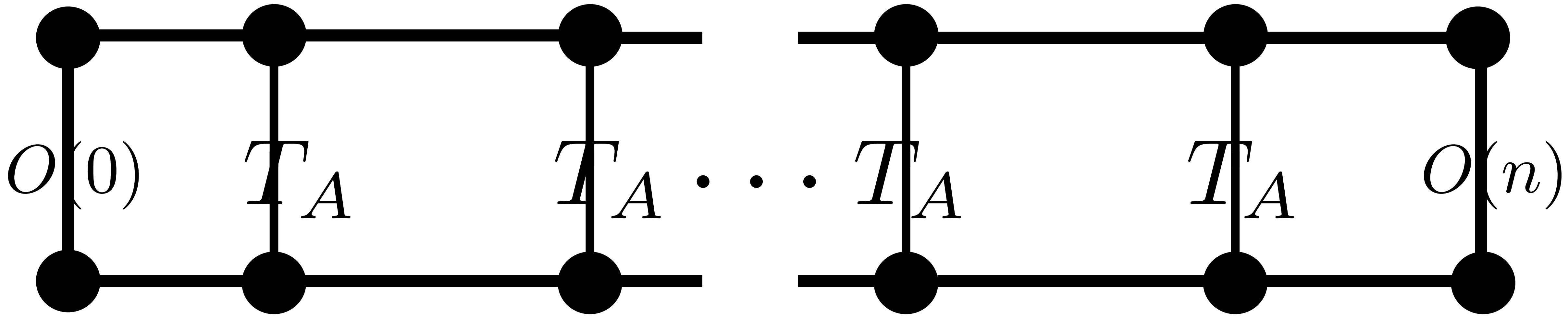
Gauge invariant variational d.o.f

Looking for transfer matrix

To see the modulation accurately, I use the translational invariance.

Two-point functions are expressed as

$$\langle O(0)O(n)\rangle =$$



If modulation exists, there should be complex phases of eigenvalues:

$$|\langle O_1(0)O_2(n)\rangle_{\text{connected}}| \sim (T_A)^n \sim \sum_{j=1}^{D^2-1} e^{-(\epsilon_j - i\phi_j)n} \quad (n \gg 1)$$

Wavenumbers of modulation can be extracted without fitting.

Bosonization

My analytic analysis (with some certain assumptions) indicates the following infrared phases. (Details are omitted.)

$$S = \int d^2x \left[-\frac{1}{4}(F^{\mu\nu\,a} F^a_{\mu\nu}) + i\bar{q}\gamma^\mu D_\mu q - \mu_q q^\dagger q + m_q \bar{q} q \right] \qquad \mu_B = \mu_q / N_c$$

$\mu_B \ll m_q$

Sine-Gordon model (Gapped)

$$S = \int \mathrm{d}^2x \left[\frac{1}{4\pi}(\partial\phi_B)^2 + M^2(1 - \cos(\phi_B)) \right]$$

$\mu_B \gg m_q$

Tomonaga-Luttinger liquid (Gapless)

$$S = \int \mathrm{d}^2x \, \frac{K}{4\pi} \left[-\frac{1}{v}(\partial_t \phi_B)^2 + v(\partial_x \phi_B)^2 \right]$$

$$\partial_x \phi_B \sim q^\dagger q \quad \phi_B \sim \phi_B + 2\pi$$

Ground state property at $\mu_B = 0$

From the simulation, I obtain the following behavior.

This is in agreement with the analytic study based on the bosonization.

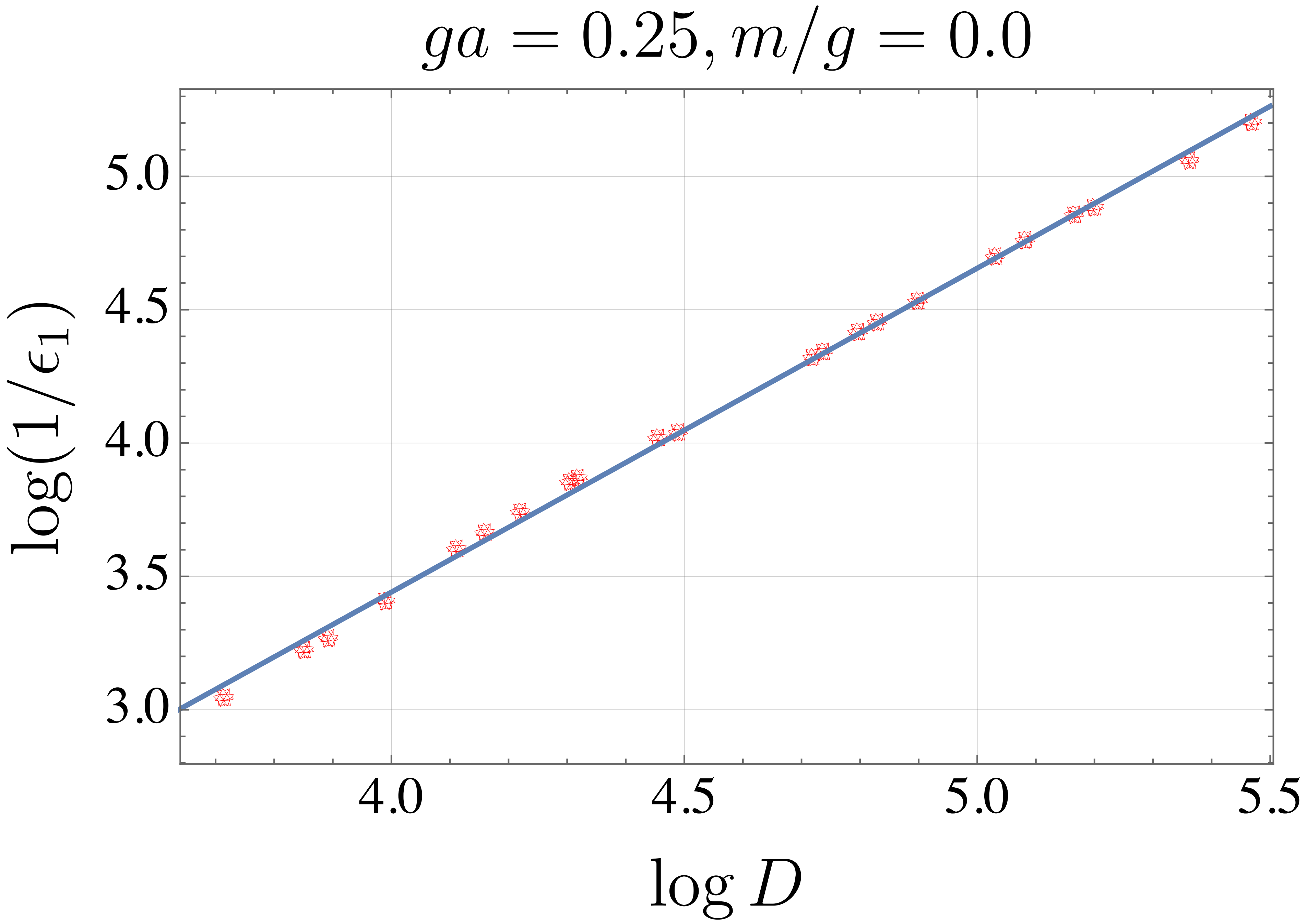
$$S = \int \mathrm{d}^2 x \left[\frac{1}{2} (\partial \phi)^2 + M \cos(\# \phi) \right] \qquad M \propto m$$

(Anti-)Baryon is (anti-)kink for $m \neq 0$ in the bosonic language.

Ground state property at $\mu_B = 0$

Baluni and Steinhardt say that there exists the critical point at $m = 0$, where the baryon becomes gapless.

Correlation length (Logarithmic scale)



Our simulation shows the critical behavior, $\xi \propto D^{\kappa}$. (D: bond dimension)

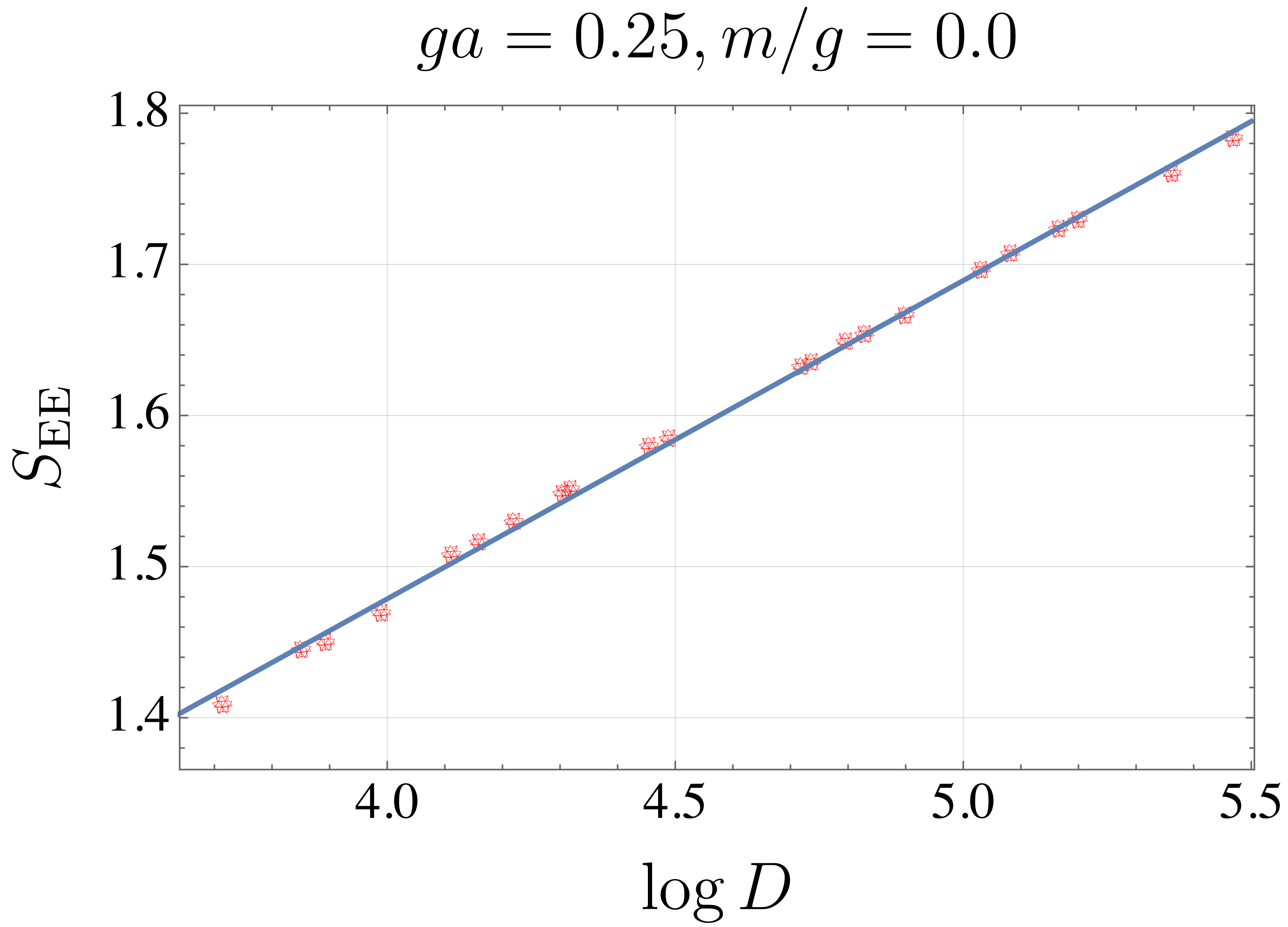
The correlation length is extracted from the transfer matrix.

Ground state property at $\mu_B = 0$

The entanglement entropy tells us the central charge of CFT.

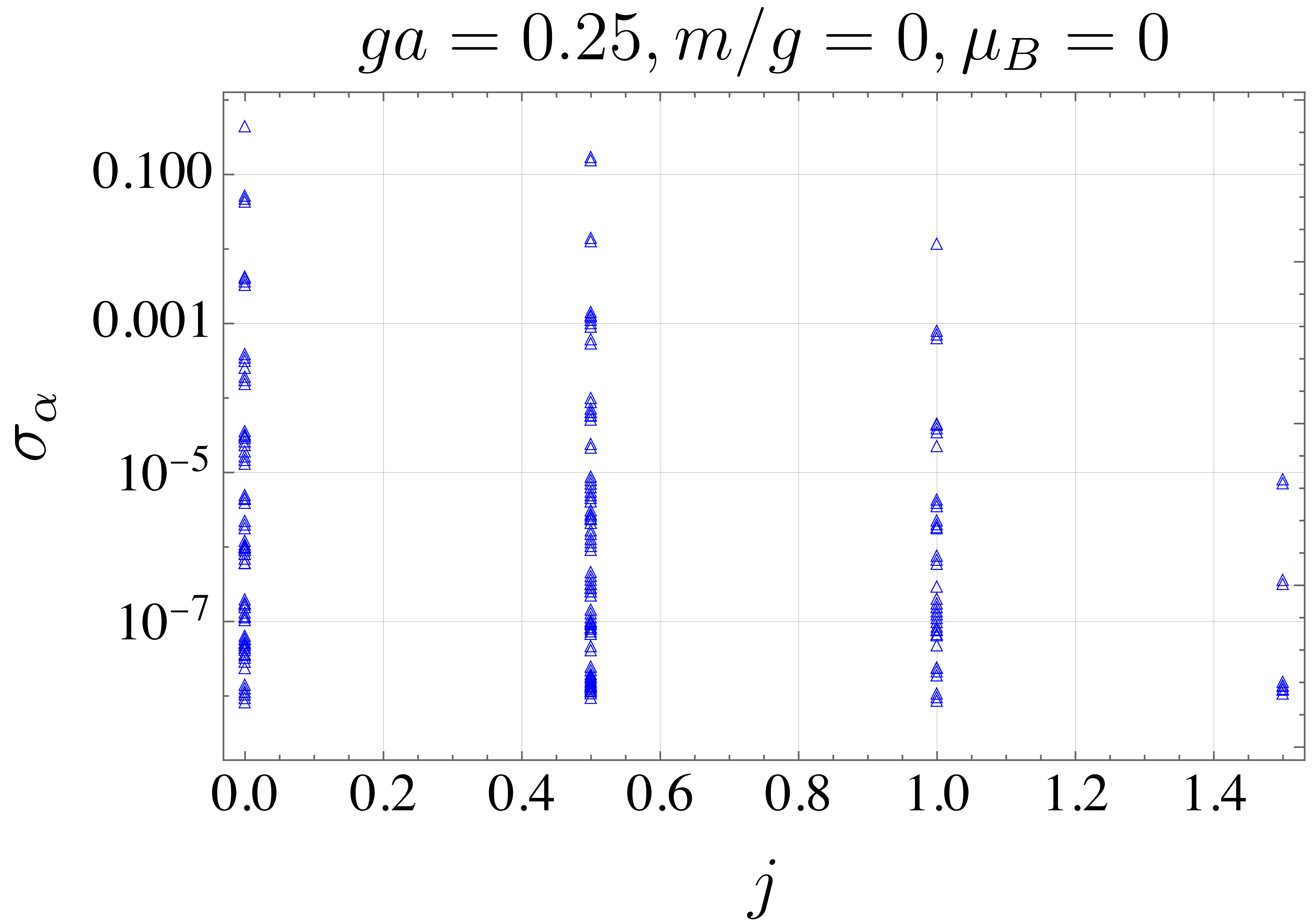
$$S_{\mathrm{EE}} \sim \frac{c}{6} \log(\xi/a)$$

[P. Calabrese, J. Cardy (2009)]

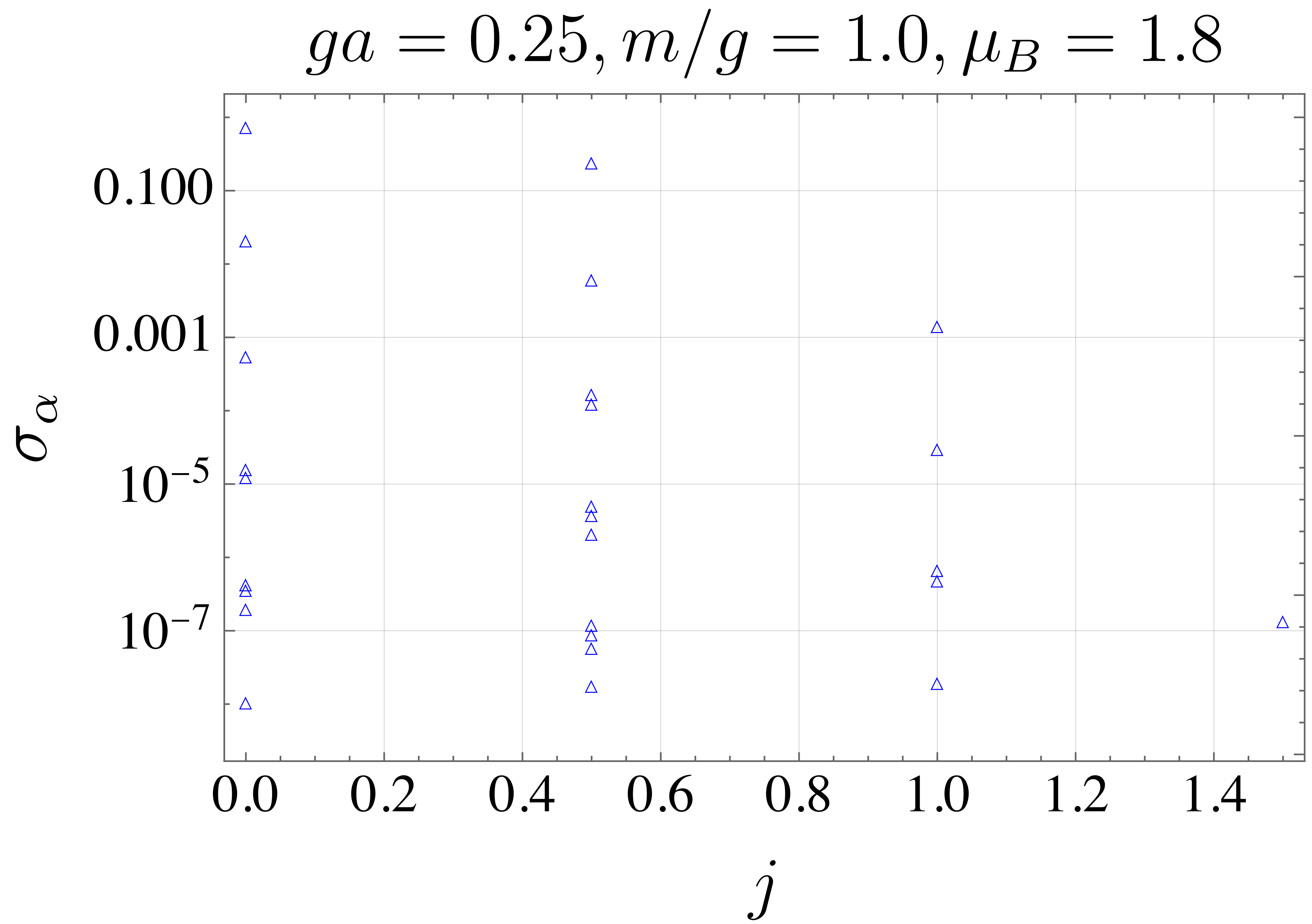


The central charge is in agreement with the free compact boson $c = 1$.

Truncation on electric flux



$ga = 0.25, m/g = 0, \mu_B = 0$



$ga = 0.25, m/g = 1.0, \mu_B = 1.8$

In uniform MPS, gauge field remains.
We make a certain truncation on the electric flux (SU(2) spin).