Simulating the critical dynamics of stochastic fluids

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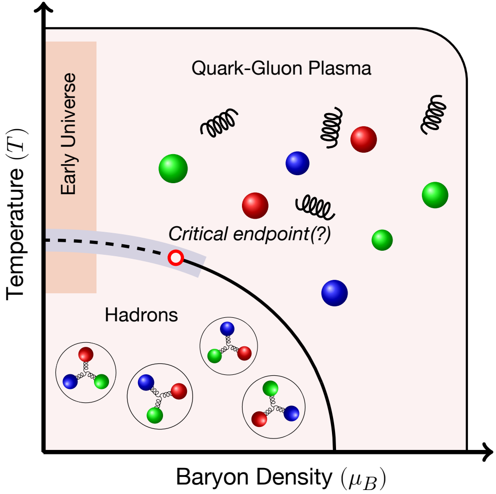
Dynamics and universality

- Start from fluid dynamics as description of large scales
- Fluctuations required for critical phenomena
- Turn to theories of **stochastic hydrodynamics**
- Relevant to us are models G (chiral transition) and model H (QCD critical endpoint)

Methods to study these models include Hydro+ (Misha) and functional renormalization group.



Model A

The simplest model: stochastic relaxation.
The order parameter ϕ gets the equation of motion

$$\partial_t \phi = \underbrace{-\Gamma \frac{\delta \mathcal{F}}{\delta \phi}}_{\text{Relaxation}} + \underbrace{\zeta}_{\text{Noise}}, \quad \text{where } \langle \zeta(t, \vec{x}) \zeta(t, \vec{x}') \rangle = T \Gamma \delta(\vec{x} - \vec{x}') \delta(t - t'),$$

with free energy functional

$$\mathcal{F} = \int \mathrm{d}^d x \left[\frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4} \phi^4 \right].$$

Just a field-theoretic generalization of Brownian motion with Einstein relation.

Metropolis algorithm

Standard hydrodynamic solvers face infinite noise problem due to stochastic term:

$$\phi(t+\Delta t) = \phi(t) + \Delta t \left[-\Gamma \frac{\delta \mathcal{F}}{\delta \phi} + \sqrt{\frac{\Gamma T}{(\Delta t) a^3}} \theta \right].$$

Address this by employing a **Metropolis algorithm** where the update

$$\phi(\vec{x}, t+\Delta t) = \phi(\vec{x}, t) + \sqrt{2T\Gamma(\Delta t)}\theta,$$

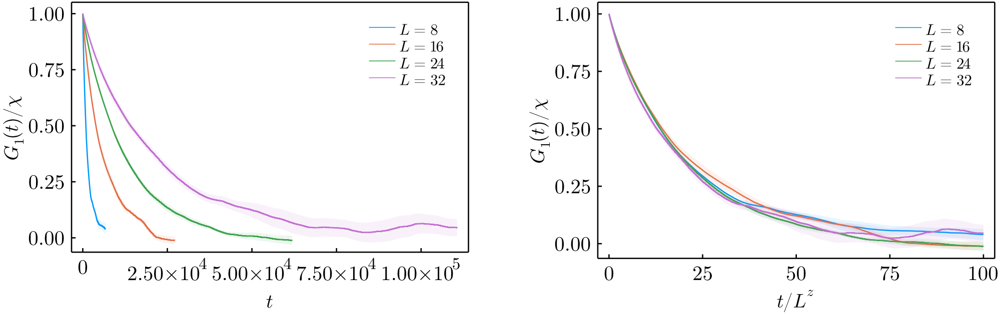
is accepted with probability $\min(1, e^{-\beta \Delta \mathcal{F}})$. This handles **both** diffusion and noise:

$$\begin{aligned} \langle \phi(t+\Delta t) - \phi(t) \rangle &= -(\Delta t)\,\Gamma\,\frac{\delta \mathcal{F}}{\delta \phi} + \mathcal{O}\left((\Delta t)^2\right) \\ \langle [\phi(t+\Delta t) - \phi(t)]^2 \rangle &= 2(\Delta t)\Gamma\,T + \mathcal{O}\left((\Delta t)^2\right) \end{aligned}$$

We thus end up with $P[\phi] \sim \exp\left(-\beta \mathcal{F}[\phi]\right)$.

Correlation functions

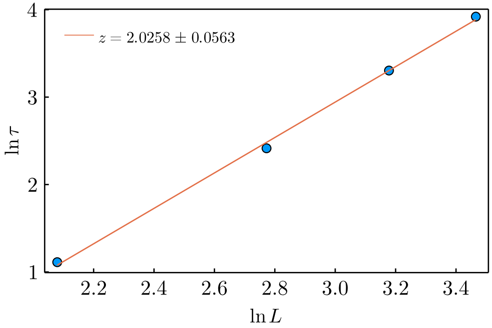
Look for dynamical scaling in $G_1(t) = \langle \phi(0)\phi(t) \rangle \sim f(t/\tau)$.



Find the dynamic critical exponent z by fitting the relaxation time to $\tau \sim L^z$.

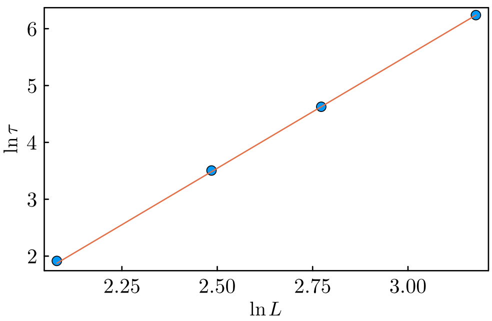
Dynamical scaling in models A and B

Model A (2204.02433)



$$z \simeq 2.03$$

Model B (2304.07279)



$$z \simeq 3.91$$

Model H

The order parameter ϕ gets the equation of motion

$$\partial_t \phi = \underbrace{\Gamma \nabla^2 \frac{\delta \mathcal{F}}{\delta \phi}}_{\text{Diffusion}} - \underbrace{\left(\vec{\nabla}\phi\right) \frac{\delta \mathcal{F}}{\delta \vec{\pi}}}_{\text{Advection}} + \underbrace{\zeta}_{\text{Noise}} ,$$

and the momentum density $\vec{\pi}$ gets the equations of motion

$$\partial_t \vec{\pi}^T = \eta \nabla^2 \frac{\delta \mathcal{F}}{\delta \vec{\pi}^T} + \left(\vec{\nabla}\phi\right) \cdot \frac{\delta \mathcal{F}}{\delta \phi} - \left(\frac{\delta \mathcal{F}}{\delta \vec{\pi}^T} \cdot \vec{\nabla}\right) \vec{\pi}^T + \vec{\xi}.$$

Based on dispersion relations, we only consider the transverse component π^T.
The free energy now just includes a term quadratic in π:

$$\mathcal{F} = \int \mathrm{d}^d x \left[\frac{1}{2\rho}\vec{\pi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4\right].$$

Conserving Metropolis step

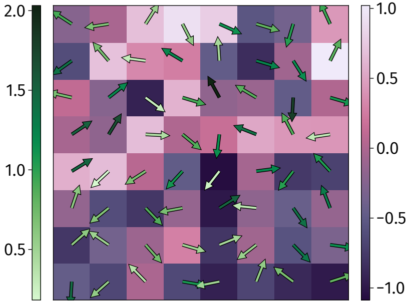
Now we need to enforce a conservation law with fluctuation-dissipation relations

$$\langle\zeta(t,\vec{x})\zeta(t,\vec{x}')\rangle = -2T\Gamma\nabla^2\delta(\vec{x}-\vec{x}')\delta(t-t')$$
$$\langle\xi_i(t,\vec{x})\xi_j(t,\vec{x}')\rangle = -2T\eta\nabla^2 P^T_{ij}\delta(\vec{x}-\vec{x}')\delta(t-t').$$

To ensure conservation, we work with pairs of sites and propose fluxes

$$\phi(\vec{x},t+\Delta t) = \phi(\vec{x},t) + \sqrt{2T\Gamma(\Delta t)}\theta,$$
$$\phi(\vec{x}+\hat{\mu},t+\Delta t) = \phi(\vec{x}+\hat{\mu},t) - \sqrt{2T\Gamma(\Delta t)}\theta.$$

Likewise for the momentum density, propose a flux $r_{\nu\mu} = \sqrt{2T\eta(\Delta t)}\zeta_\nu$ between $\pi_\nu(\vec{x})$ and $\pi_\nu(\vec{x}+\hat{\mu})$.



Advection

We need to solve the ideal part of the equations of motion

$$\partial_t \phi = -\left(\vec{\nabla}\phi\right)\frac{\delta \mathcal{F}}{\delta \vec{\pi}}, \quad \partial_t \vec{\pi}^T = \underbrace{\left(\vec{\nabla}\phi\right)\cdot\frac{\delta \mathcal{F}}{\delta \phi}}_{\text{mode coupling}} - \underbrace{\left(\frac{\delta \mathcal{F}}{\delta \vec{\pi}^T}\cdot\vec{\nabla}\right)\vec{\pi}^T}_{\text{self-advection}}$$

Neglecting self-advection leaves universal properties intact: we call this model H0. Then we solve the advection equations for ϕ and $\vec{\pi}$

$$\begin{aligned}\dot{\phi} &= -\frac{\pi_i^T}{\rho}\nabla_i\phi,\\ \dot{\pi}_i^T &= -P_{ij}^T\Bigg[\underbrace{\frac{1}{2}\nabla_k\left(\frac{1}{\rho}\pi_k^T\pi_j^T\right) + \frac{1}{2\rho}\pi_k^T\nabla_k\pi_j^T}_{\text{skew-symmetric derivative}} + \nabla_j\phi\nabla^2\phi\Bigg]\end{aligned}$$

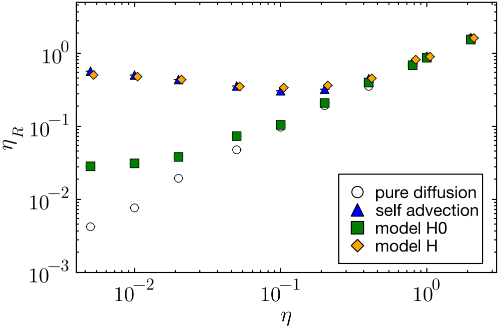
as one would treat Navier-Stokes. (note: $\pi_i^T = P_{ij}^T\pi_j$)

Viscosity

A one-loop calculation for the self-advection of π predicts a renormalization to the viscosity

$$\eta_R = \eta + \frac{7}{60\pi^2}\frac{\rho T \Lambda}{\eta}.$$

Here we also consider Model H0, which neglects this self-coupling. Such a truncation permits lower physical viscosities.



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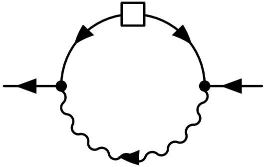
Dynamical scaling

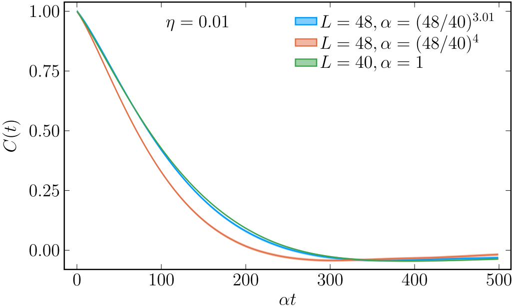
A one-loop calculation with the advection of ϕ by π predicts



$$\Gamma_k = \frac{\Gamma}{\xi^4}(k\xi)^2\left(1+(k\xi)^2\right) + \frac{T}{6\pi\eta_R\xi^3}K(k\xi).$$

This suggests the dynamic critical exponent crosses over from $z \simeq 4$ to $z \simeq 3$ for $\xi \gg 6\pi\eta_R/T$.

With lower η_R permitted by Model H0, we can observe this crossover.



We find $z \simeq 3.013 \pm 0.058$ at $\eta = 10^{-2}$.

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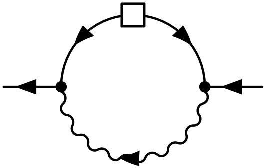

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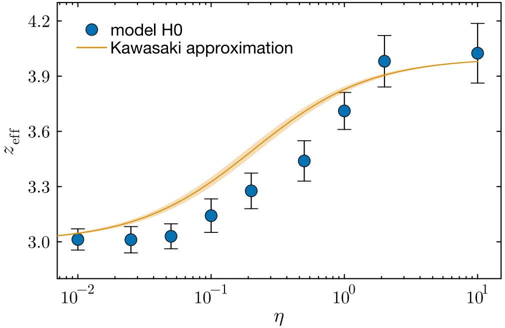

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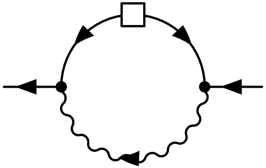

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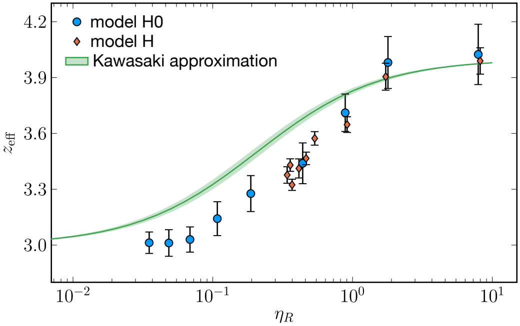

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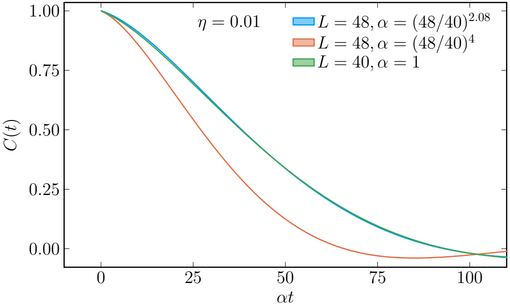
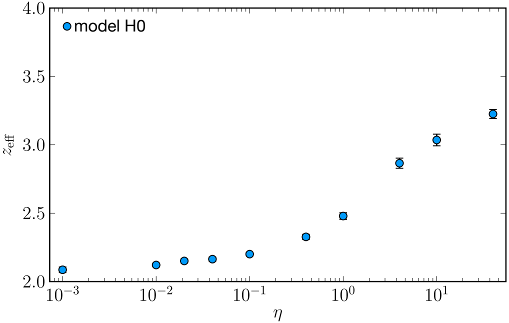
Scaling in two-dimensional fluids

The ϵ-expansion predicts $z = 2.179$. For $\eta = 10^{-2}$, we get $z = 2.11 \pm 0.015$.





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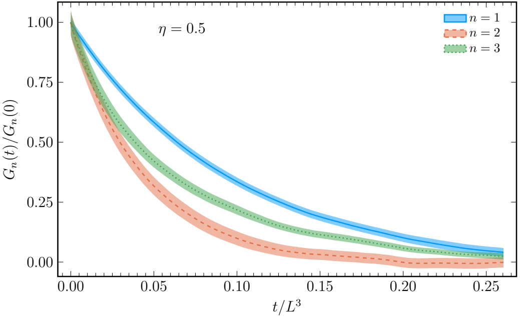
Non-Gaussian moments

Non-Gaussian fluctuations are particularly important as experimental observables for the critical point search.
Integrate over a subvolume V of the lattice

$$G_n(t) = \langle M^n(0) M^n(t) \rangle$$
$$M(t) = \int_V \mathrm{d}^3 x \, \phi(\vec{x}, t).$$

This method is only restricted by statistics and compute. We find:

- Dynamical scaling still holds
- Nontrivial dependence of τ on n



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Look ahead

What we have studied with this method:

- Renormalization of parameters
- Dynamical scaling
- Non-Gaussian moments

To connect this with experiment, we need to put it on a relativistic, expanding background.