Fluctuation Signatures from the Equation of State to Proton Factorial Cumulants

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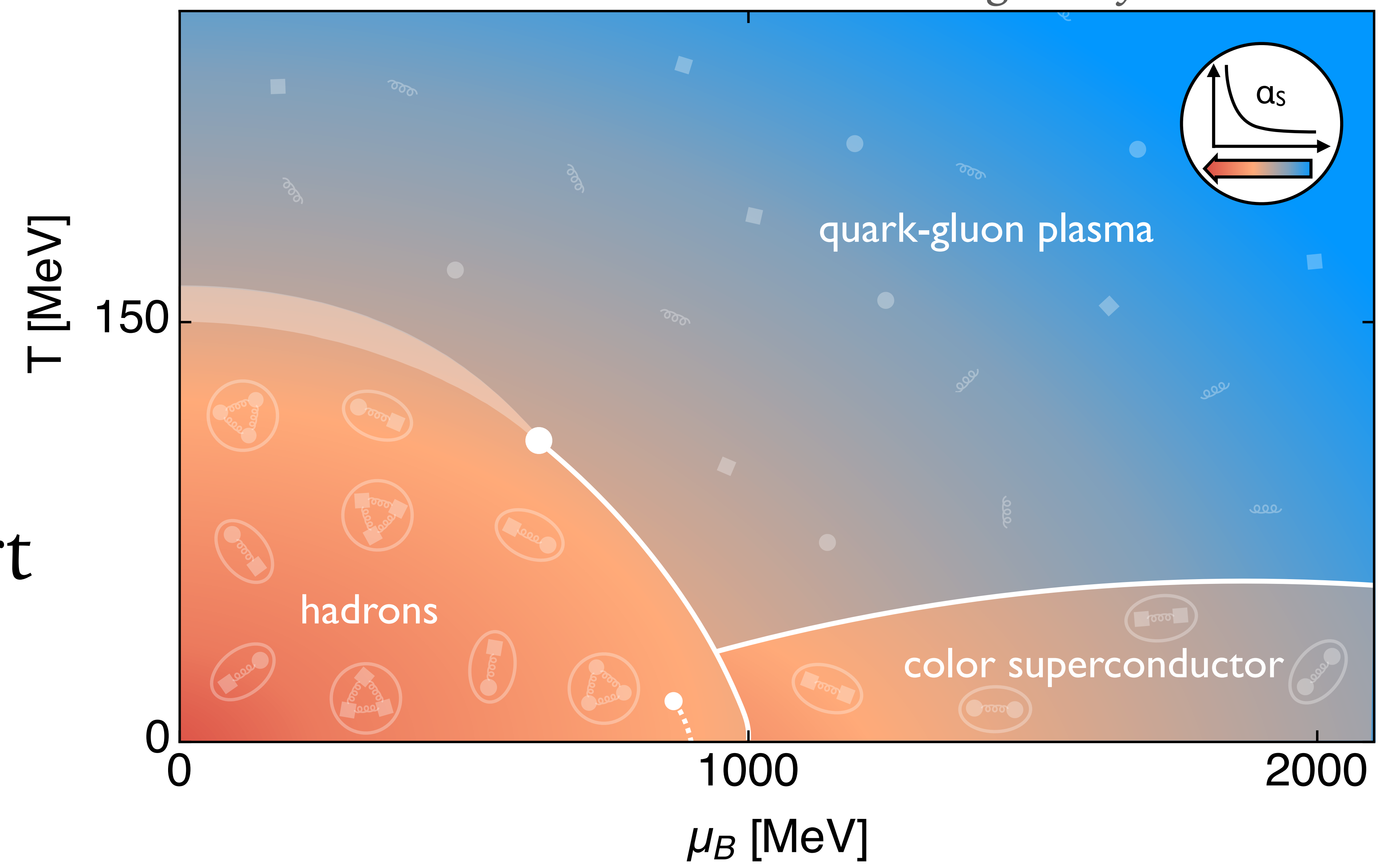
QCD Critical Point and Hydrodynamic Evolution at the Yukawa Institute for Theoretical Physics

Phase Diagram (Sketch)

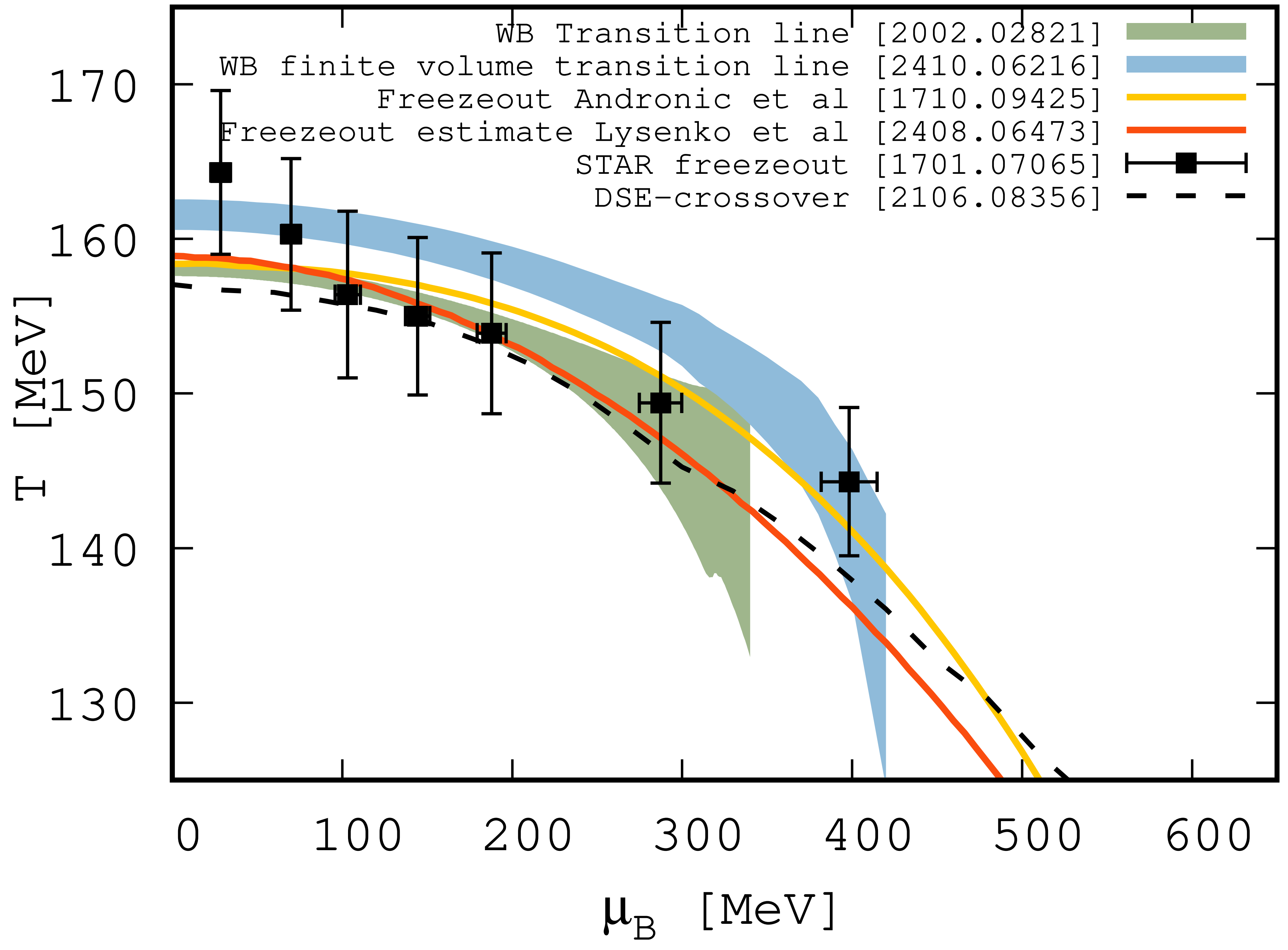
- Knowledge of the equation of state of strongly-interacting matter in equilibrium is crucial for:
  - Fluctuations, via derivatives of the pressure
  - The hadronic spectrum, i.e. the composition of the system in HICs, via thermal models
  - Hydrodynamic simulations
  - Hadronic transport simulations
  - Merger simulations
  - The behavior of the bulk viscosity & transport
  - The interior composition of neutron stars
  - …

Phase diagram by F. Rennecke



(Less Sketchy) Phase Diagram

- Much of the current knowledge of the QCD phase diagram from *ab initio* theory and experiment from QCD crossover transition & freeze-out

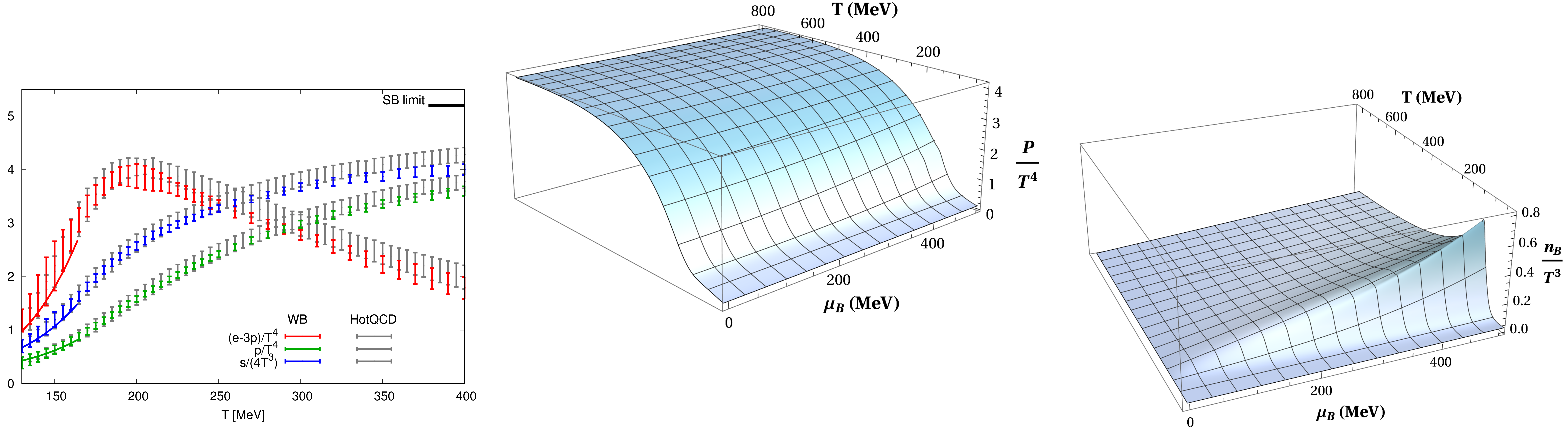


S. Borsanyi et al, arXiv:2502.10267

Lattice EoS at Finite T & μ_B

- Equilibrium thermodynamics calculated from first principles lattice QCD computations are well-established with good agreement amongst techniques

$$\frac{p(T,\mu_B)}{T^4} = \frac{p(T,0)}{T^4} + \sum_{n=1}^{\infty} \frac{1}{(2n)!} \frac{\mathrm{d}^{2n}(p/T^4)}{d(\frac{\mu_B}{T})^{2n}}\bigg|_{\mu_B=0} \left(\frac{\mu_B}{T}\right)^{2n} = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \chi^B_{2n} \left(\frac{\mu_B}{T}\right)^{2n}$$



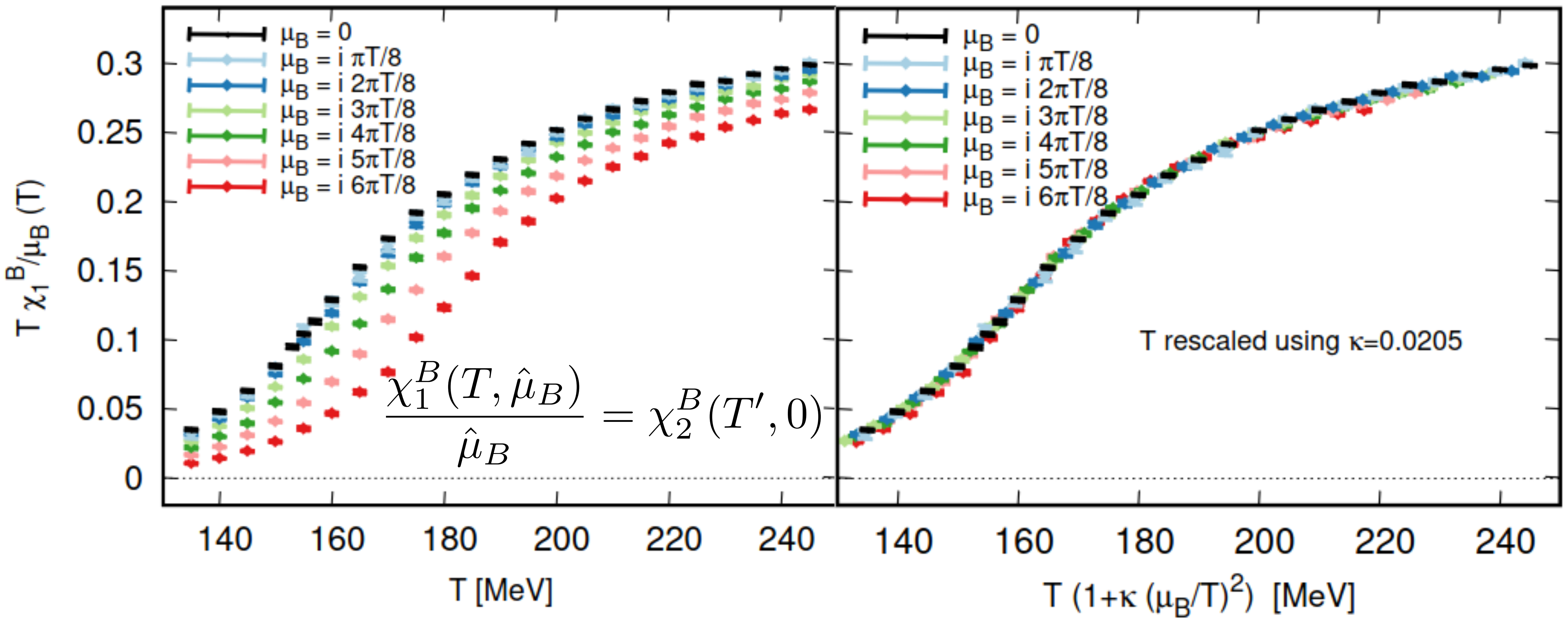
A. Bazavov PRD (2014), S. Borsanyi PLB (2014)

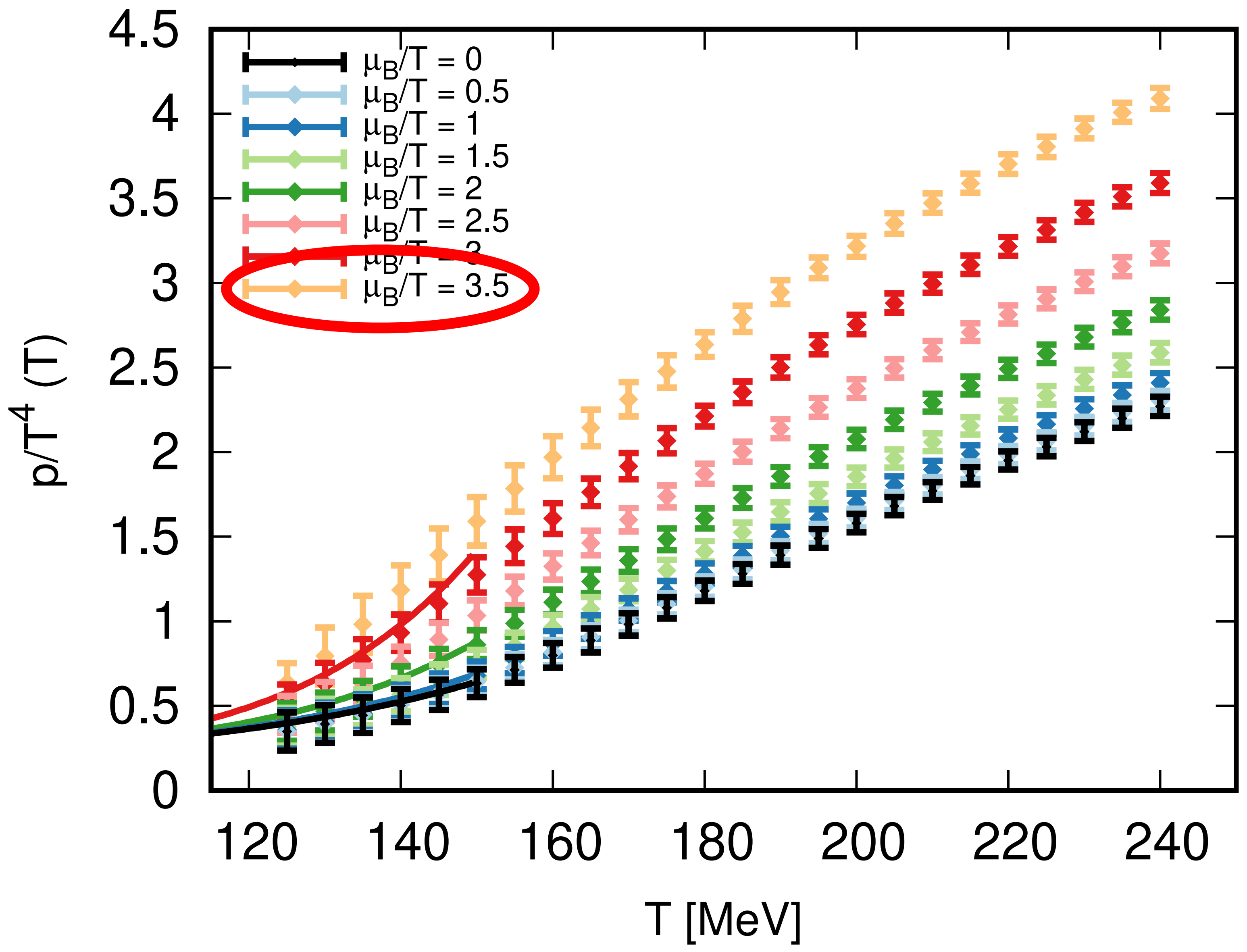
Alternative Expansion Scheme: TExS

- Extend the coverage at finite density with the T'-Expansion Scheme (TExS) via a rescaled μ_B-dependent temperature T'

$$T'(T,\hat{\mu}_B) = T\left(1+\kappa_2^{BB}(T)\hat{\mu}_B^2+\kappa_4^{BB}(T)\hat{\mu}_B^4+\mathcal{O}(\hat{\mu}_B^6)\right)$$

$$T \to T'$$





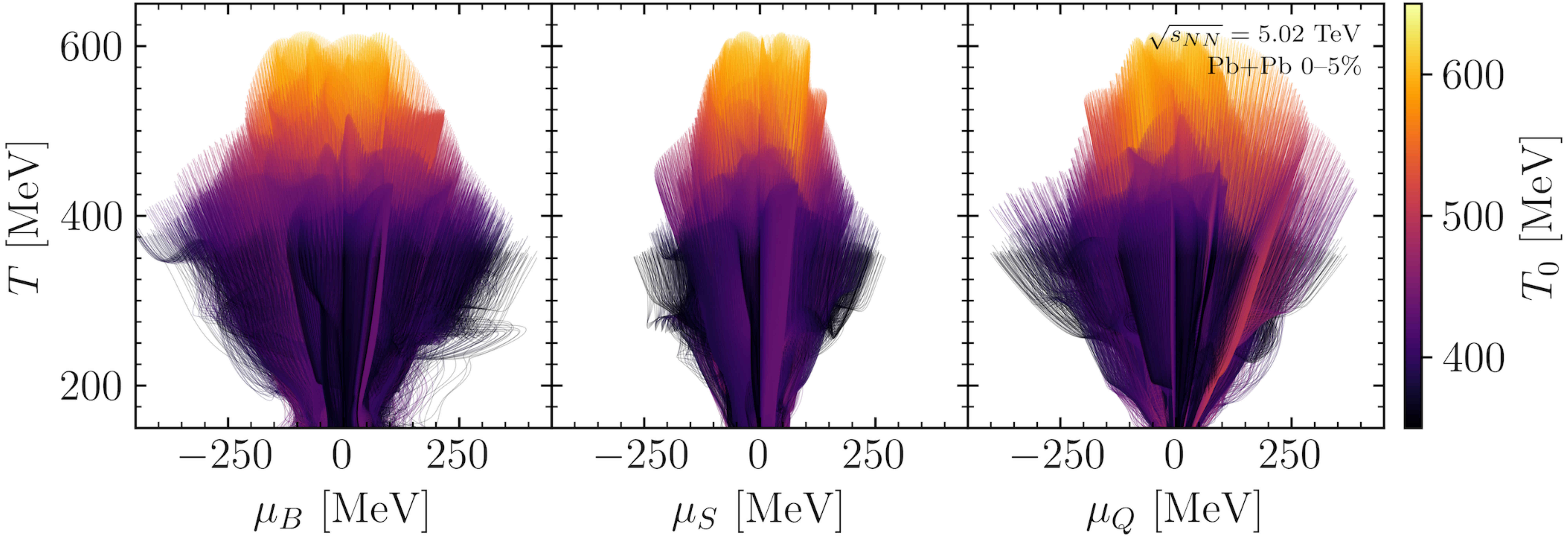
T' expansion coefficients are reshuffling of Taylor terms:

$$\kappa_2^{BB}(T) = \frac{1}{6T}\frac{\chi_4^B(T)}{\chi_2^{B\prime}(T)} \qquad \kappa_4^{BB}(T) = \frac{1}{360\chi_2^{B\prime}(T)^3}\left(3\chi_2^{B\prime}(T)^2\chi_6^B(T) - 5\chi_2^{B\prime\prime}(T)\chi_4^B(T)^2\right)$$

S. Borsanyi et al, PRL (2021)

BQS Fluctuations at the LHC

- Hydrodynamic fluid cell trajectories show the range across which all three conserved charge chemical potentials can fluctuate for PbPb collisions at the LHC



C. Plumberg, D. Almaalol, T. Dore et al, 2405.09648

Outline

I. Lattice QCD-based equations of state

II. A first study of equilibrium proton factorial cumulants with MaxEnt

I. Lattice QCD-based equations of state

Equation of State with Three Conserved Charges

- During HICs the system is not only confined to the T-μ_B plane: determine the equations that depend on μ_B, μ_Q, μ_S

$$\frac{P(T,\mu_B,\mu_Q,\mu_S)}{T^4} = \sum_{i,j,k} \frac{1}{i!j!k!} \chi_{ijk}^{BQS}(T) \left(\frac{\mu_B}{T}\right)^j \left(\frac{\mu_Q}{T}\right)^k \left(\frac{\mu_S}{T}\right)^i$$

where: $$\chi_{ijk}^{BQS}(T) = \left.\frac{\partial^{i+j+k}(p/T^4)}{\partial(\frac{\mu_B}{T})^i \partial(\frac{\mu_Q}{T})^j \partial(\frac{\mu_S}{T})^k}\right|_{\mu_B,\mu_Q,\mu_S=0}$$

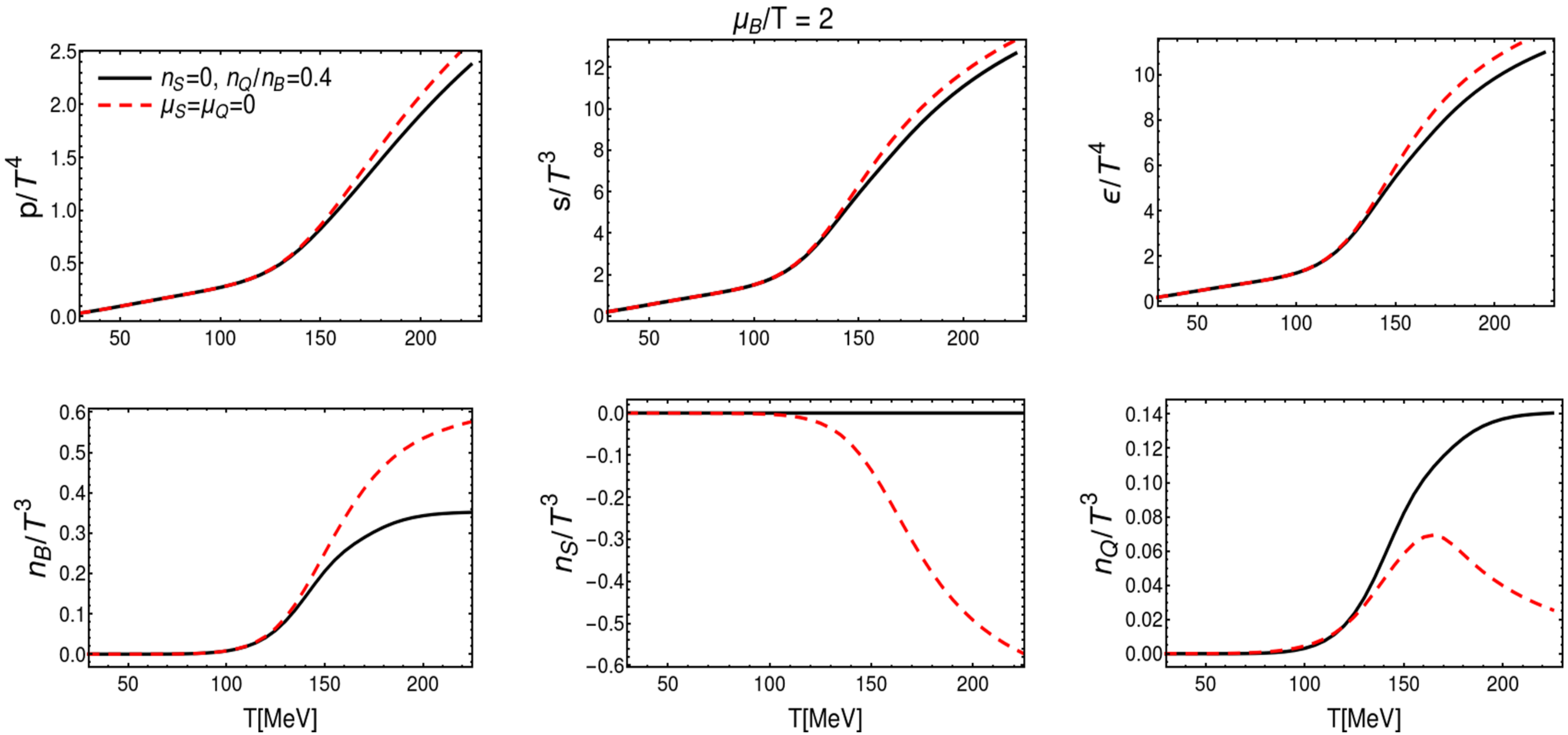
Lattice results only between T ~ 135 - 220 MeV for all 22 coefficients
- Utilize HRG for low T
- Impose Stefan-Boltzmann limit at high T

See also: A. Monnai et al, PRC (2019)

*J. Noronha-Hostler, JS=**JMK** et al, PRC (2019)*

EoS with Conserved Charge Constraints

- ➤ Observe the effect of imposing strangeness neutrality and a fixed ratio of baryon number to electric charge as expected for the densities

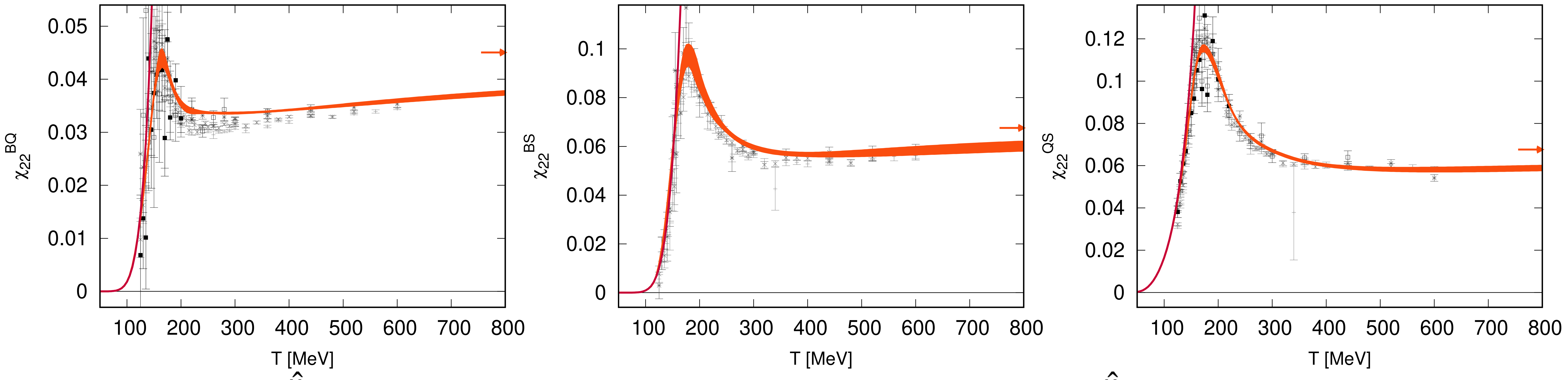


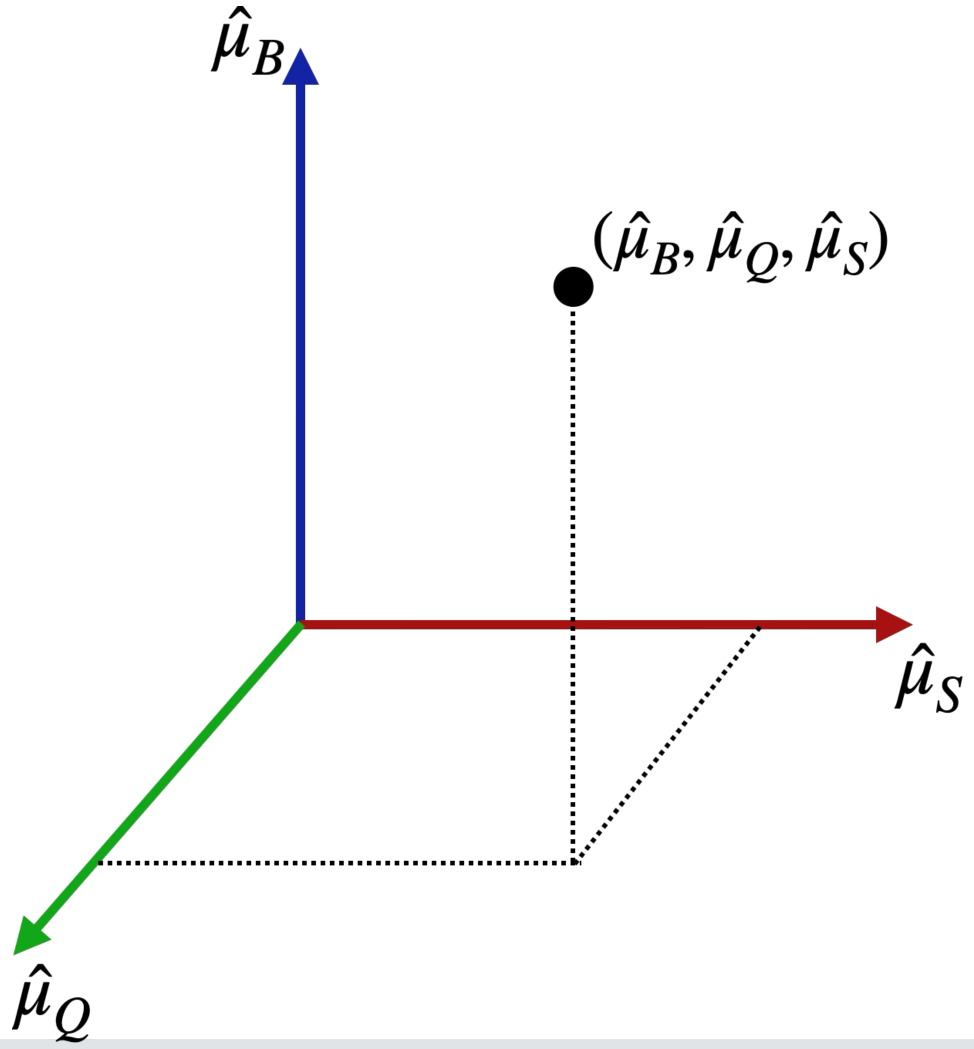
See also: A. Monnai et al, PRC (2019)

*J. Noronha-Hostler, JS=**JMK** et al, PRC (2019)*

New Lattice QCD EoS in T-μ_B-μ_Q-μ_S: 4D TExS

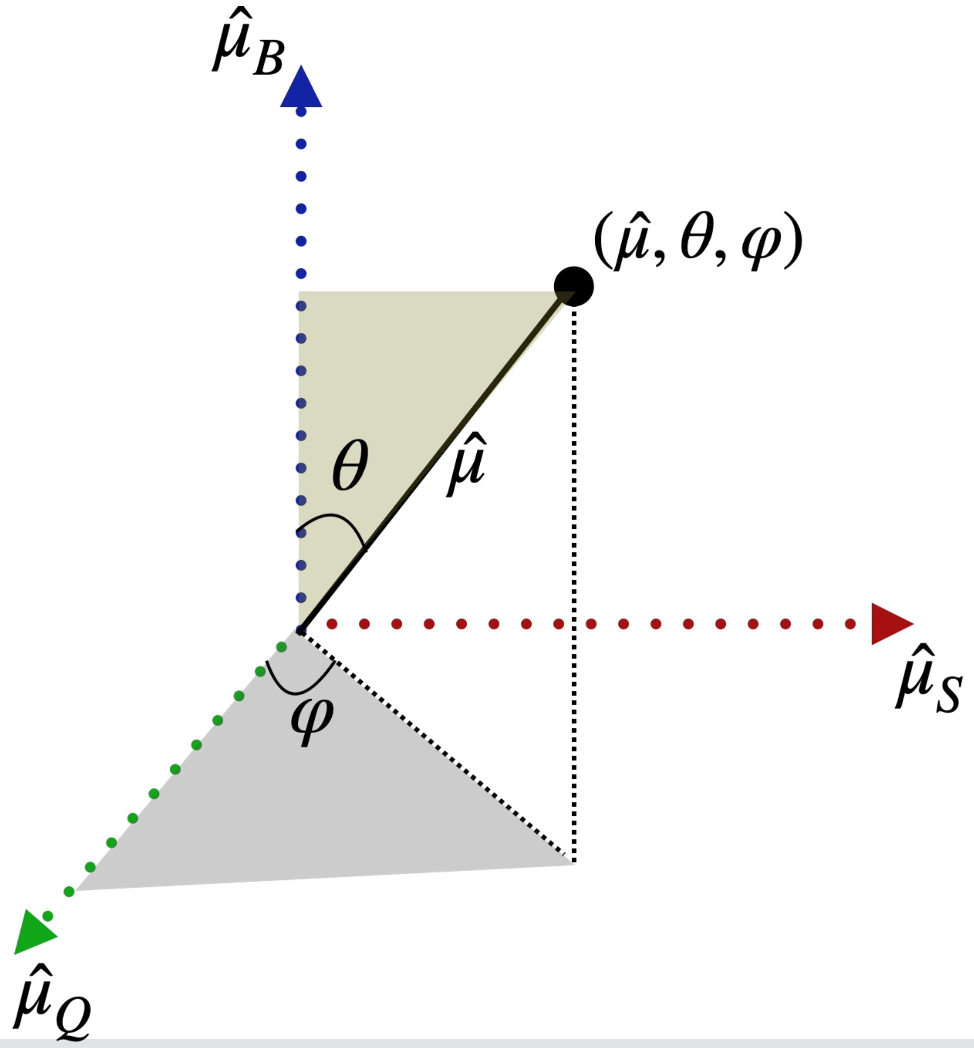
- New continuum estimated lattice results in full 4D phase diagram of temperature and conserved charges by transforming to spherical coordinates





$$\hat{\mu}_B = \hat{\mu}\cos(\theta)$$
$$\hat{\mu}_Q = \hat{\mu}\sin(\theta)\cos(\varphi)$$
$$\hat{\mu}_S = \hat{\mu}\sin(\theta)\sin(\varphi)$$

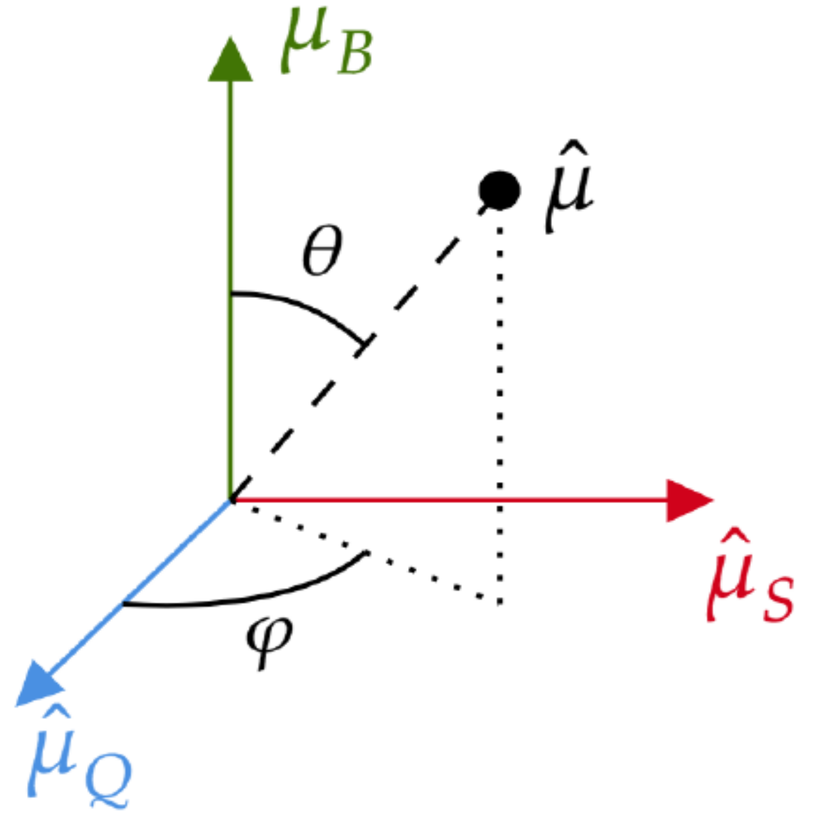
$$\hat{\mu} = \sqrt{\hat{\mu}_B^2 + \hat{\mu}_Q^2 + \hat{\mu}_S^2}$$
$$\theta = \arccos\left(\hat{\mu}_B/\hat{\mu}\right)$$
$$\varphi = \mathrm{sgn}(\hat{\mu}_S)\arccos\left(\hat{\mu}_Q/\sqrt{\hat{\mu}_Q^2 + \hat{\mu}_S^2}\right)$$



A. Abuali et al, PRD (2025)

New Lattice QCD EoS in T-μ_B-μ_Q-μ_S: 4D TExS

- Extended coverage via TExS along lines of constant μ/T at fixed (θ, ϕ) and define the generalized susceptibilities



$$X_2^{\theta,\varphi}(T) = c_\theta^2 \cdot \chi_2^B(T) + s_\theta^2 c_\varphi^2 \cdot \chi_2^Q(T) + s_\theta^2 s_\varphi^2 \cdot \chi_2^S(T) + \dots$$

$$X_4^{\theta,\varphi}(T) = c_\theta^4 \cdot \chi_4^B(T) + s_\theta^4 c_\varphi^4 \cdot \chi_4^Q(T) + s_\theta^4 s_\varphi^4 \cdot \chi_4^S(T) + \dots$$

to determine the expansion with constrained high T by approach to SB limit with coefficient λ:

$$T'_F(T, \hat{\mu}_B) = T\left(1 + \lambda_{2,F}(T)\hat{\mu}_B^2 + \dots\right)$$

$$\lambda_2^{\theta,\varphi}(T) = \frac{1}{6T}\frac{1}{X_2'^{\theta,\varphi}(T)} \times \left(X_4^{\theta,\varphi}(T) - \frac{\overline{X}_4^{\theta,\varphi}(0)}{\overline{X}_2^{\theta,\varphi}(0)} X_2^{\theta,\varphi}(T)\right)$$

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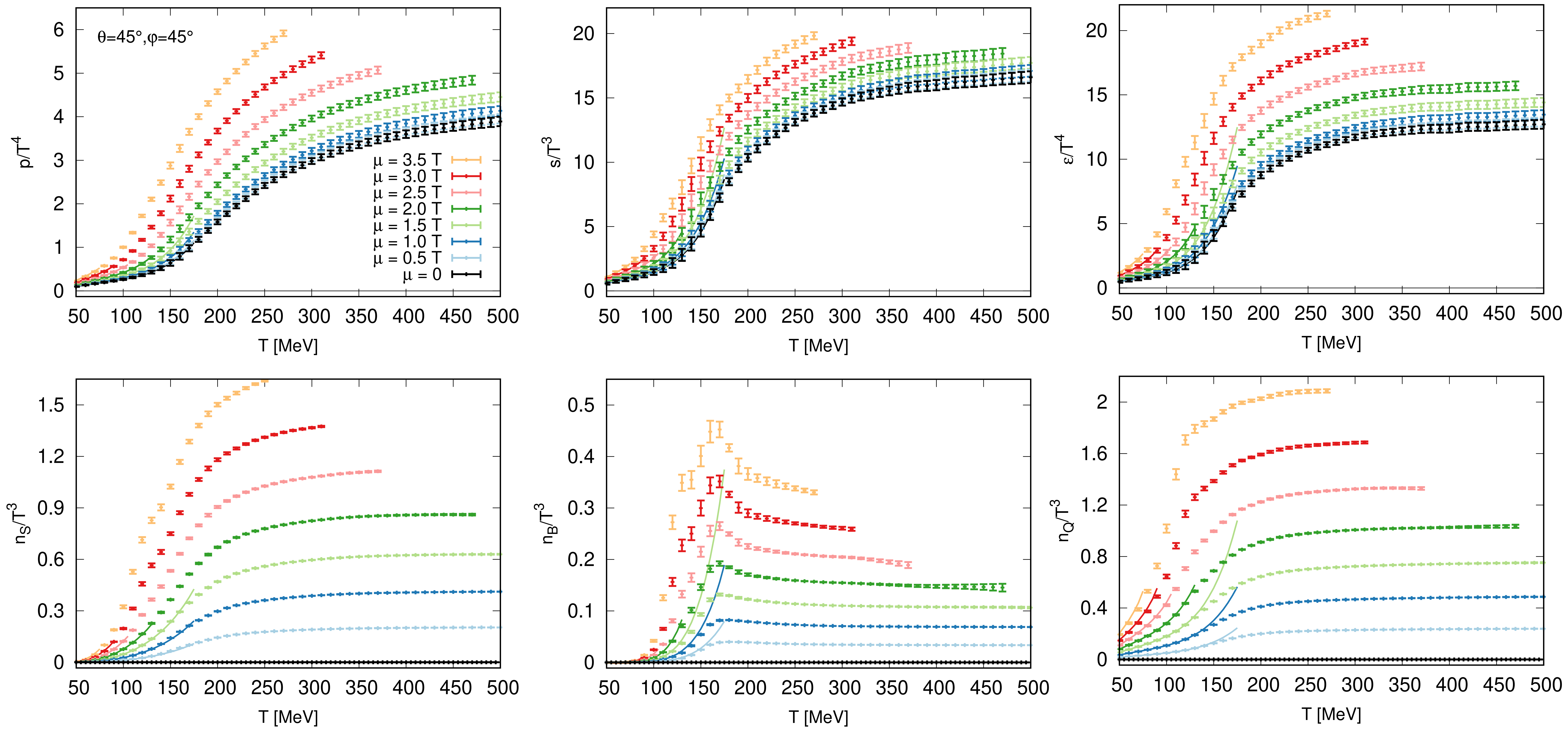

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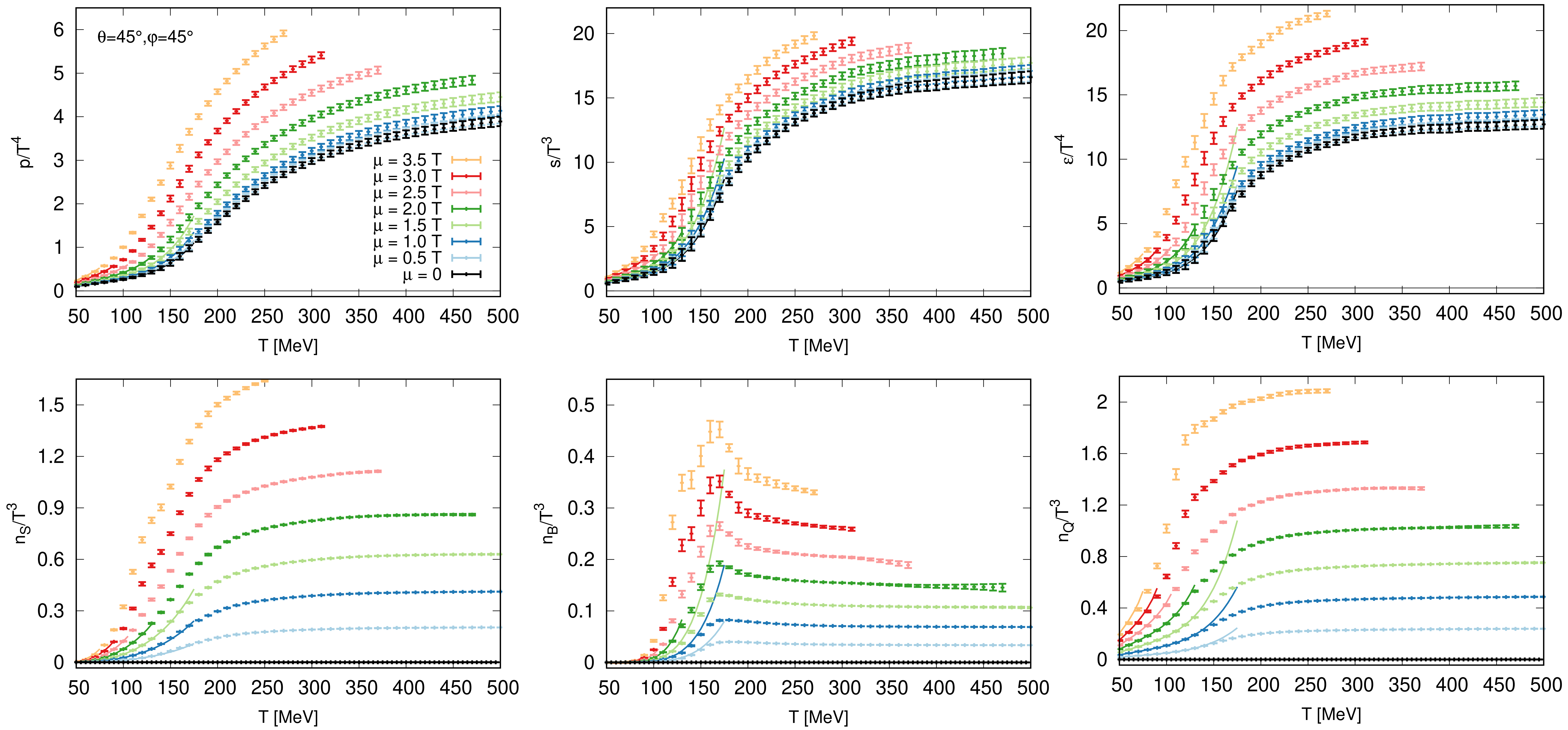

What about the critical point?

A. Abuali et al, PRD (2025)

BEST Critical EoS

- Original work from the BEST collaboration to implement critical features into EoS for heavy-ion collisions:

$$P(\mu, T) = P^{\mathrm{reg}}(\mu, T) + P^{\mathrm{sing}}(\mu, T)$$

Nonaka & Asakawa, PRC (2005)
P. Parotto et al, PRC (2020)

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Matched to lattice QCD Taylor coefficients

3D Ising model

$$T^4 c_n^{\text{LAT}}(T) =$$
$$T^4 c_n^{\text{Non-Ising}}(T) + c_n^{\text{Ising}}(T)$$

P. Parotto et al, PRC (2020)

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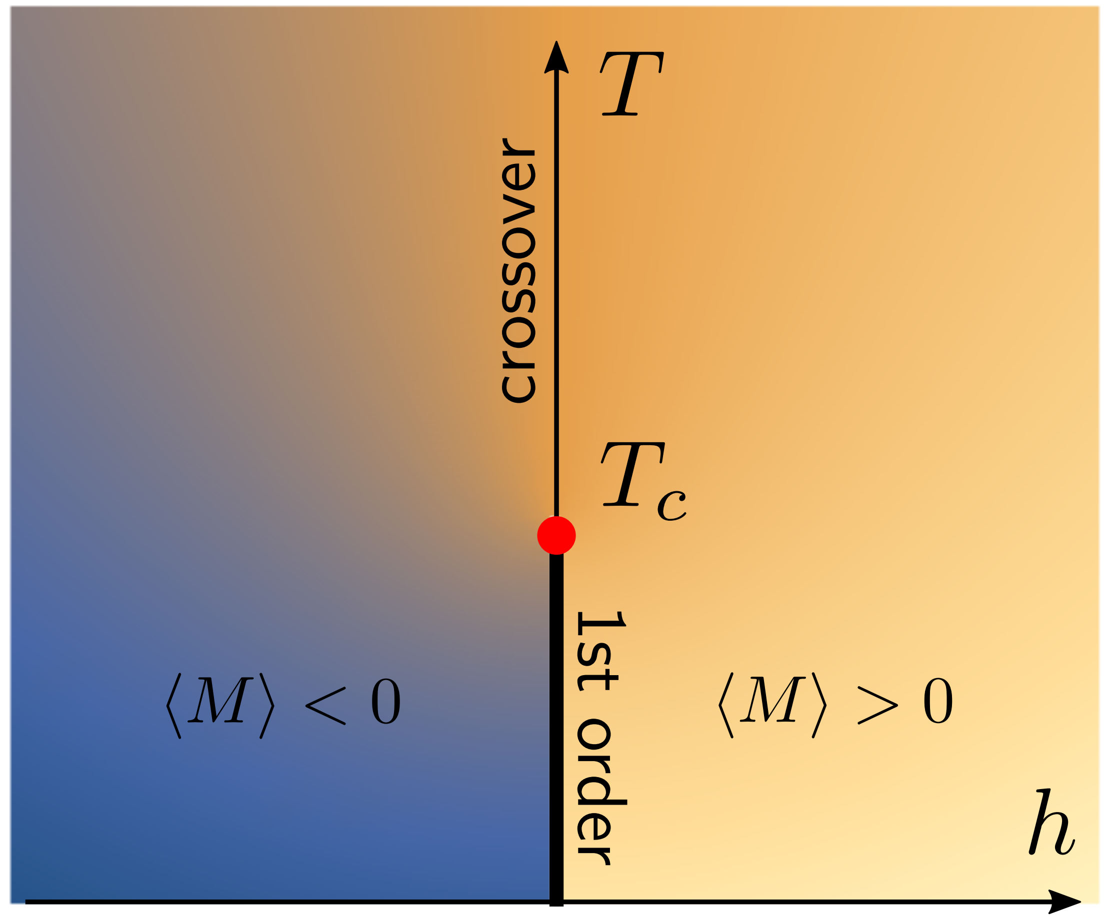

$$h = h_0 R^{\beta\delta} H(\theta)$$
$$r = \frac{T - T_c}{T_c} = R(1 - \theta^2)$$
$$G = h_0 M_0 R^{2-\alpha}[g(\theta) - \theta H(\theta)] = -P$$

P. Parotto et al, PRC (2020)
Fig: A. Bzdak et al, Phys.Rept. (2020)

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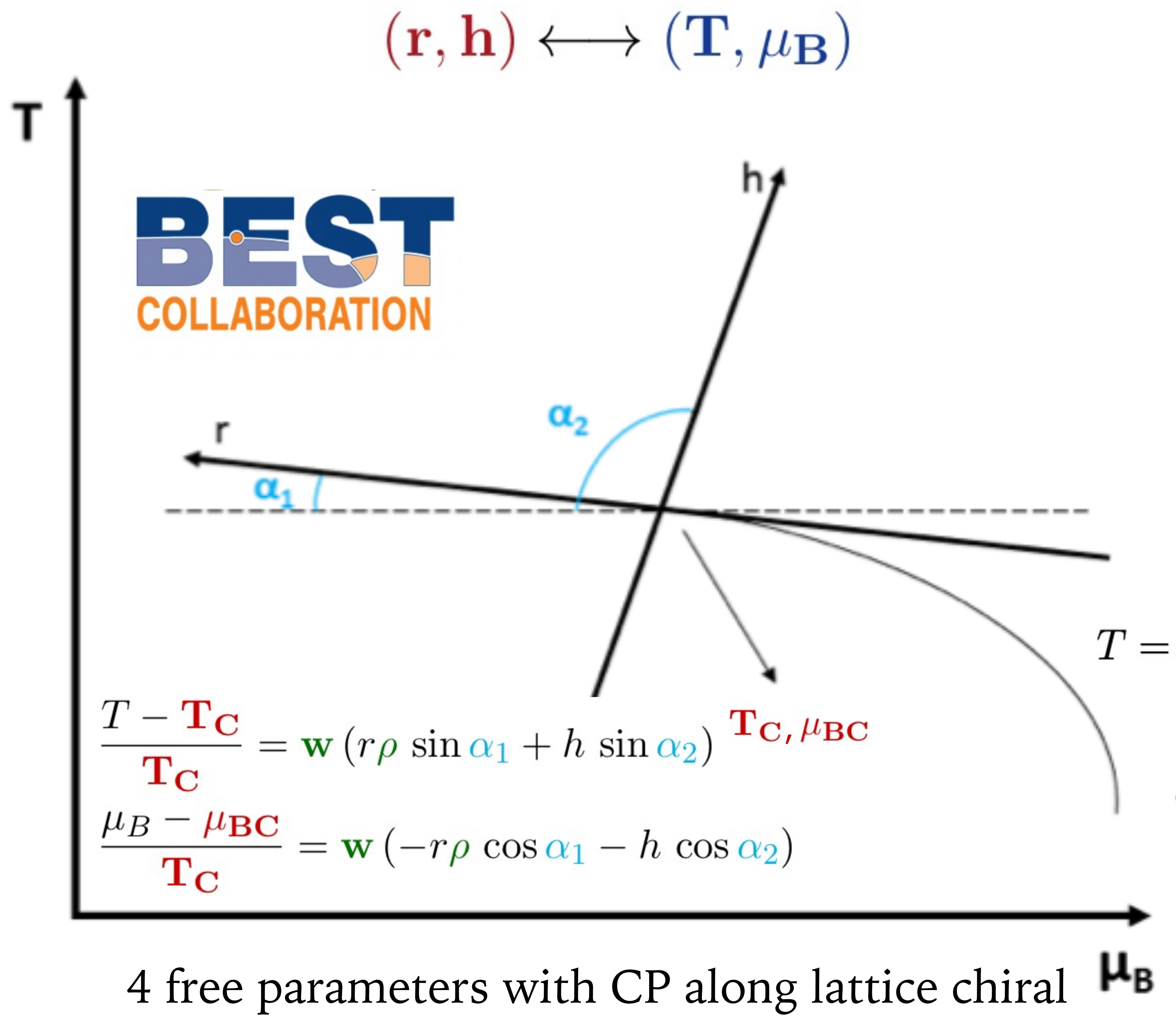
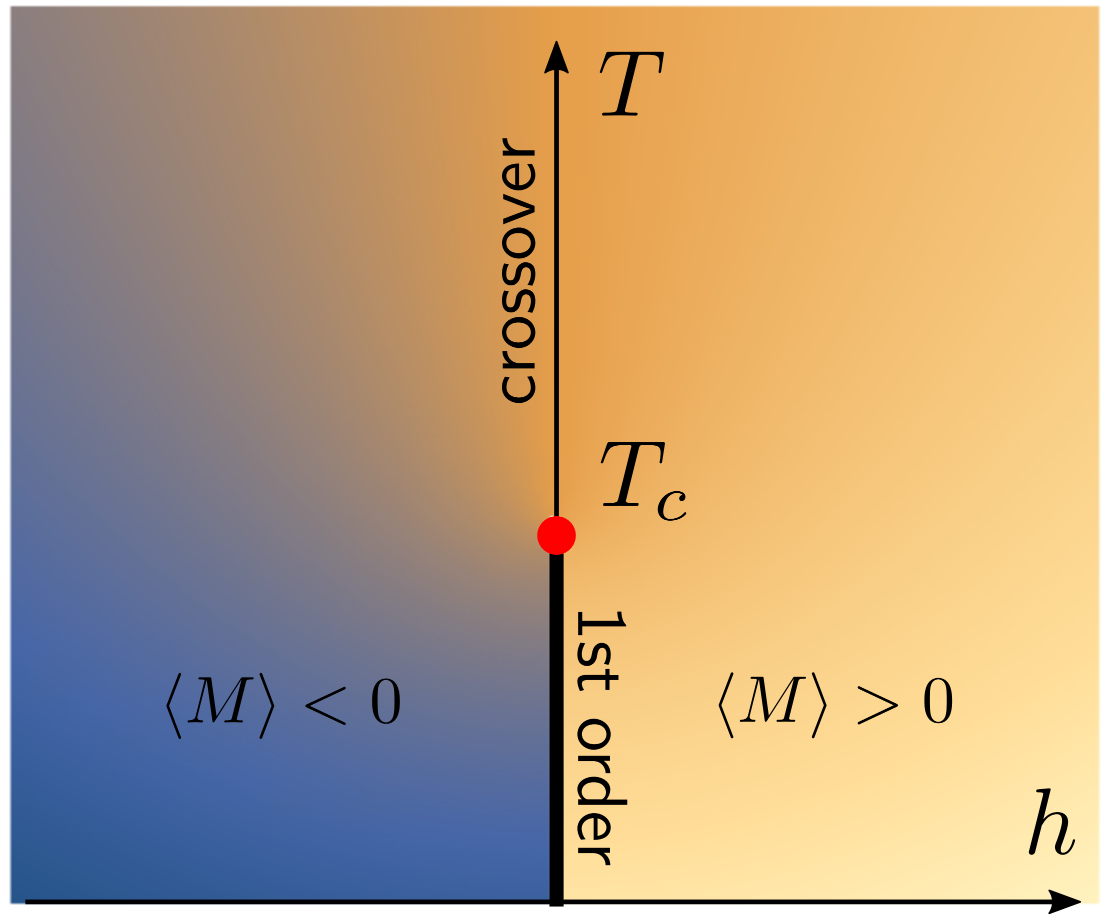


4 free parameters with CP along lattice chiral PT line

$$T^4 c_n^{\mathrm{LAT}}(T) =$$
$$T^4 c_n^{\mathrm{Non-Ising}}(T) + c_n^{\mathrm{Ising}}(T)$$

$$T = T_0 + \kappa T_0 \left(\frac{\mu_B}{T_0}\right)^2 + O(\mu_B^4)$$

$$\alpha_1 = \tan^{-1}\left(2\frac{\kappa}{T_0}\mu_{BC}\right)$$



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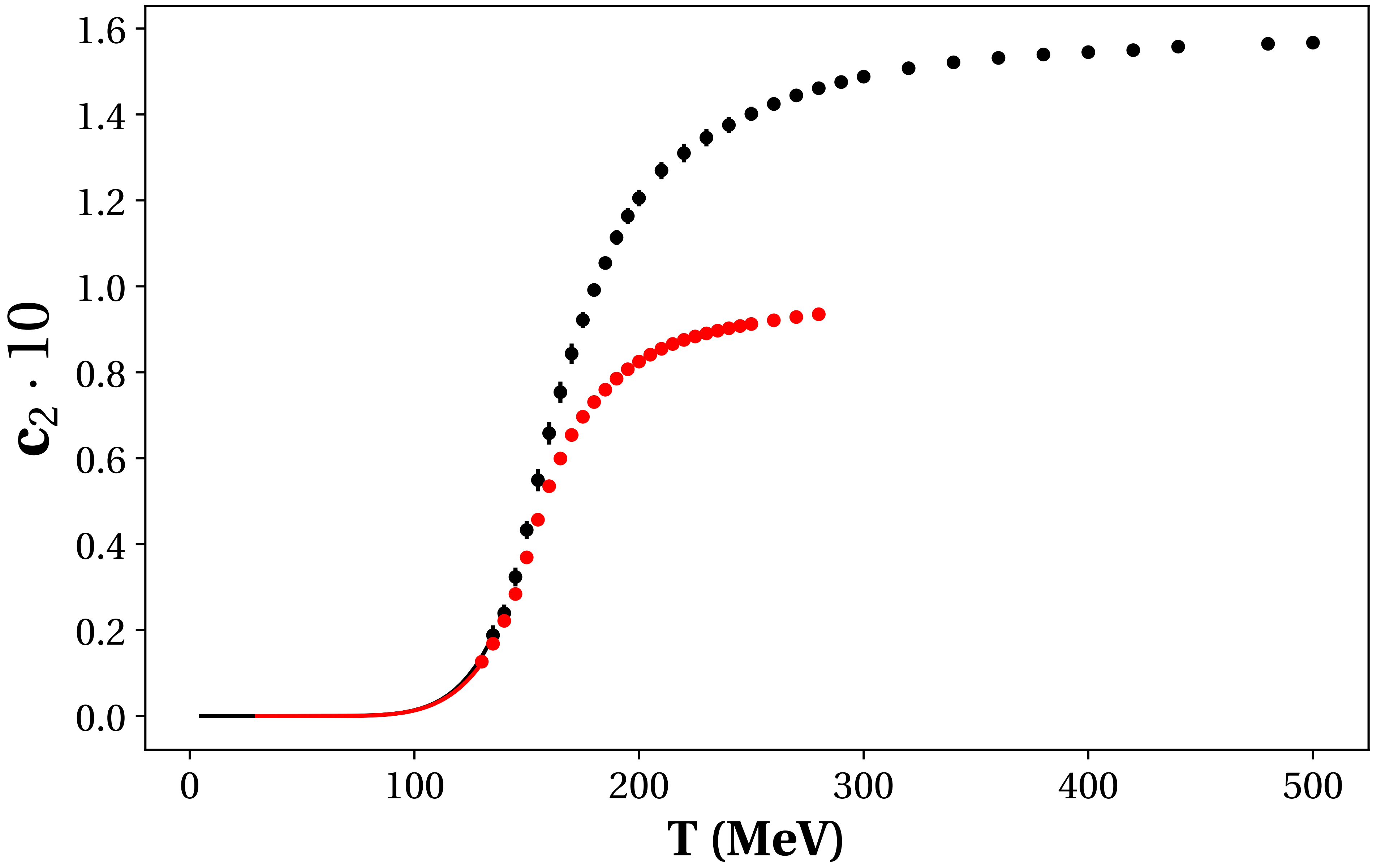
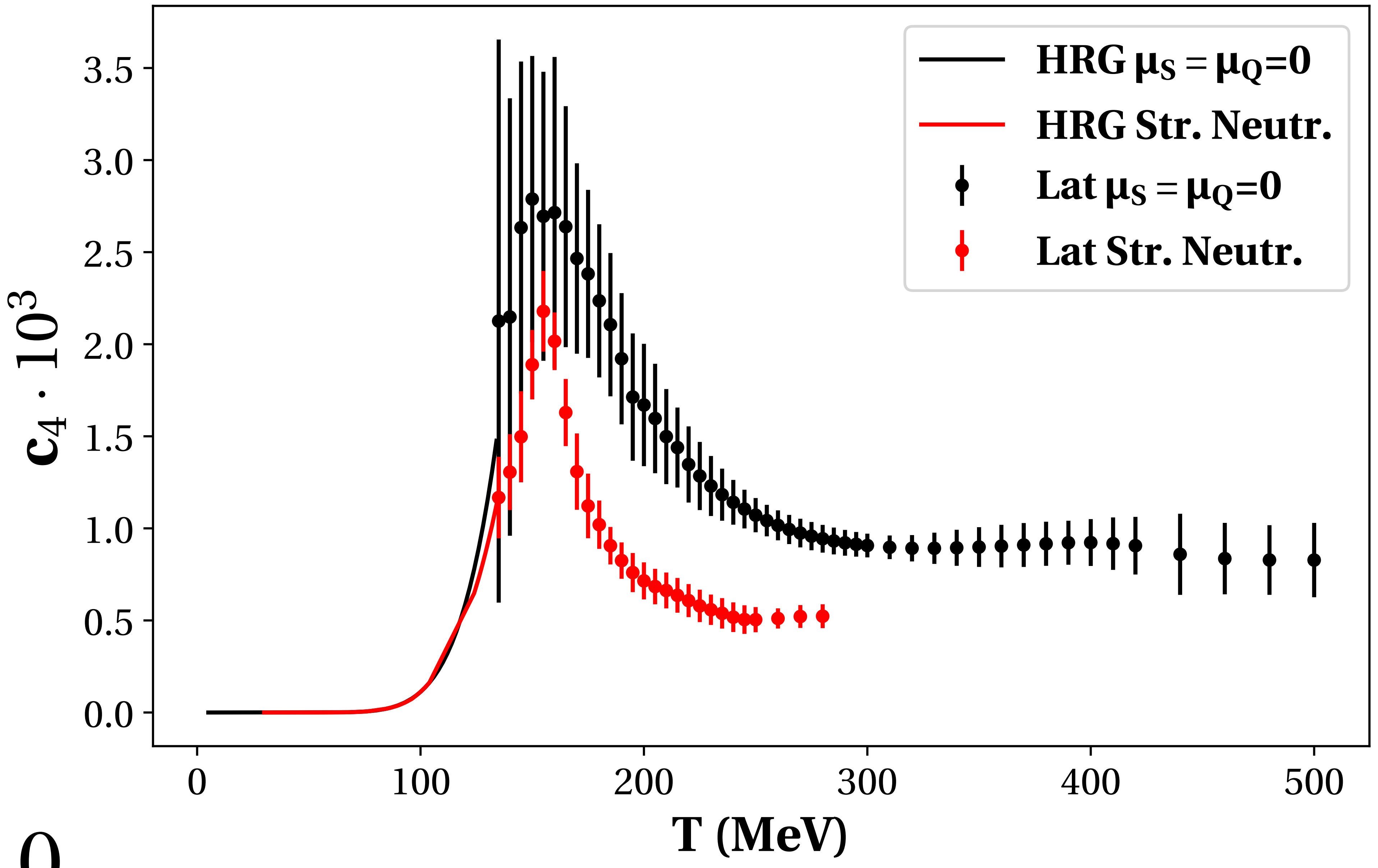
P. Parotto et al, PRC (2020)
Fig: A. Bzdak et al, Phys.Rept. (2020)

Taylor Coefficients from LQCD

- Lattice results for Taylor expansion of pressure around $\mu_B = 0$ up to $\mathcal{O}(\mu_B^4)$ are the backbone of the procedure for creating this equation of state

$$\frac{P(T,\mu_B)}{T^4} = \sum_n c_{2n}(T)\left(\frac{\mu_B}{T}\right)^{2n}$$





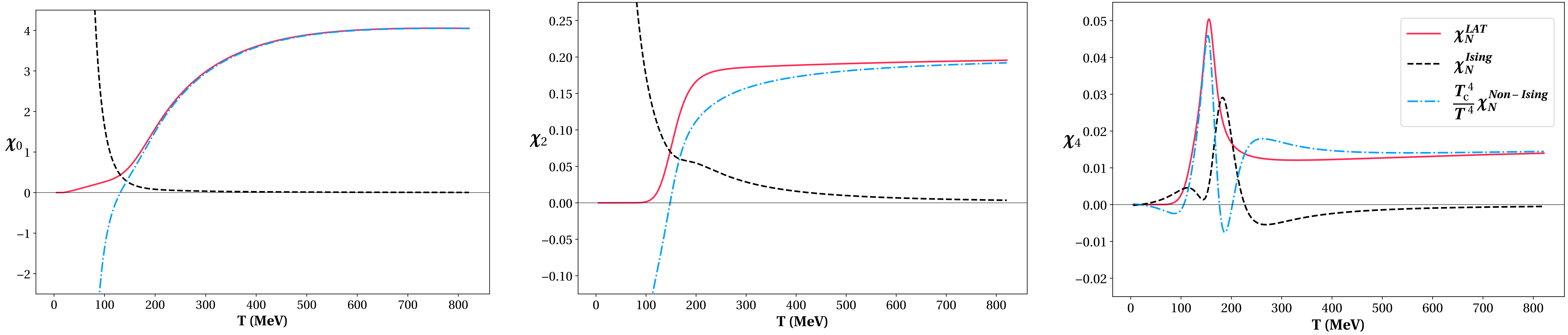
- $\mu_Q = \mu_S = 0$
- strangeness-neutral

JMK et al, EPJ+ (2021)
J. Guenther et al, NPA (2017)
R. Bellweid et al, PRD (2015)
See also: A. Bazavov et al, PRD (2017)

Singular and Non-singular Contributions

- Require that the total free energy (pressure) is the one from the lattice, so order-by-order we have:

$$\chi_N^{Lat}(T) = \chi_N^{Ising}(T) + \chi_N^{Non-Ising}(T)$$



$$P(T,\mu_B) = T^4 \sum_{n=0}^{2} c_{2n}^{\text{non-Ising}}(T)\left(\frac{\mu_B}{T}\right)^{2n} + T_C^4\, P_{\text{symm}}^{\text{Ising}}(T,\mu_B)$$

JMK et al, EPJ+ (2021)

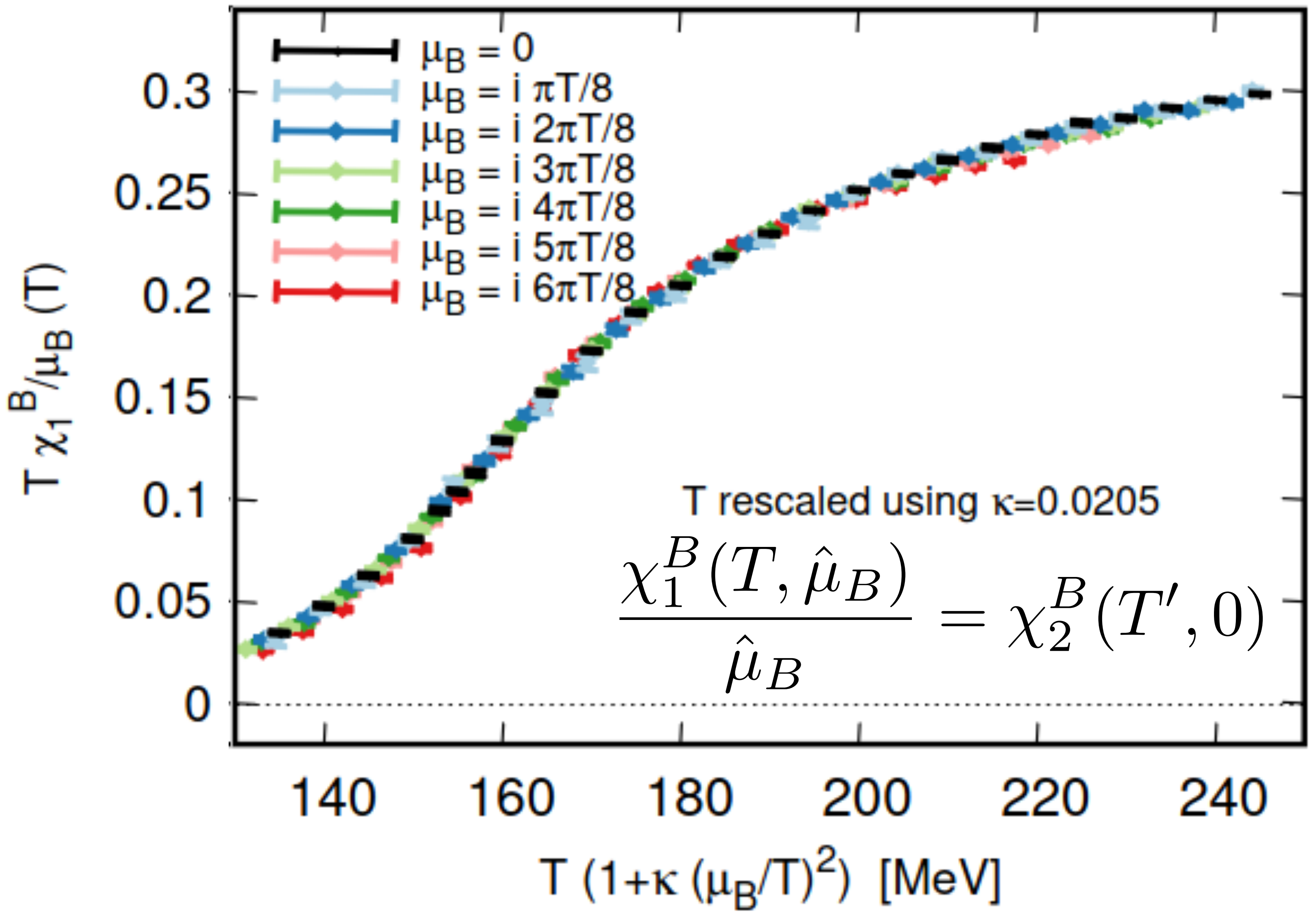
Ising TExS EoS

- Recent development of critical EoS to extend coverage to $\mu_B \simeq 700$ MeV in BES-II range: Ising TExS EoS is successor to BEST EoS
  - Utilize lattice results from T'-Expansion Scheme (TExS) and incorporate singularity in n_B via a singularity in T'

$$T'(T,\mu_B) = \underbrace{T'_{\text{lat}}(T,\mu_B)}_{\text{lowest orders in } (\mu_B/T)} + \underbrace{T'_{\text{crit}}(T,\mu_B) - \text{Taylor}_{n\leq 2}[T'_{\text{crit}}(T,\mu_B)]}_{\text{higher orders in } (\mu_B/T)}$$

which has the same singularity as T'_{crit} and the same truncated Taylor expansion as T'_{lat}

$T \to T'$



M. Kahangirwe et al, PRD (2024)

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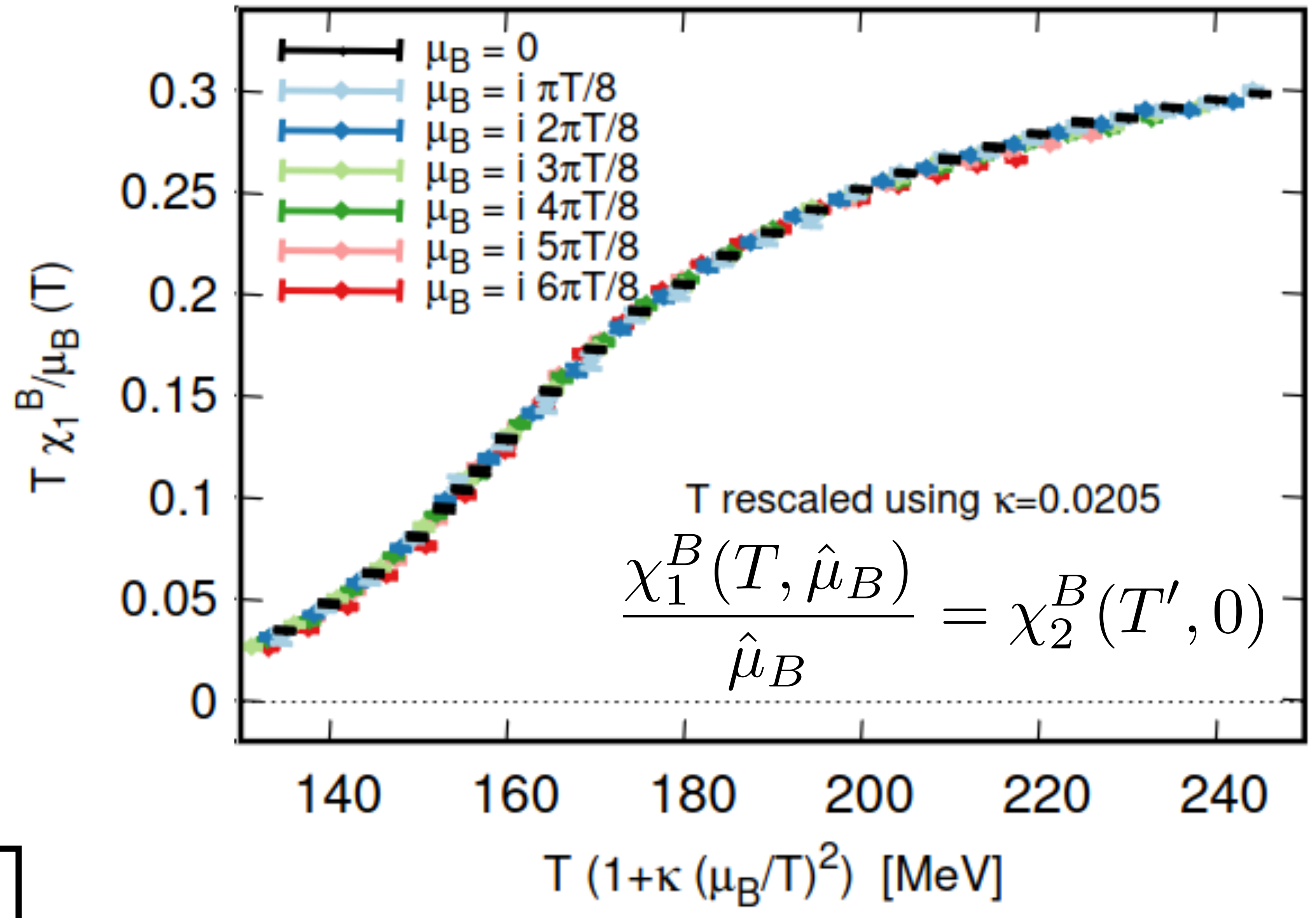
which has the same singularity as $T'_{\rm crit}$ and the same truncated Taylor expansion as $T'_{\rm lat}$

Incorporate critical physics into higher orders in μ_B by subtracting the 2nd order Taylor term

$T \to T'$



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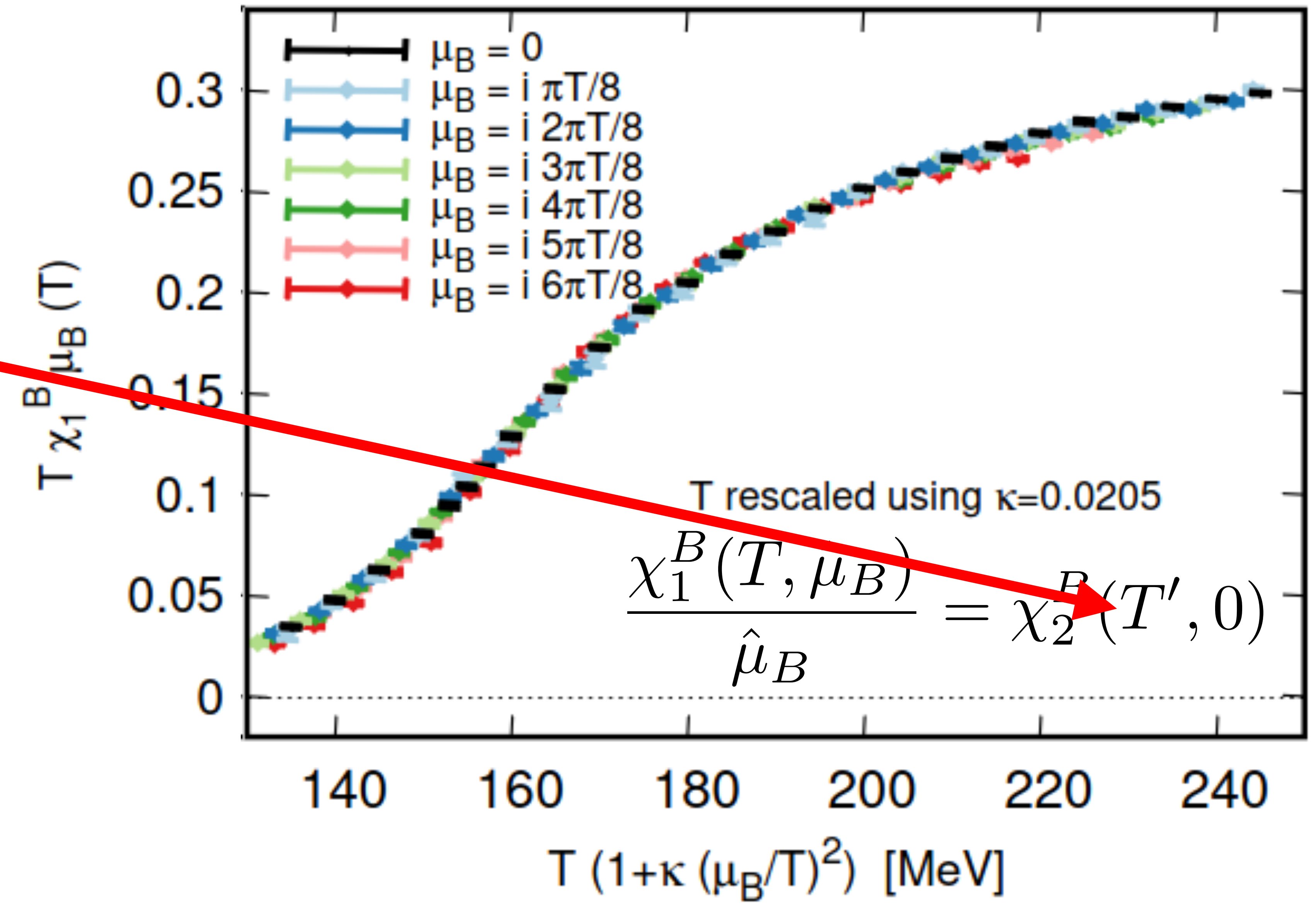


T' completely defines the baryon density with a critical point → converges faster

M. Kahangirwe et al, PRD (2024)

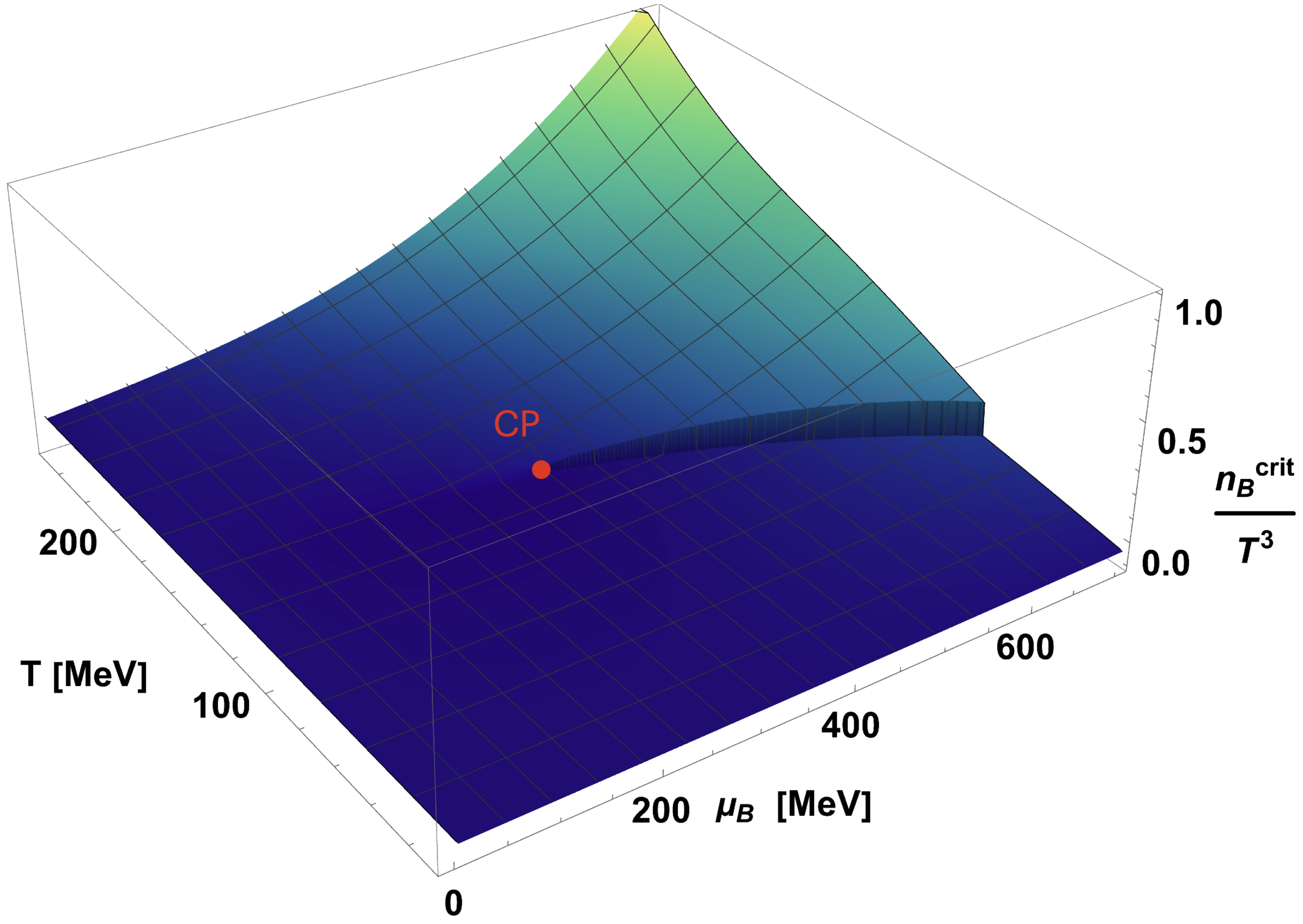
Ising TExS EoS

- Recent development of critical EoS to extend coverage to $\mu_B \simeq 700$ MeV in BES-II range: Ising TExS EoS is successor to BEST EoS
  - Map Ising transition line quadratically along chiral phase transition rather than tangent to the critical point with a linear mapping

$$(\mathbf{r},\mathbf{h}) \longleftrightarrow (\mathbf{T},\mu_\mathbf{B}) : \quad h(\mu,T) = -\frac{\Delta T' \cos\alpha_1}{T_c w \, \sin(\alpha_1-\alpha_2)}$$

$$r(\mu,T) = -\frac{\mu^2-\mu_c^2}{2\mu_c T_c \, \rho w \, \cos\alpha_1} + \frac{\Delta T' \cos\alpha_2}{T_c \, \rho w \, \sin(\alpha_1-\alpha_2)}$$

$$\Delta T'(\mu,T) \equiv \frac{T'(\mu,T)-T_0}{(\partial T'/\partial T)_{\rm cp}}$$



New quadratic mapping puts Ising phase transition line directly on chiral crossover from lattice

M. Kahangirwe et al, PRD (2024)

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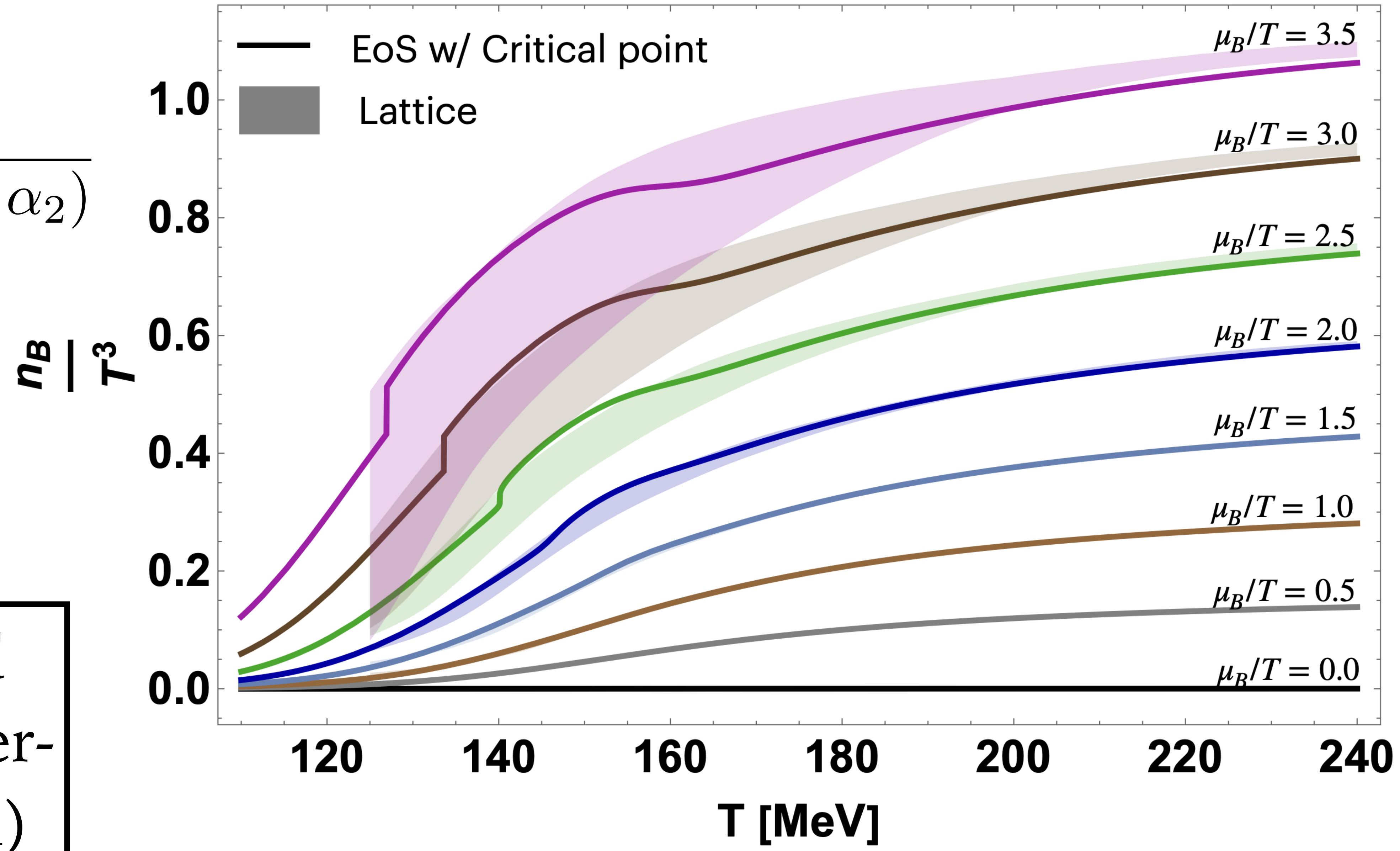
$$\Delta T'(\mu, T) \equiv \frac{T'(\mu, T) - T_0}{(\partial T'/\partial T)_{\mathrm{cp}}}$$

Full EoS agrees with lattice and includes critical features with user-chosen parameters (e.g. location)



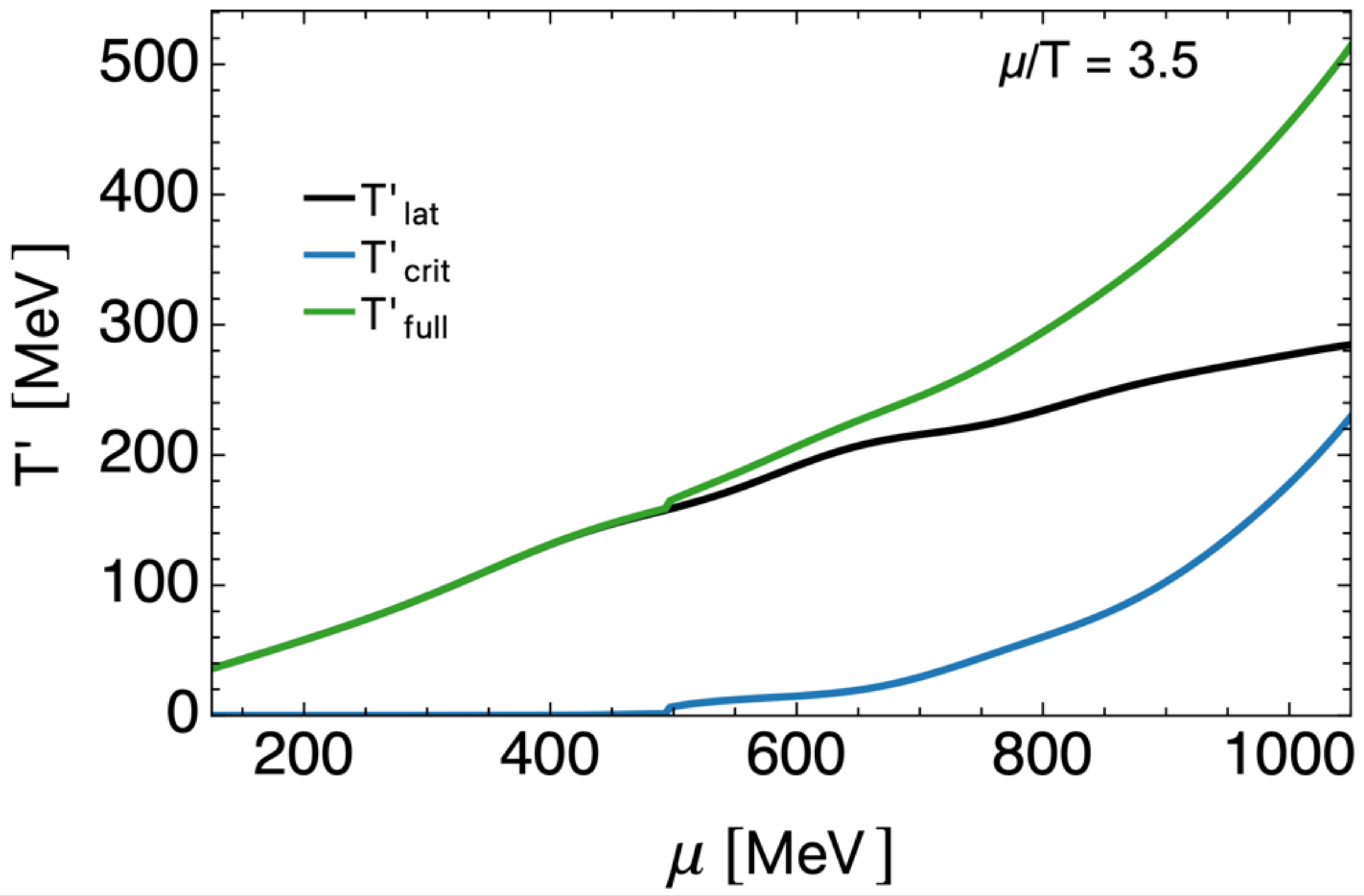
M. Kahangirwe et al, PRD (2024)

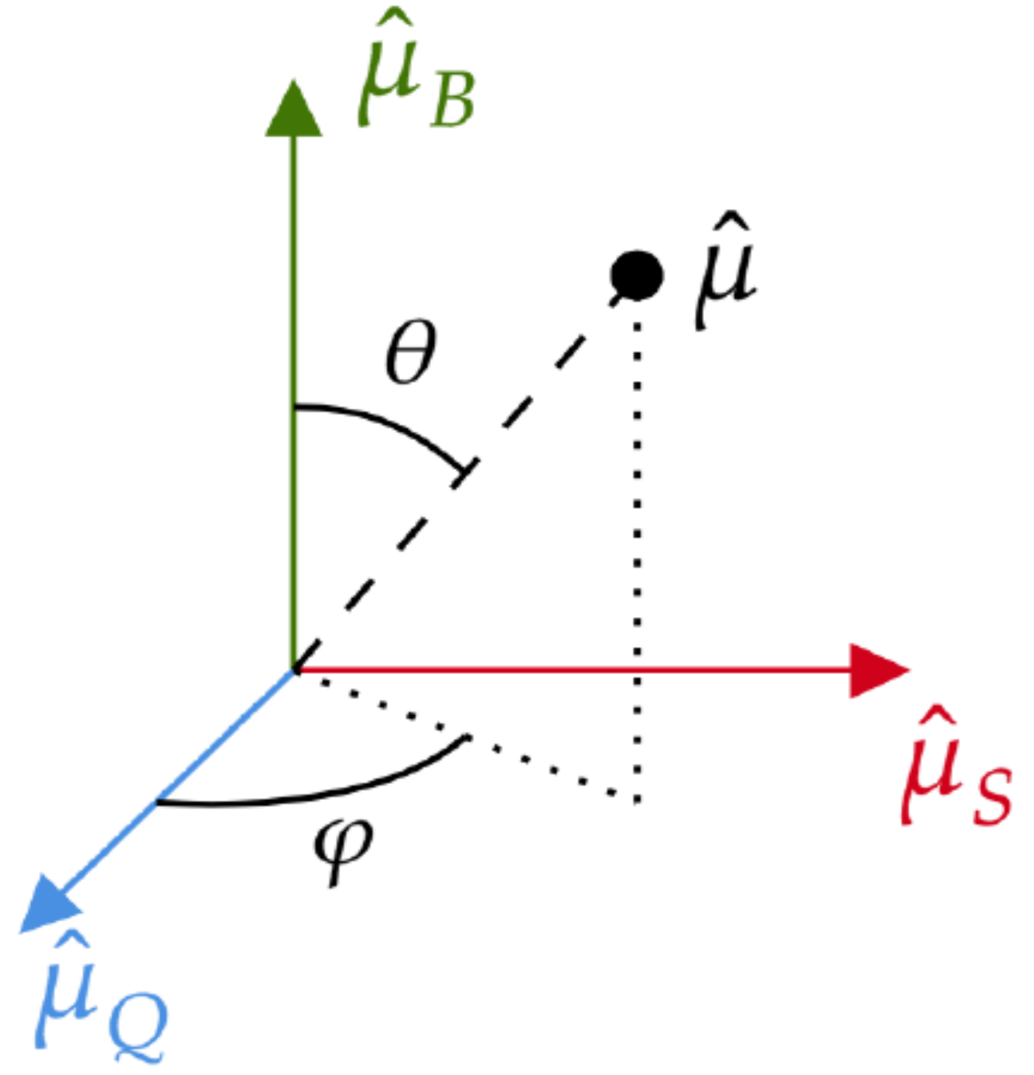
Critical EoS in T-μ_B-μ_Q-μ_S: 4D Ising TExS

- Extend to all relevant conserved charges based on the Ising TExS critical EoS approach and the 4D TExS EoS in spherical coordinates:

$$T'_{\rm full}(T,\hat{\mu},\theta,\varphi) = T'_{\rm lat}(T,\hat{\mu},\theta,\varphi) + T'_{\rm crit}(T,\hat{\mu},\theta,\varphi)$$

$$T'_{\rm crit}(T,\hat{\mu},\theta,\varphi) = \left(\left.\frac{\partial X_2}{\partial T}\right|_{T_0} \right)^{-1} \cdot \frac{n_{\rm Ising}(T,\hat{\mu},\theta,\varphi)}{\hat{\mu}} - T'^{\,\mathcal{O}\leq 2}_{\rm Taylor}$$





Critical EoS in T-μ_B-μ_Q-μ_S: 4D Ising TExS

- Adding additional dimensions in conserved charges promotes the critical point to a critical surface: first consider two chemical potentials - a critical ellipse

Parametrize the location of the critical ellipse by:

- $\{\tau, \mu_{B,c}, \mu_{S,c}\}$
- $\{\mu_c, d\mu/d\theta, d^2\mu/d\theta^2\}$ along strangeness neutrality

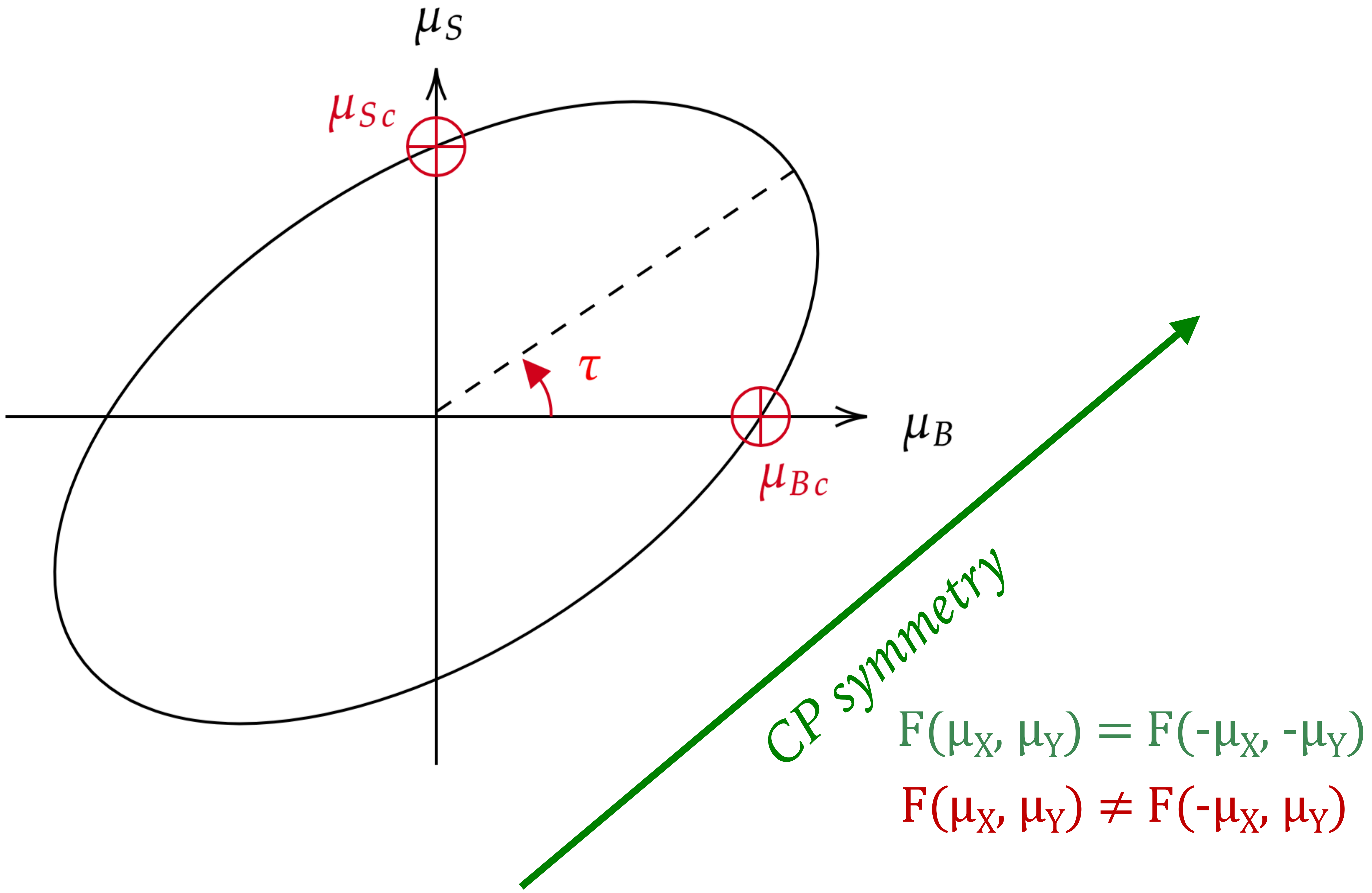


Figure adapted from F. DiClemente CPOD 2026

Critical EoS in T-μ_B-μ_Q-μ_S: 4D Ising TExS

- Adding additional dimensions in conserved charges promotes the critical point to a critical surface: first consider two chemical potentials - a critical ellipse

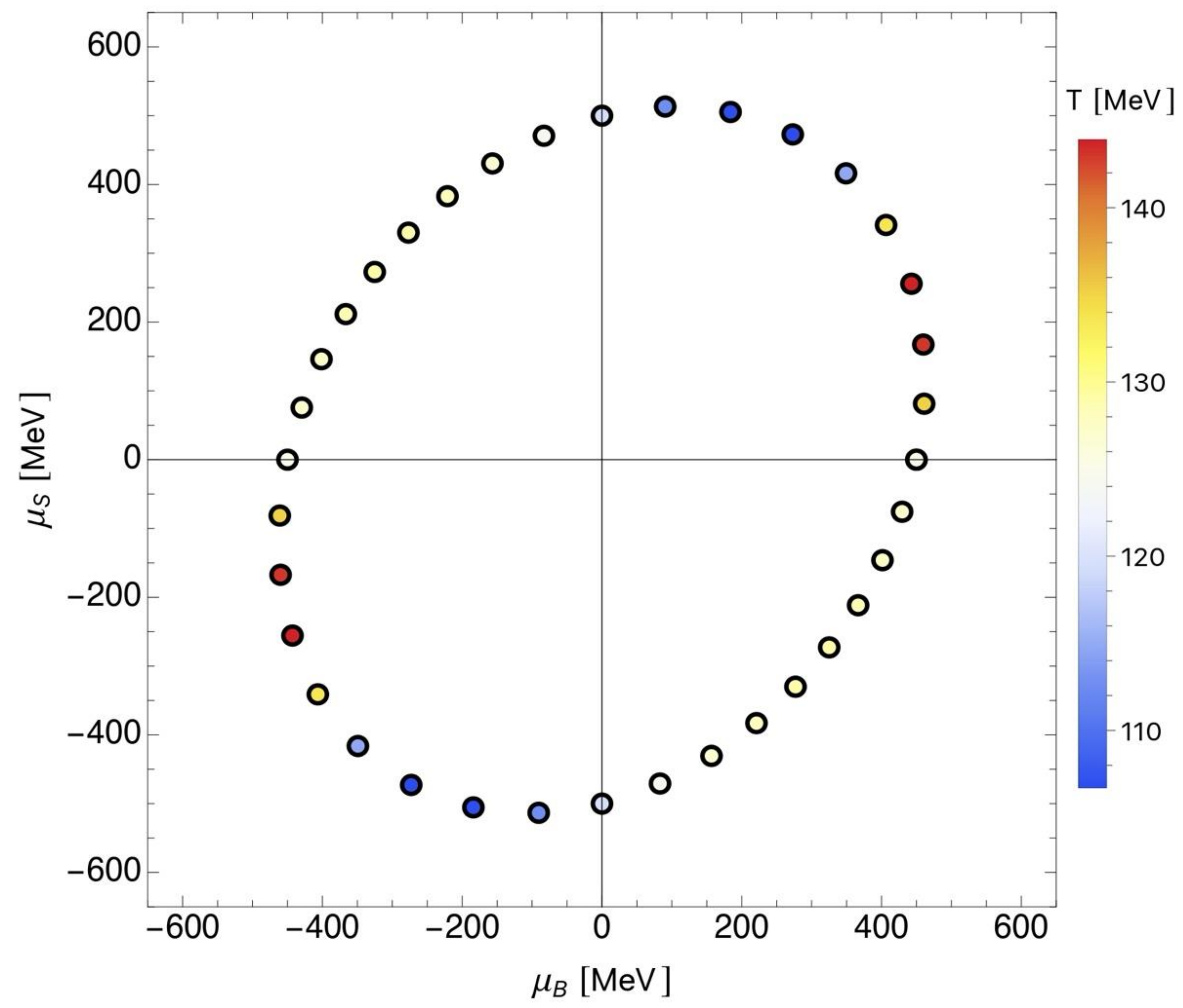
An example critical ellipse:

$\tau = 57°, \mu_{B,c} = 450$ MeV, $\mu_{S,c} = 500$ MeV

Temperature constrained by $\Delta T' = 0$

$$\Delta T'(\mu, T) \equiv \frac{T'(\mu, T) - T_0}{(\partial T'/\partial T)_{\rm cp}}$$

$$T_0 = T_c \cdot \left[1 + \lambda_2(T_c) \cdot \hat{\mu}_c^2\right] \longrightarrow \hat{\mu}_c = \sqrt{\frac{T_0/T_c - 1}{\lambda_2(T_c)}}$$



Critical EoS in T-μ_B-μ_Q-μ_S: 4D Ising TExS

- Preliminary results given an example choice of parameters and trajectory in hyperplane

$\mu_S = \mu_B/3$

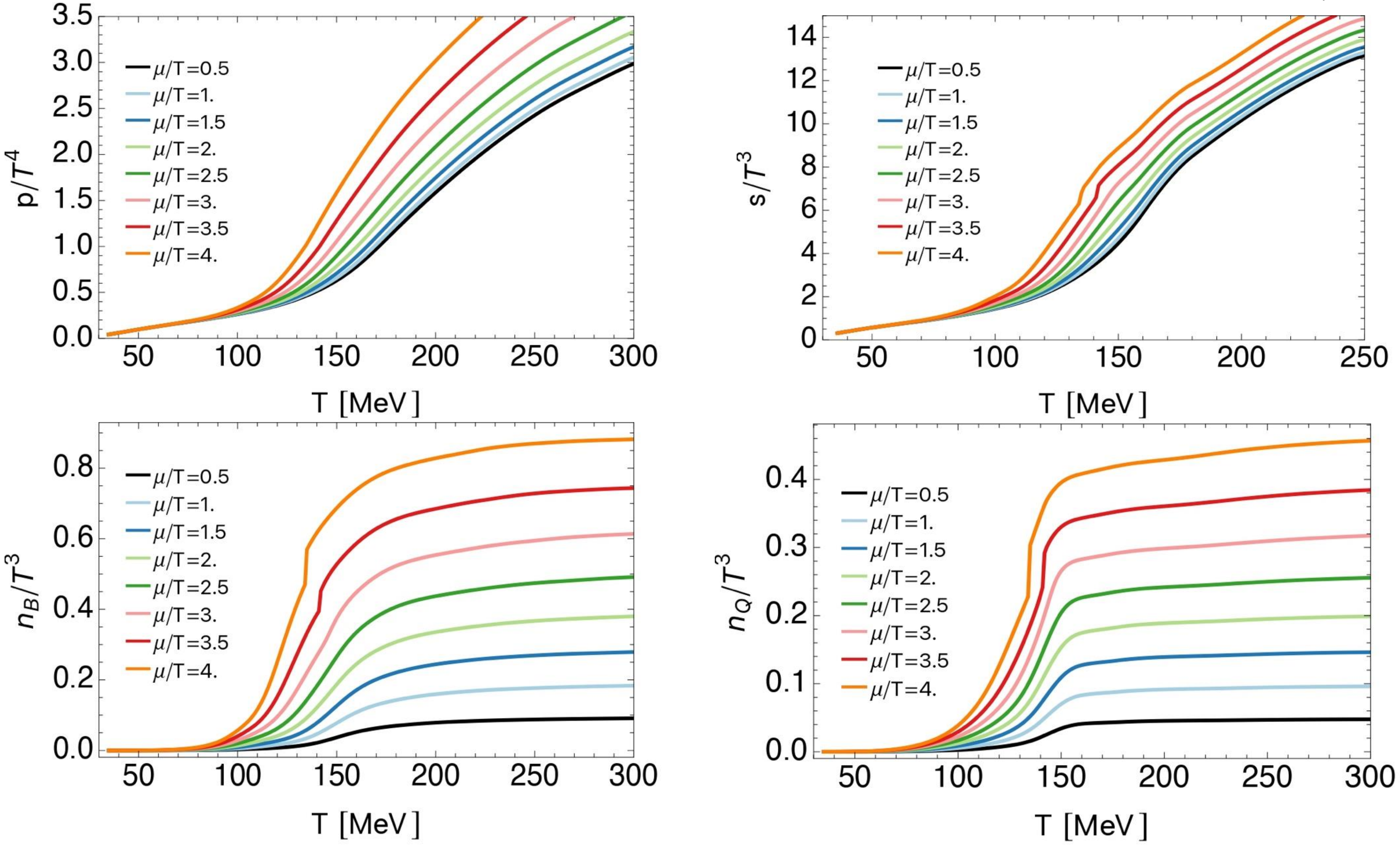
$\mu_{B,c} = 450$ MeV

$\mu_{S,c} = 500$ MeV

$\alpha_{12} = \alpha_1 - \alpha_2 = 14.6°$

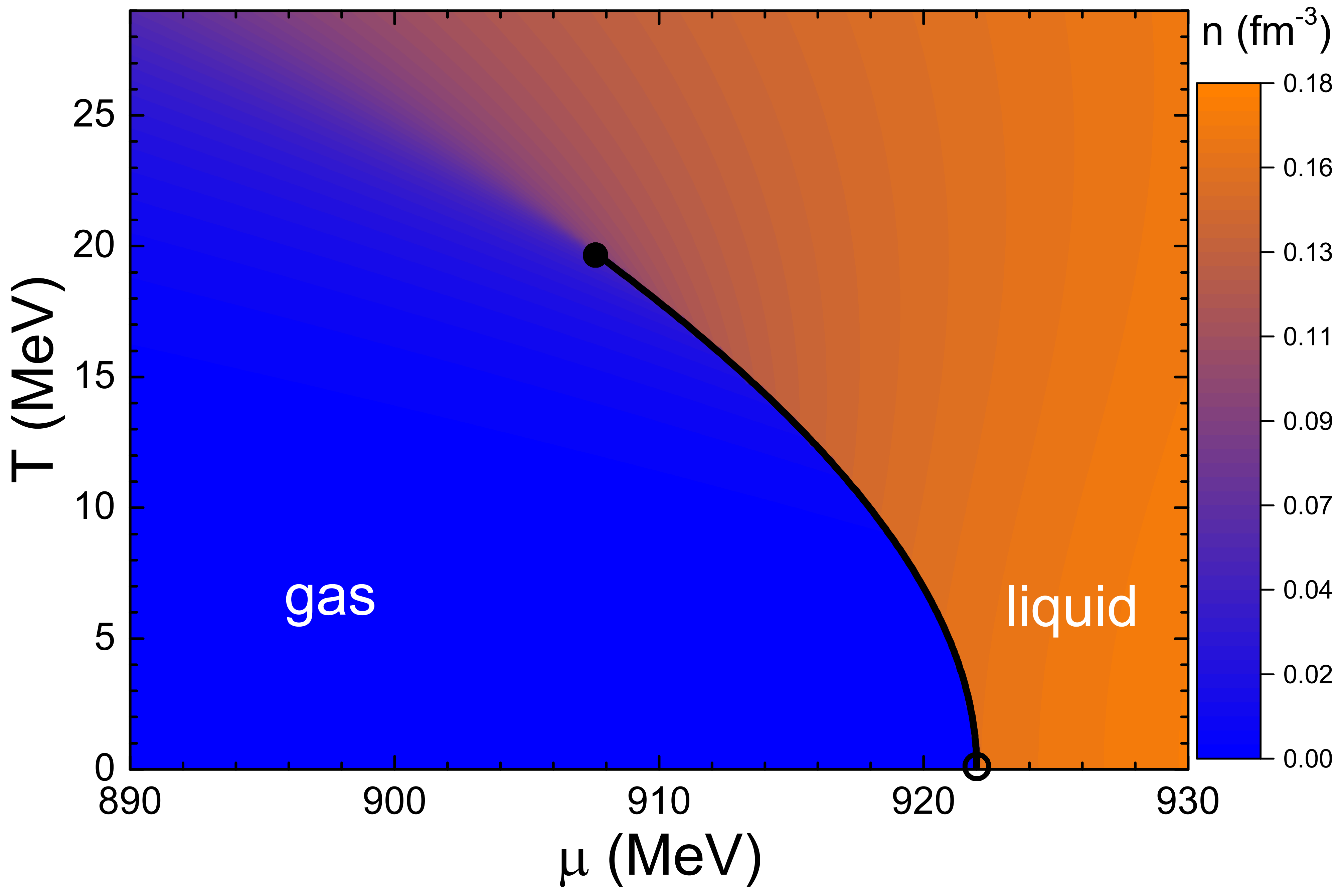
$w = 5$

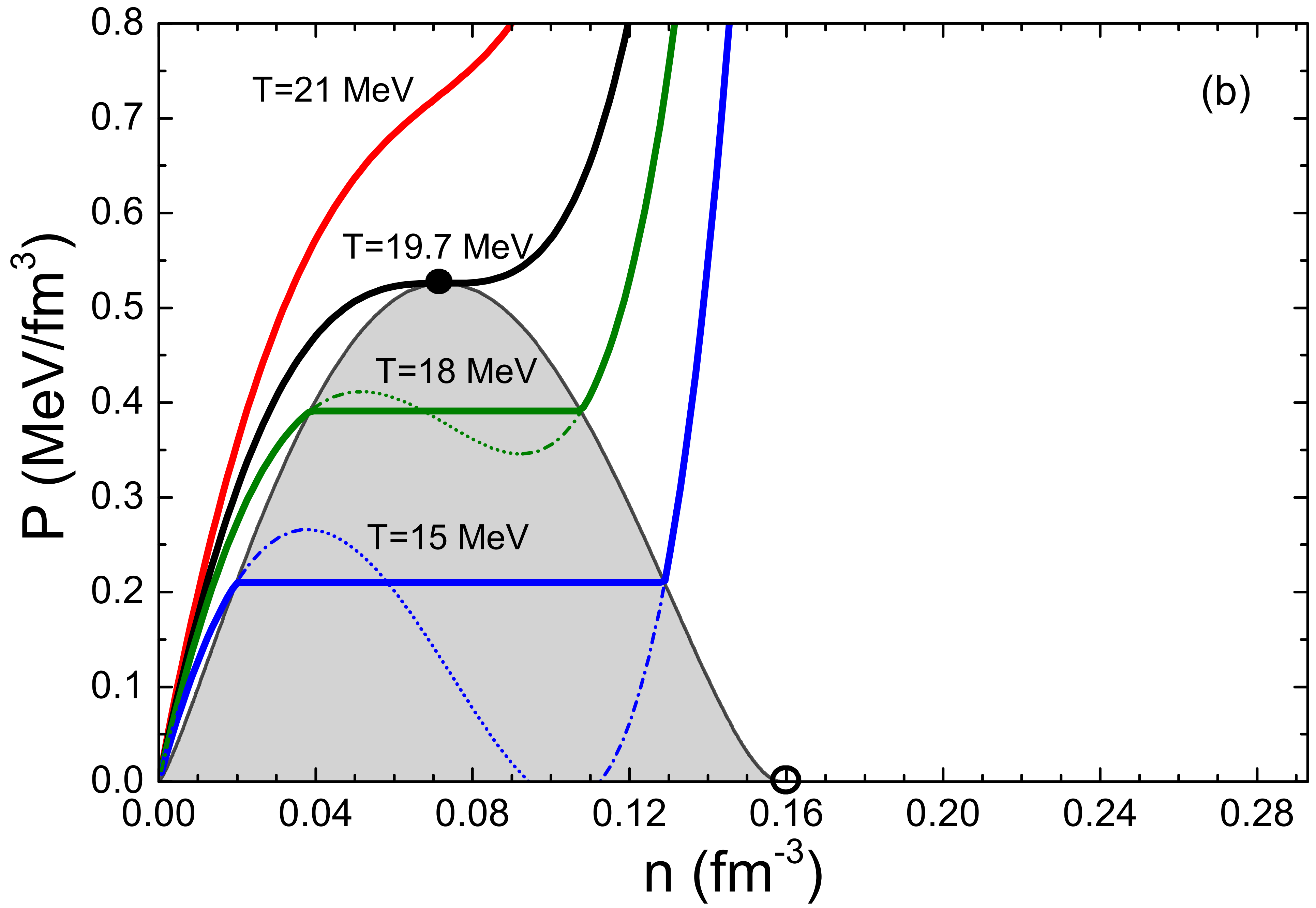
$\rho = 0.3$



First-order Phase Transition Features

- Besides the critical point itself, the features of the first order phase transition can tell us about the existence of a critical point
  - Coexistence region & spinodals





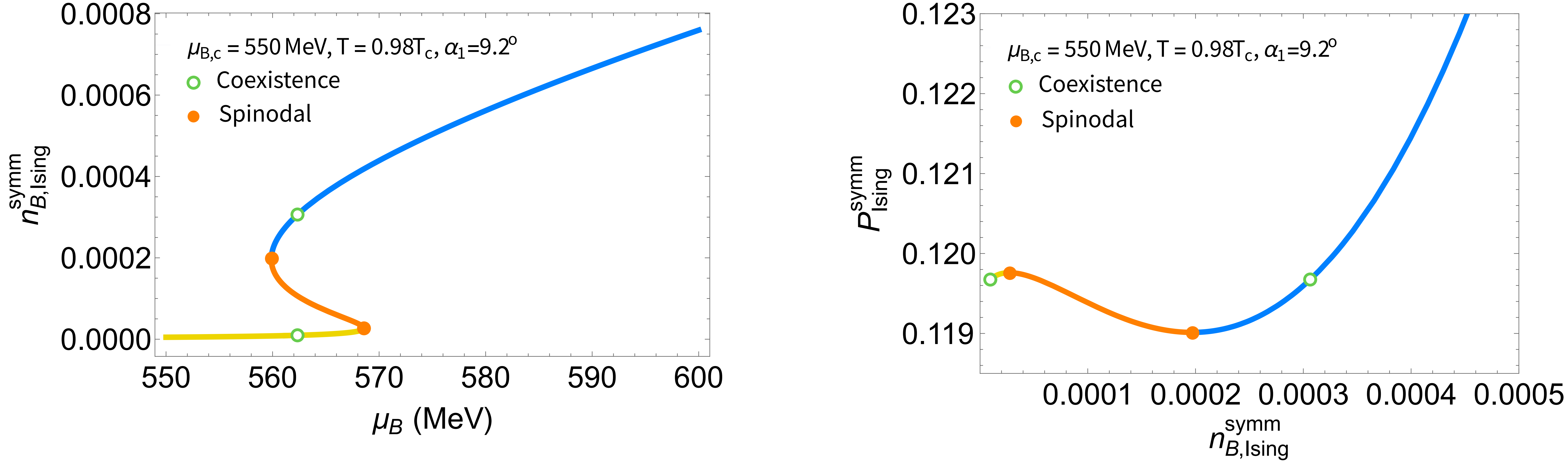
V. Vovchenko, D.V. Anchishkin, M.I. Gorenstein, PRC (2015)

EoS for First Order Regime

- Utilize mean field Ising model to incorporate first order features, e.g. spinodals, not present in 3D Ising due to shift into complex plane:

$$\Delta\phi = \pi\left(\beta\delta - \frac{3}{2}\right)$$

X. An, D. Mesterhazy, M.Stephanov, J. Stat. Mech (2018)

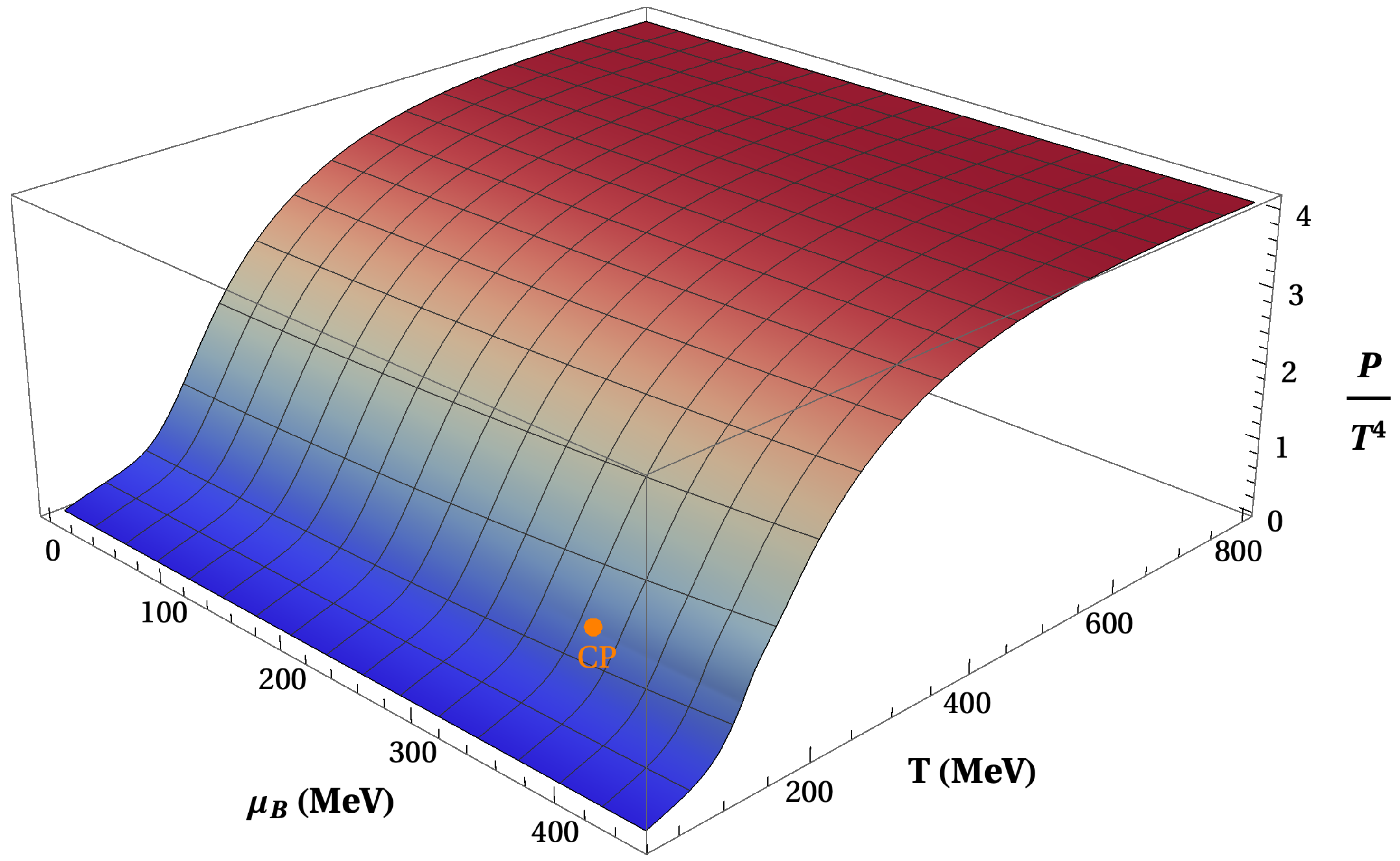


Strong dependence of the features on the angle α_1
$\rightarrow$ future work will use new quadratic mapping

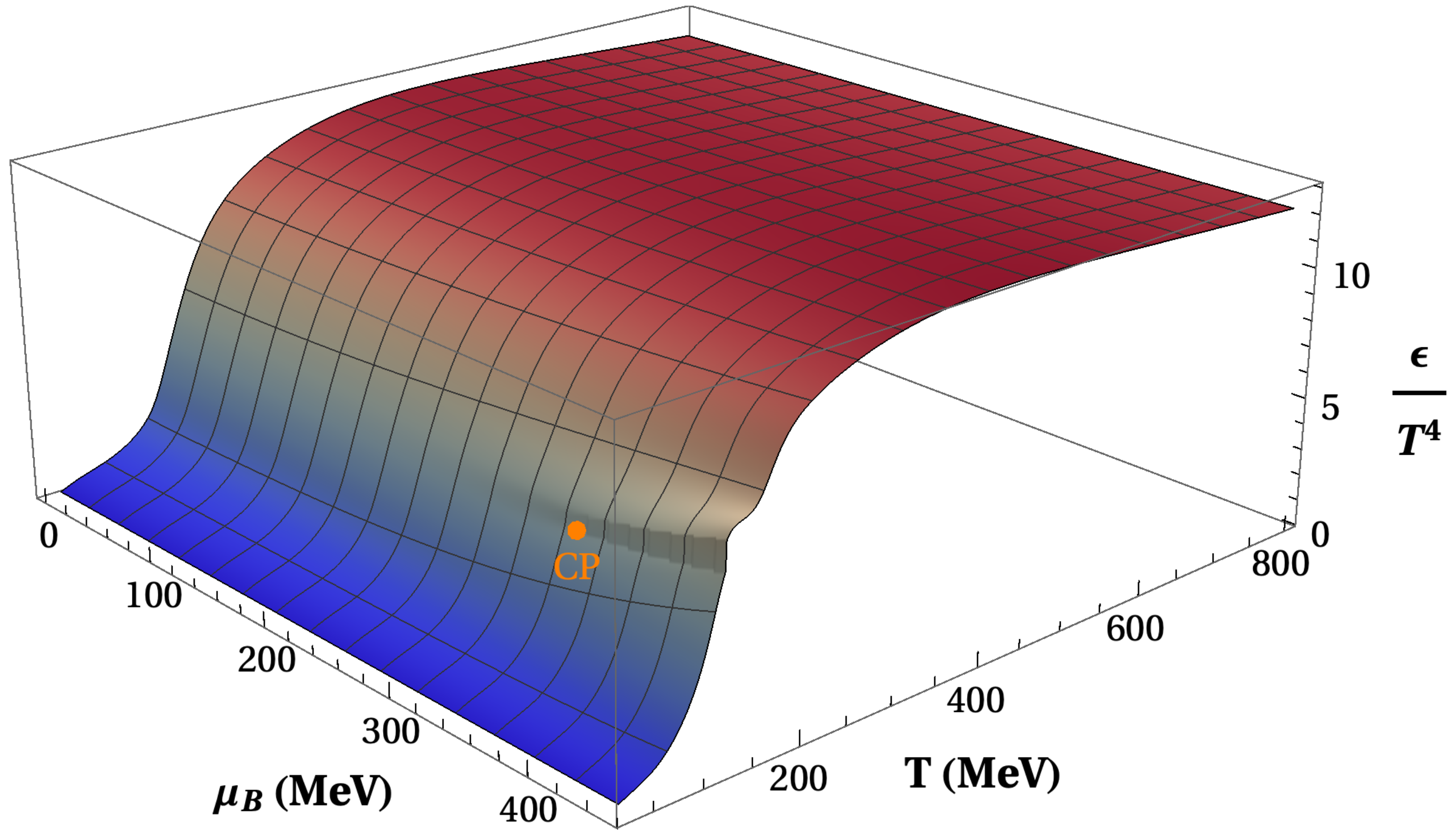
See also: M. Pradeep, N. Sogabe et al, PRC (2024) and J. Kapusta, M. Singh, T. Welle, PRC (2024)

***JMK**, V. Koch, C. Ratti, PRD (2025)*

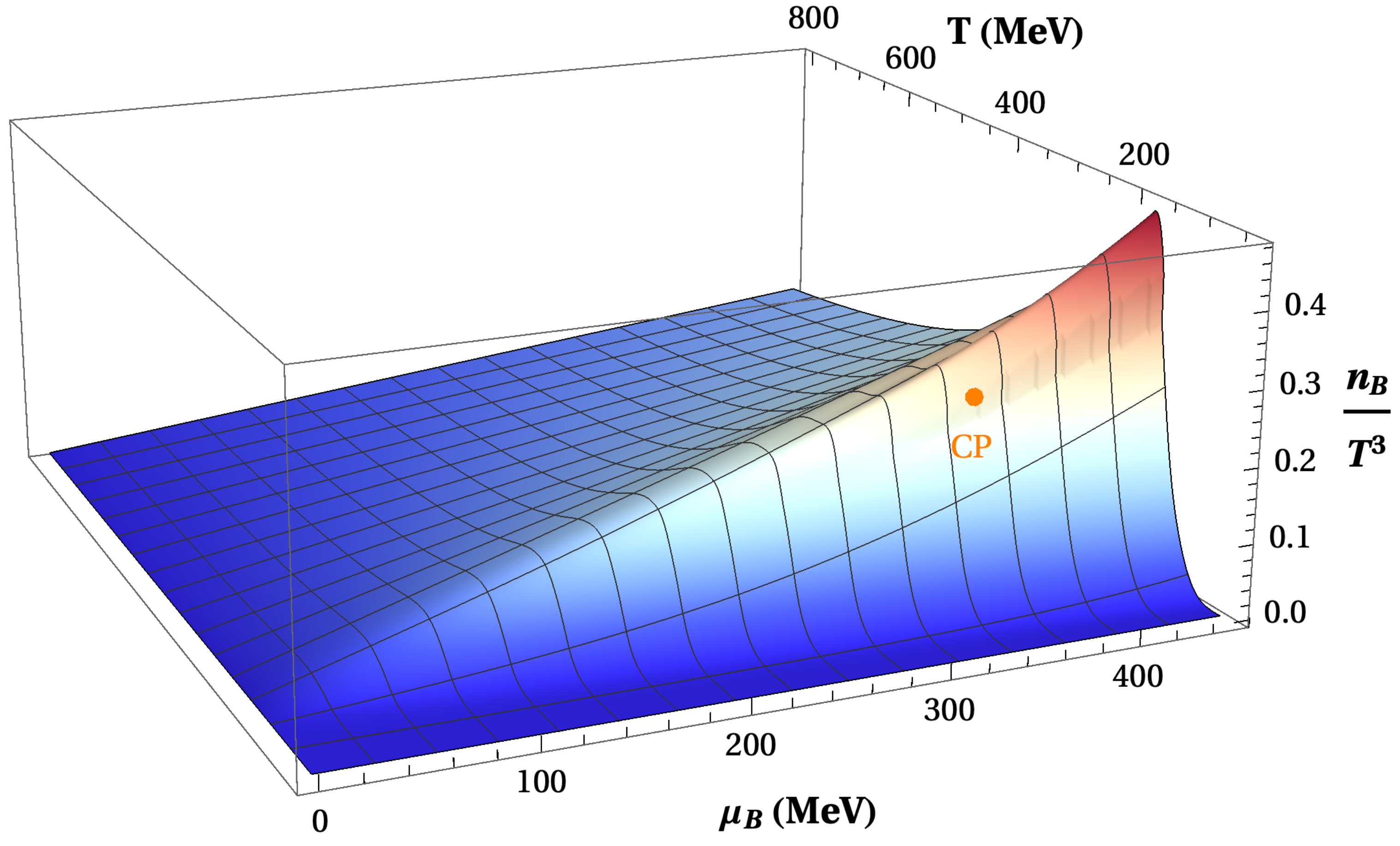
How to use EoS to connect to experiment?



Pressure



Energy density

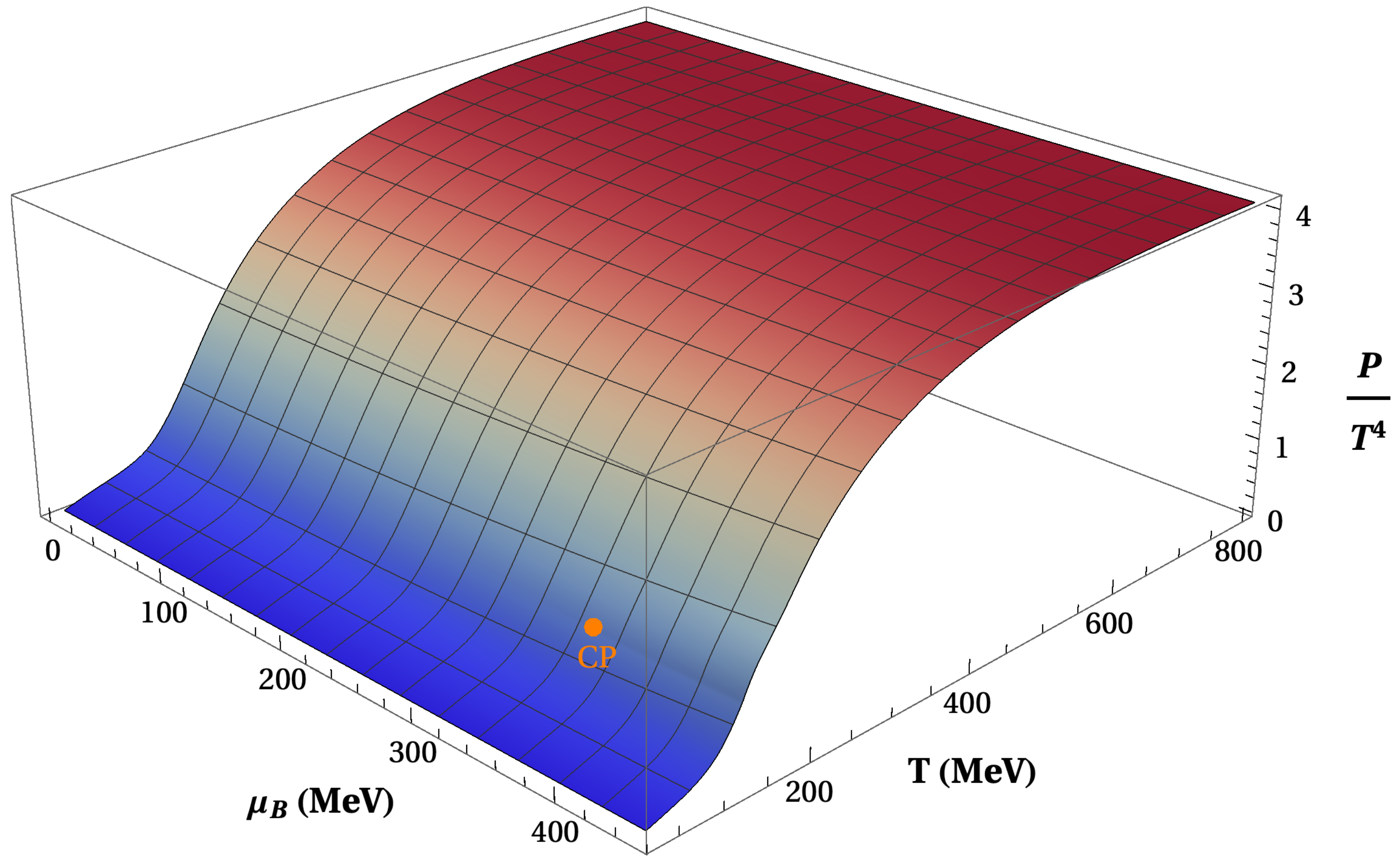


Baryon density

✅ *EoS with parametrized CP*

JMK et al, EPJ+ (2021)

How to use EoS to connect to experiment?



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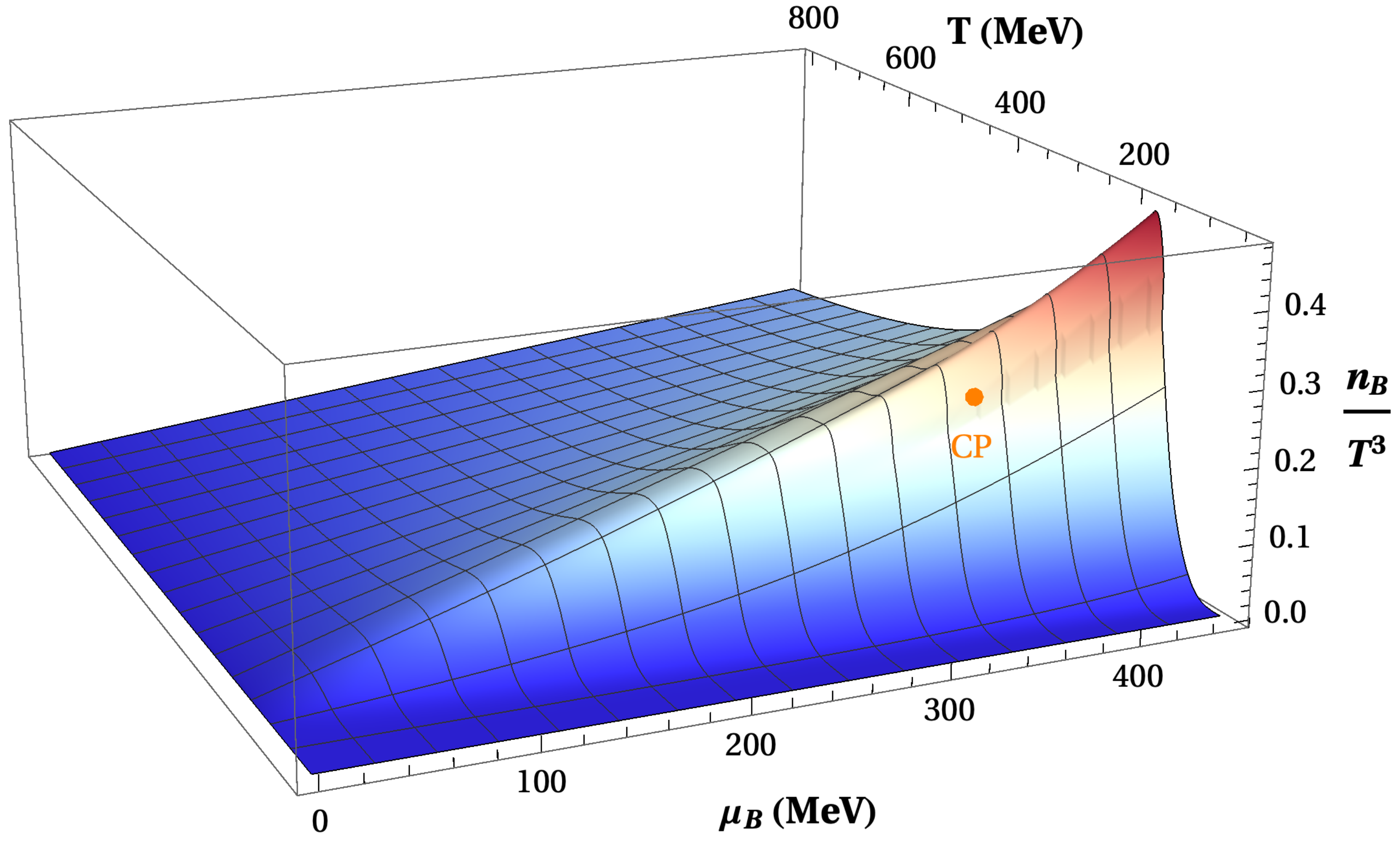
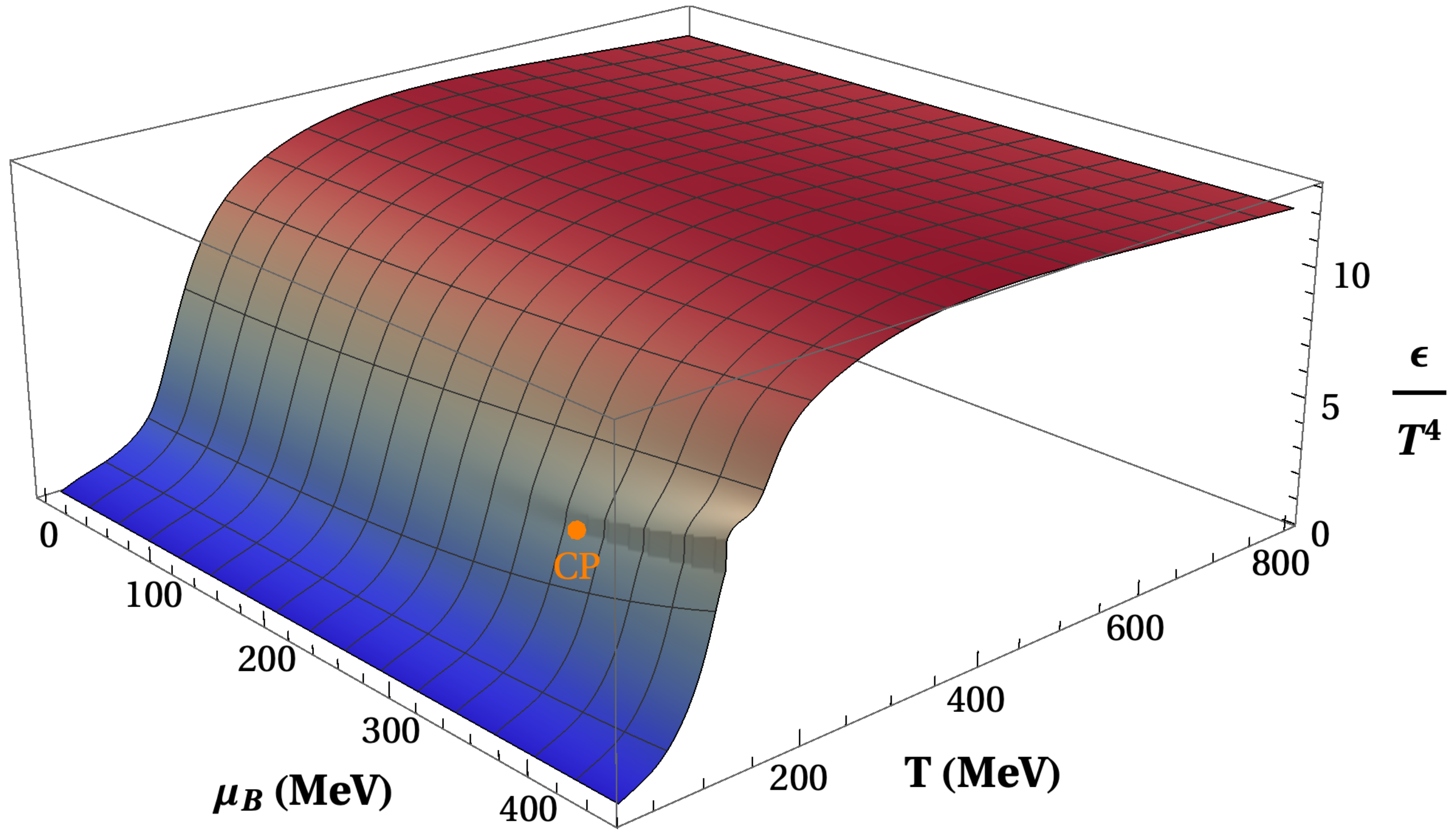
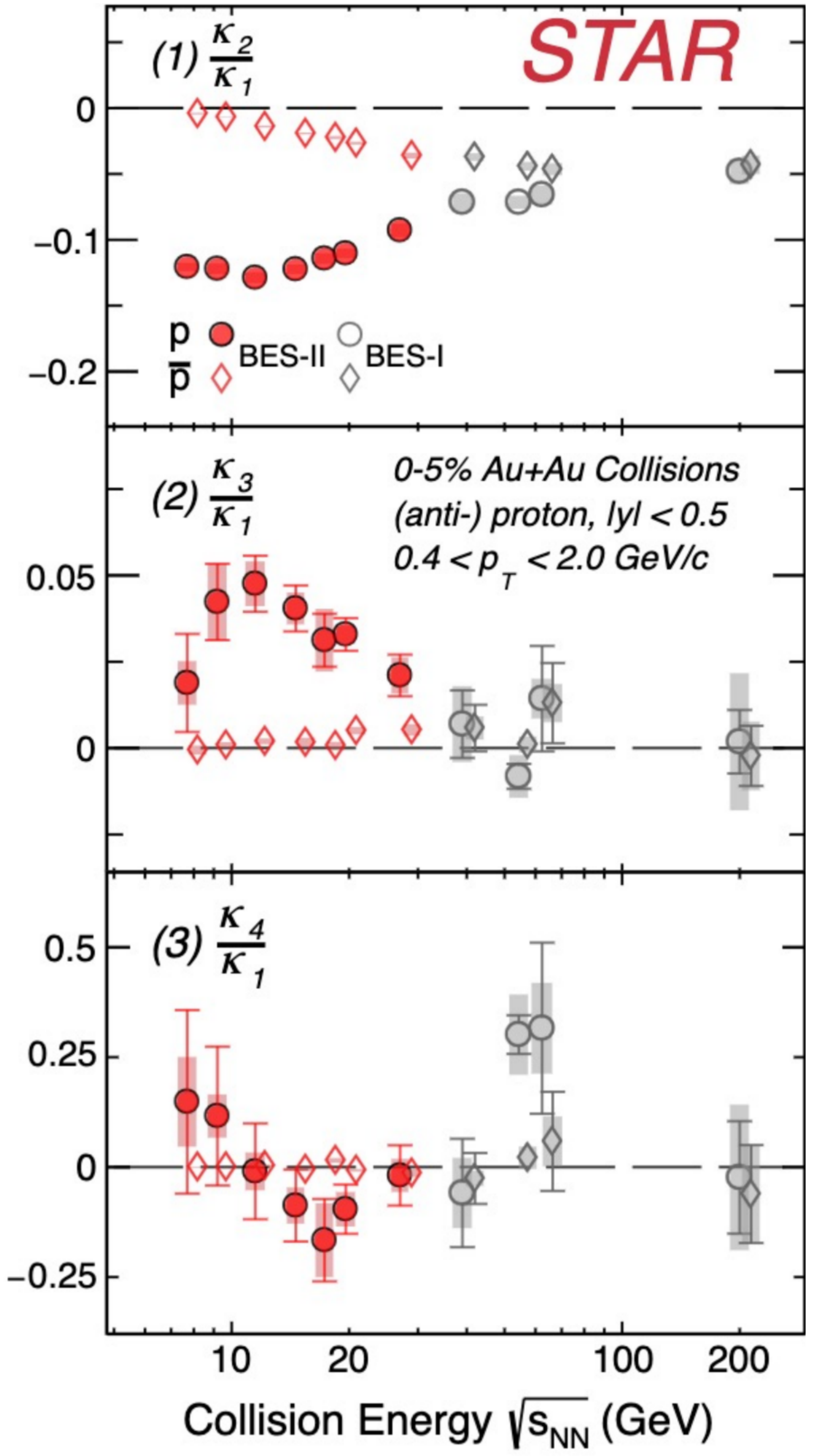


Baryon density



Energy density

?



JMK et al, EPJ+ (2021)
A. Pandav (STAR collaboration), CPOD 2024

Effective Model for Critical Fluctuations

- Linear sigma model predicts a critical point at which the sigma mass vanishes & correlation length diverges

$$\Omega = \int d^3\boldsymbol{x} \left[\frac{(\boldsymbol{\nabla}\sigma)^2}{2} + \frac{m_\sigma^2}{2}\sigma^2 + \frac{\lambda_3}{3}\sigma^3 + \frac{\lambda_4}{4}\sigma^4 + \ldots \right]$$

- Historically used to calculate particle multiplicity fluctuations

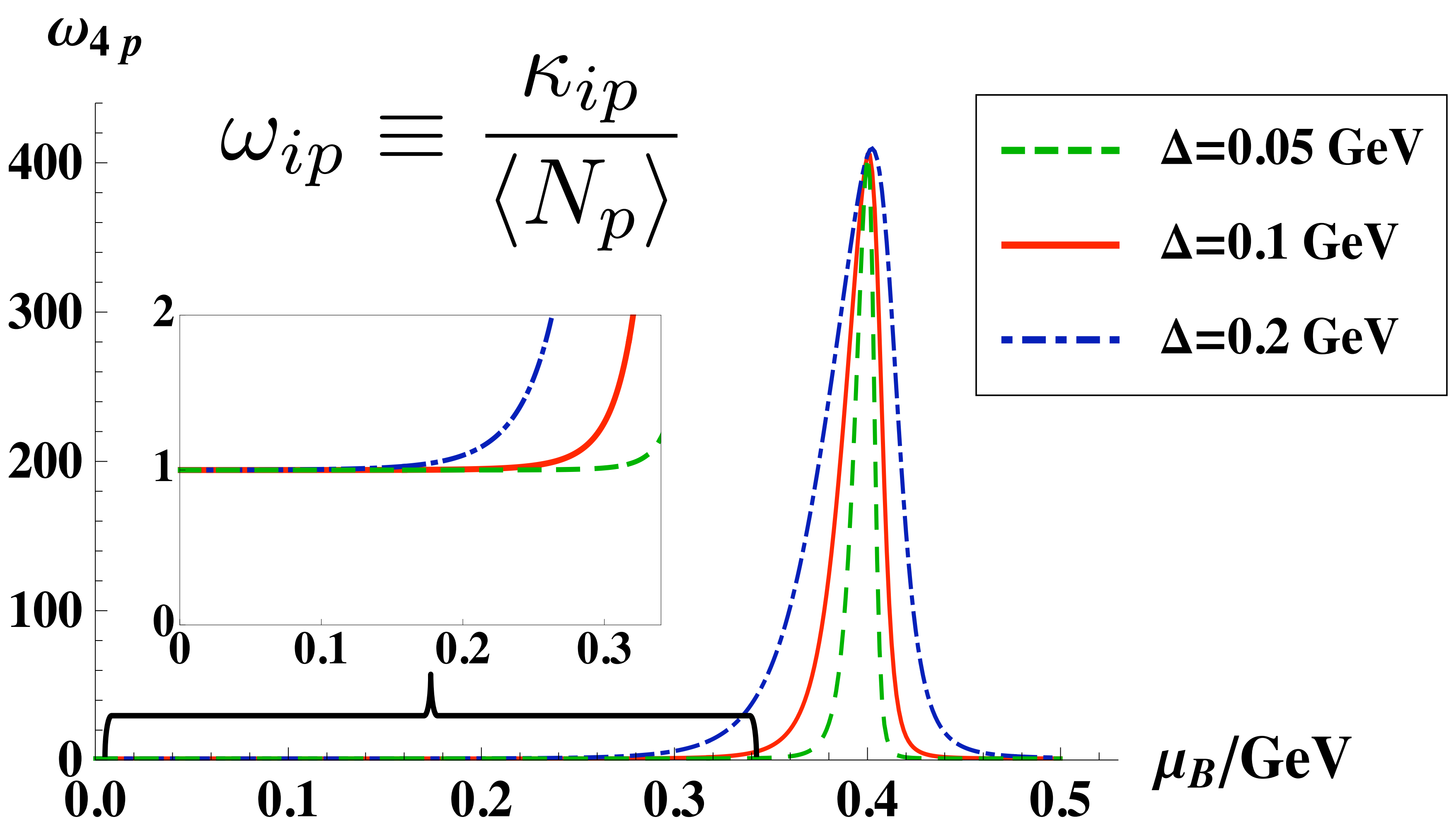
$$V\langle \delta n_{\boldsymbol{p}} \delta n_{\boldsymbol{k}} \rangle = \frac{g^2}{m_\sigma^2 T} \frac{4m^2}{E_{\boldsymbol{p}} E_{\boldsymbol{k}}} \left[n_{\boldsymbol{p}}^+(1-n_{\boldsymbol{p}}^+) - n_{\boldsymbol{p}}^-(1-n_{\boldsymbol{p}}^-) \right] \times \left[n_{\boldsymbol{k}}^+(1-n_{\boldsymbol{k}}^+) - n_{\boldsymbol{k}}^-(1-n_{\boldsymbol{k}}^-) \right]$$

  - Unknown couplings (particle, g, and higher point, λ), correlation length ($\xi = 1/m_\sigma$)

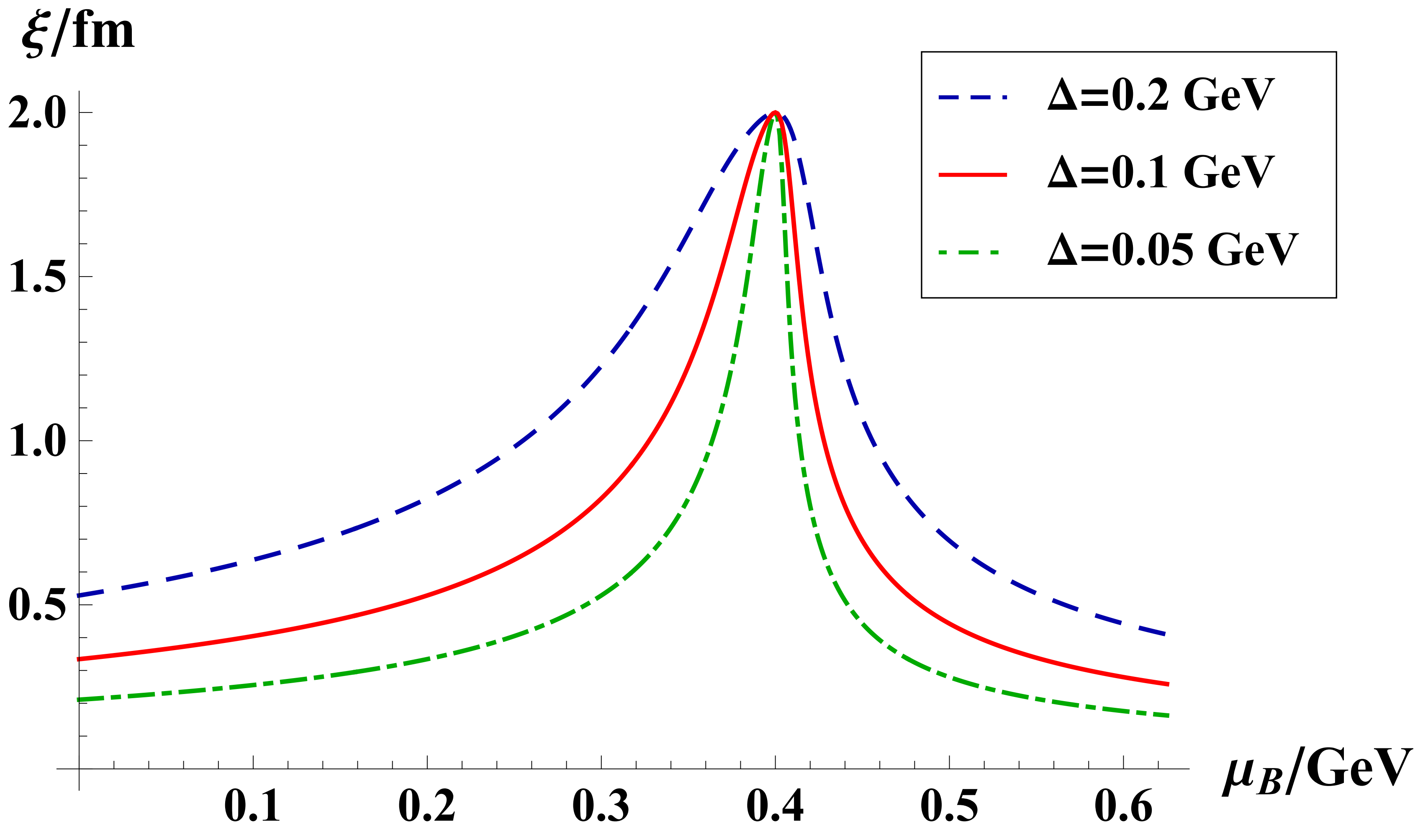
M. Gell-Mann and M. Levy, Nuovo Cim. (1960)
M. Stephanov, K. Rajagopal, E. Shuryak, PRL (1998)
O. Scavenius et al, PRC (2001)
Y. Hatta and M.A. Stephanov PRL (2003)
… + many more

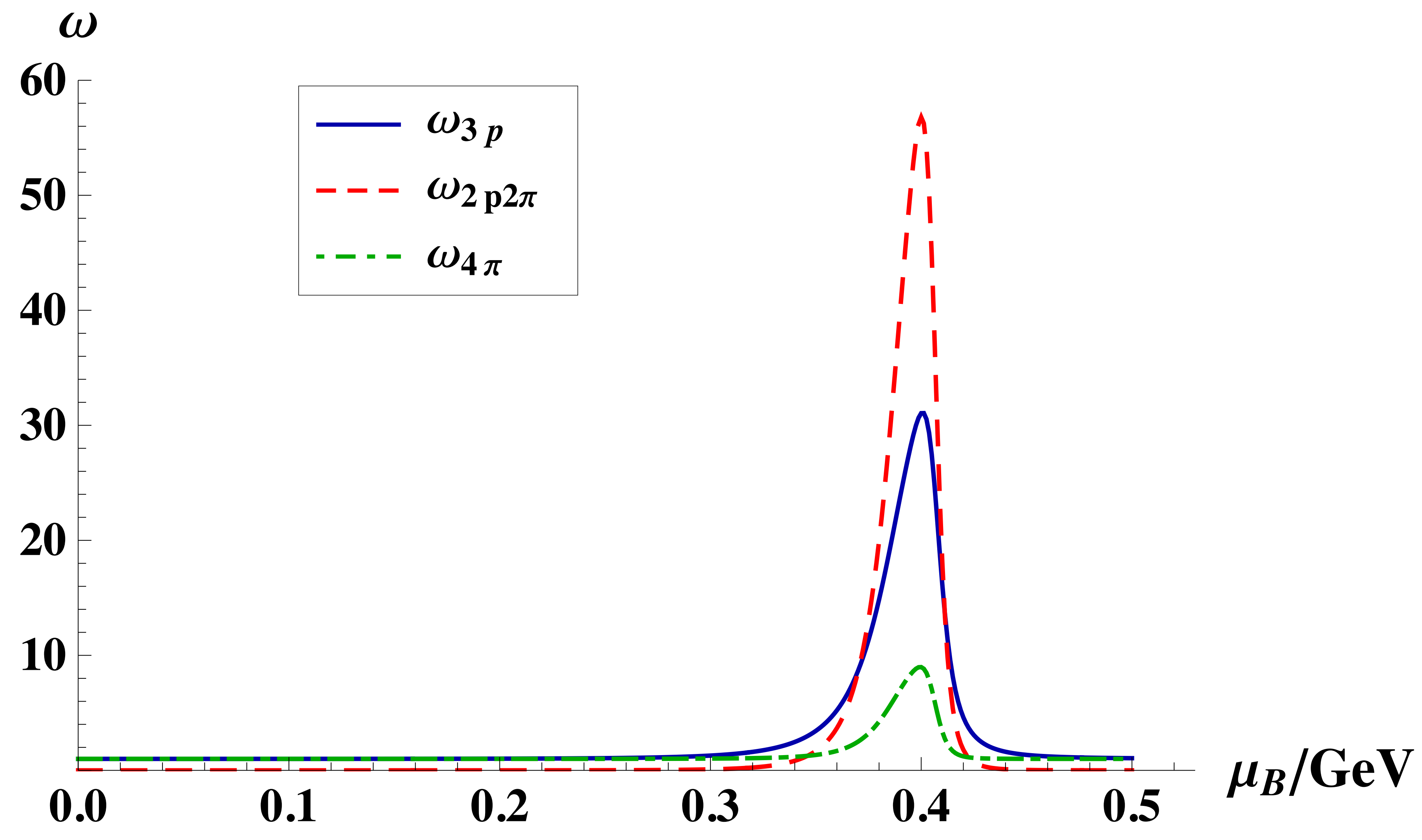
Early Estimates of Equilibrium Fluctuations

- Order-of-magnitude predictions of volume-independent normalized cumulants from 2010 relied on these unknown parameters
  - Original estimates used parametrized correlation length with width Δ



$$\xi(\mu_B) = \frac{\xi_{\max}}{\left[1 + \frac{(\mu_B - \mu_B^c)^2}{W(\mu_B)^2}\right]^{1/3}}$$

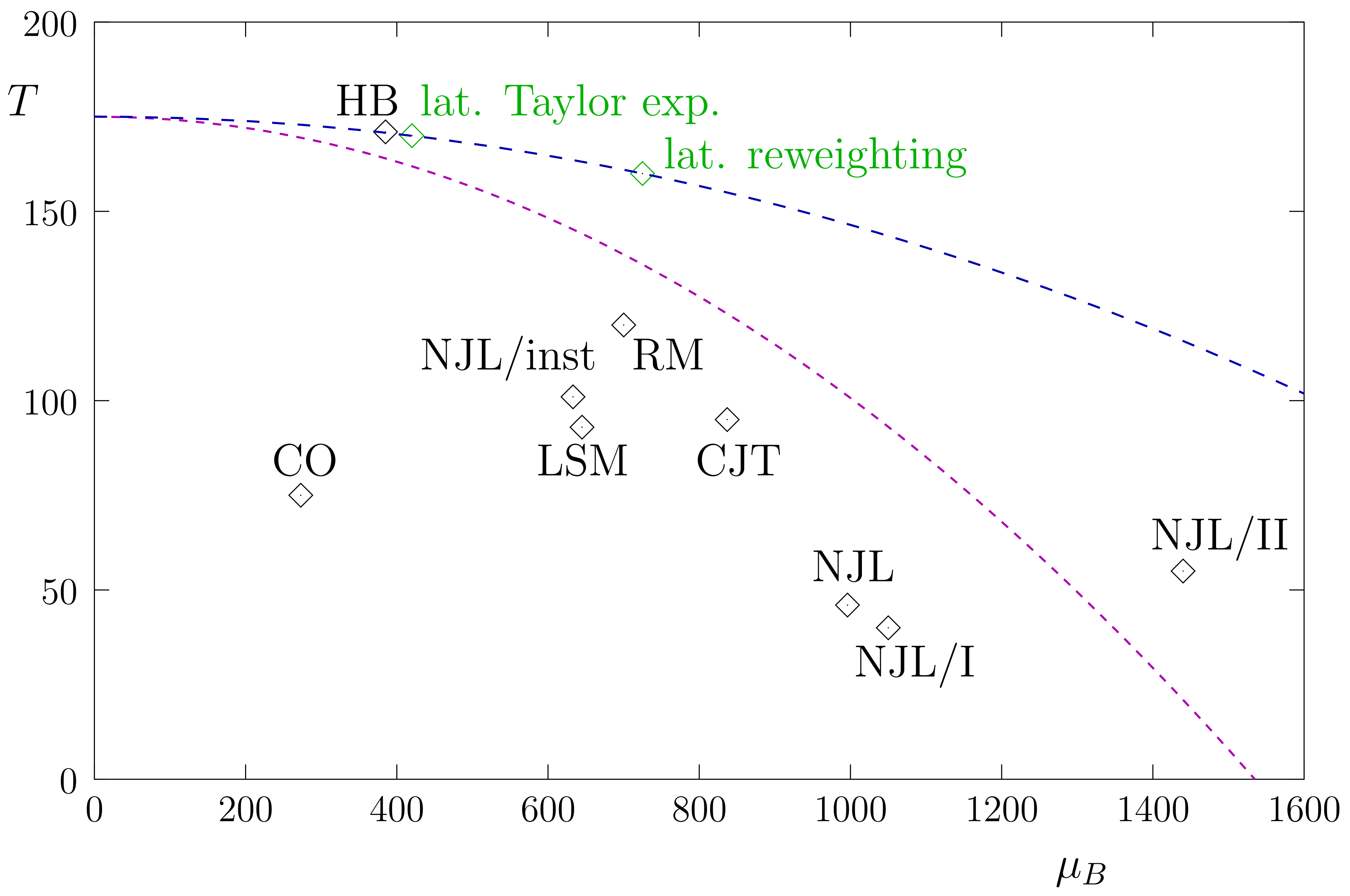




C. Athanasiou, K. Rajagopal, M. Stephanov, PRD (2010)

Brief Aside: Critical Point from LSM

- Early calculations with two light flavored quarks obtained critical point location within linear sigma model



M. Stephanov, Prog.Theor.Phys.Suppl. (2004)

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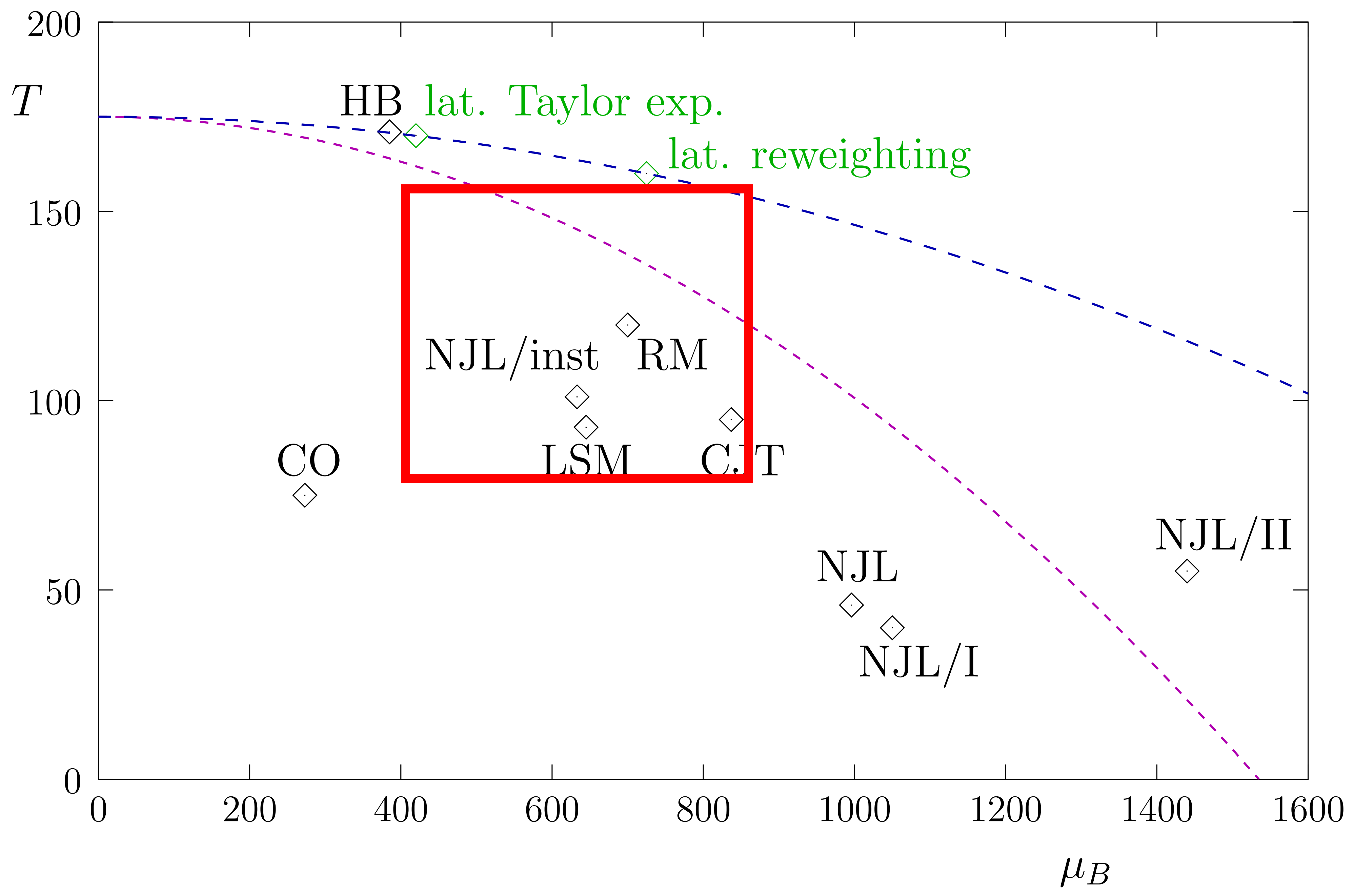

M. Stephanov, Prog.Theor.Phys.Suppl. (2004)
O. Scavenius et al, PRC (2001)

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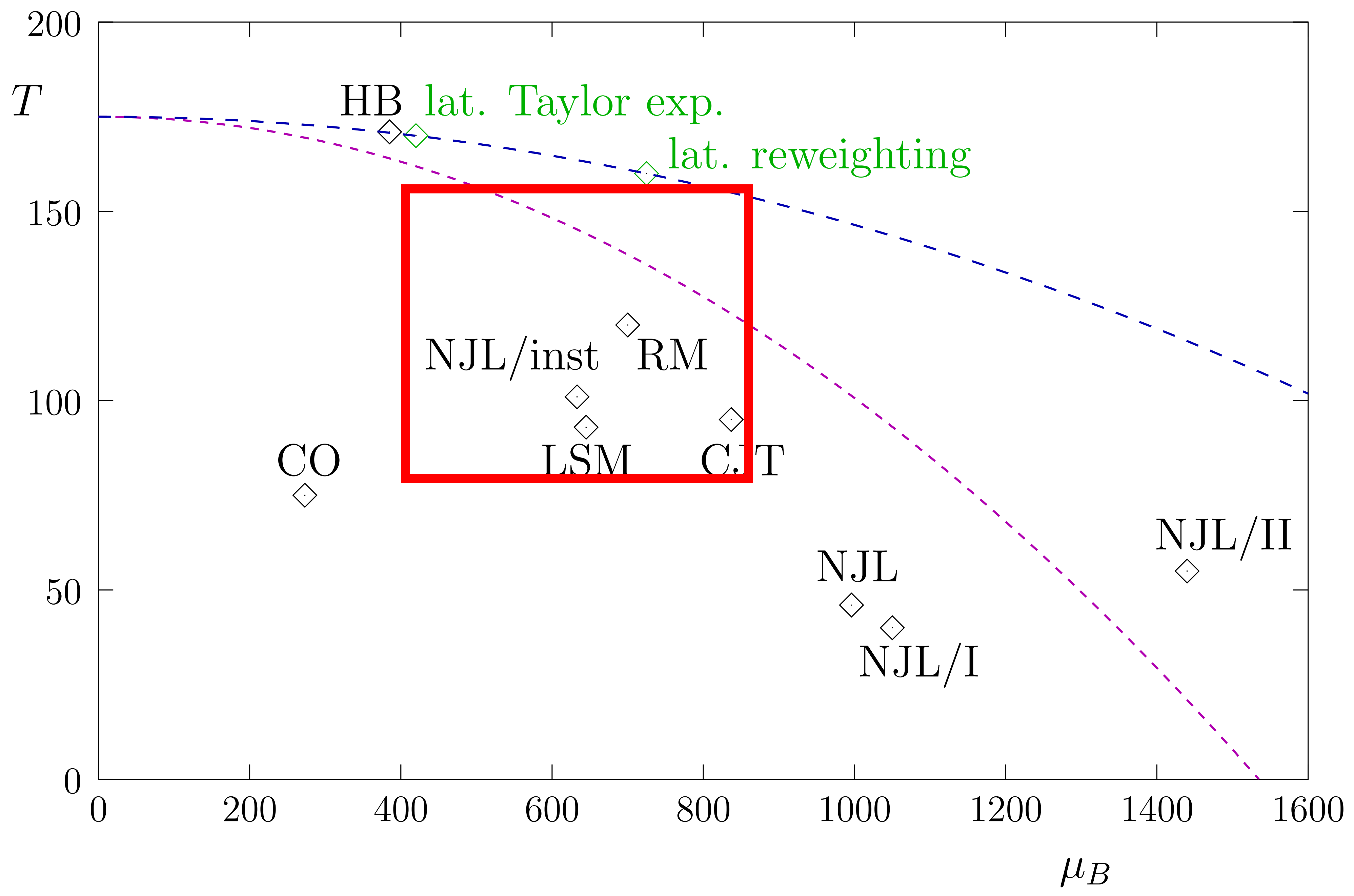
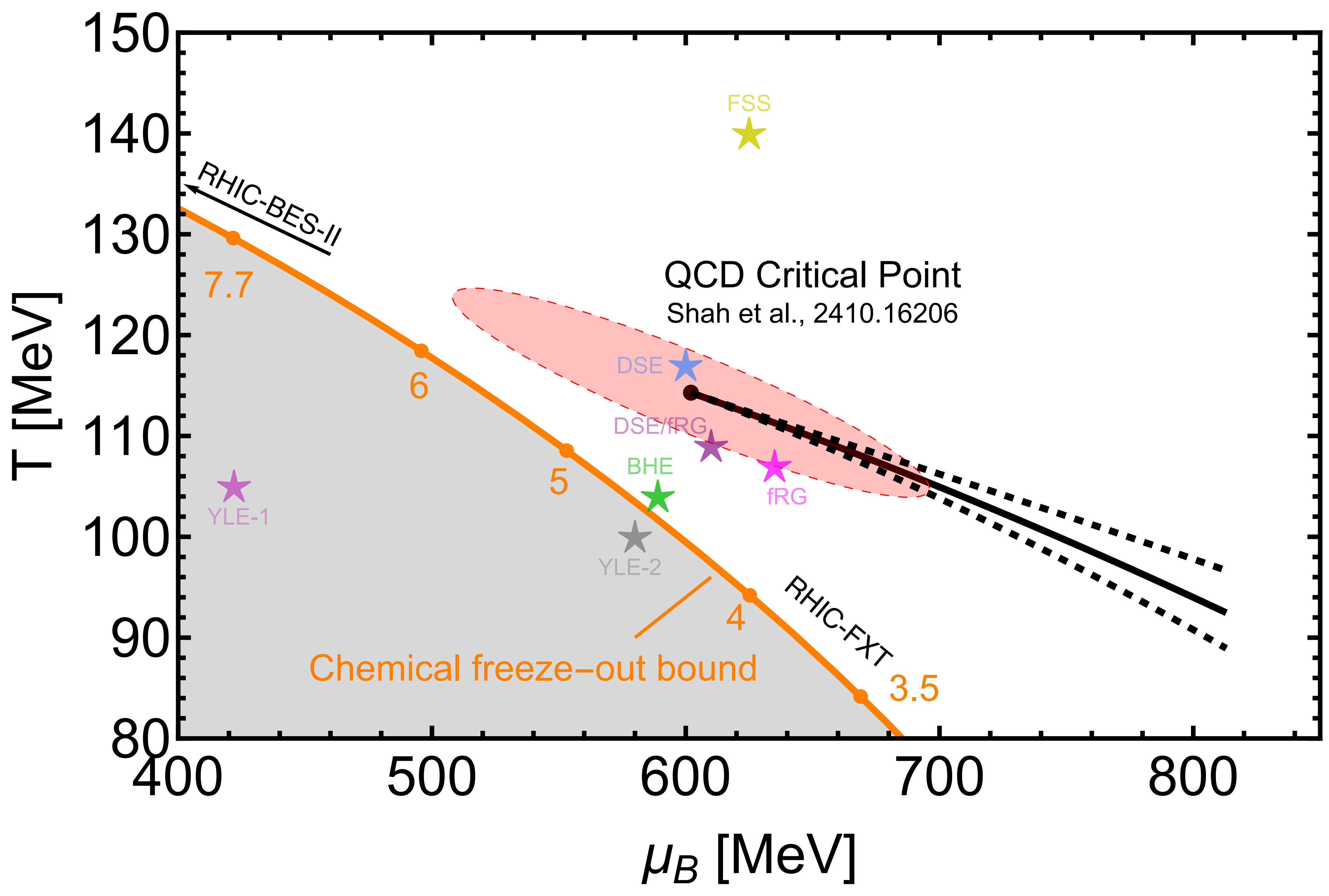
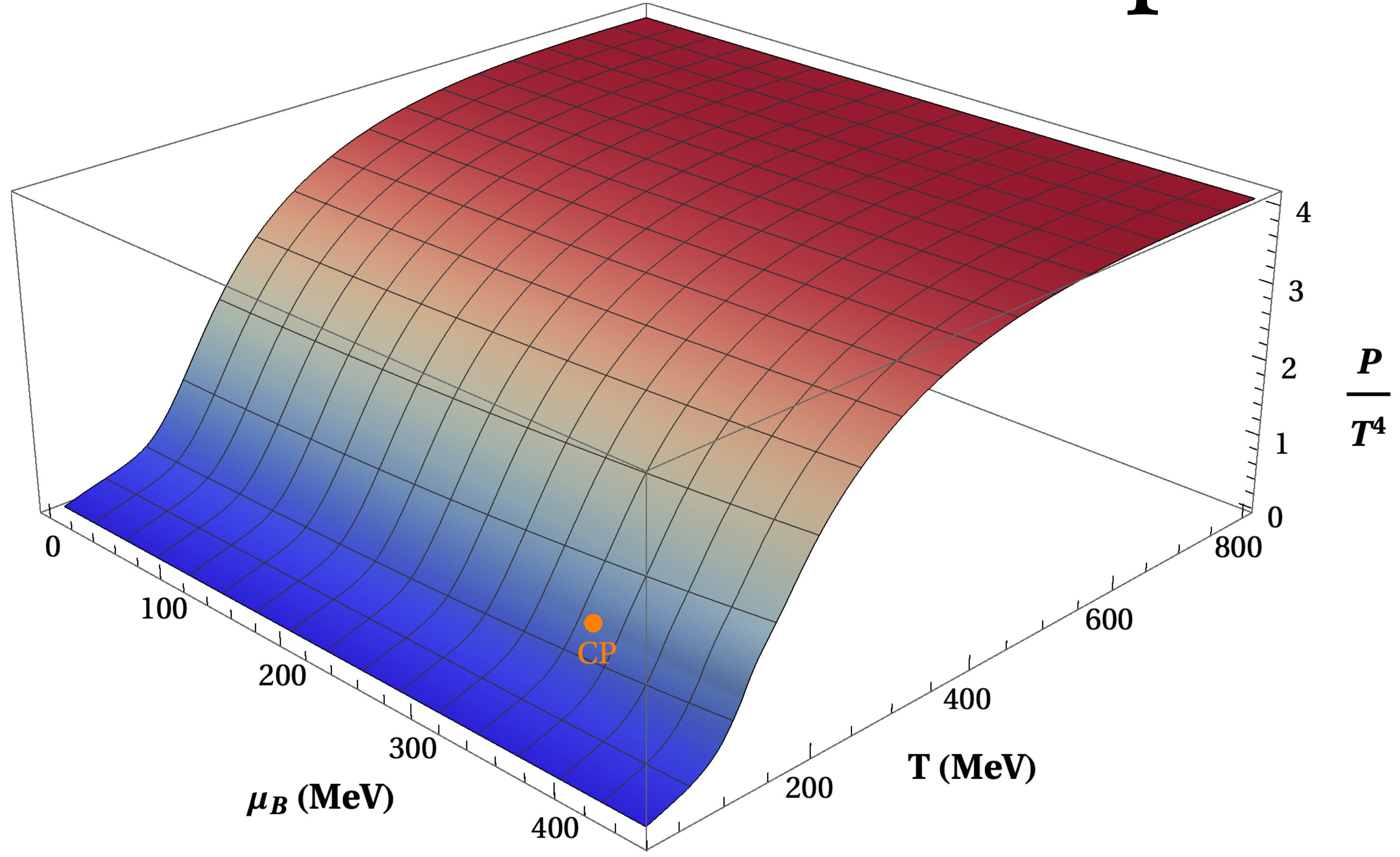




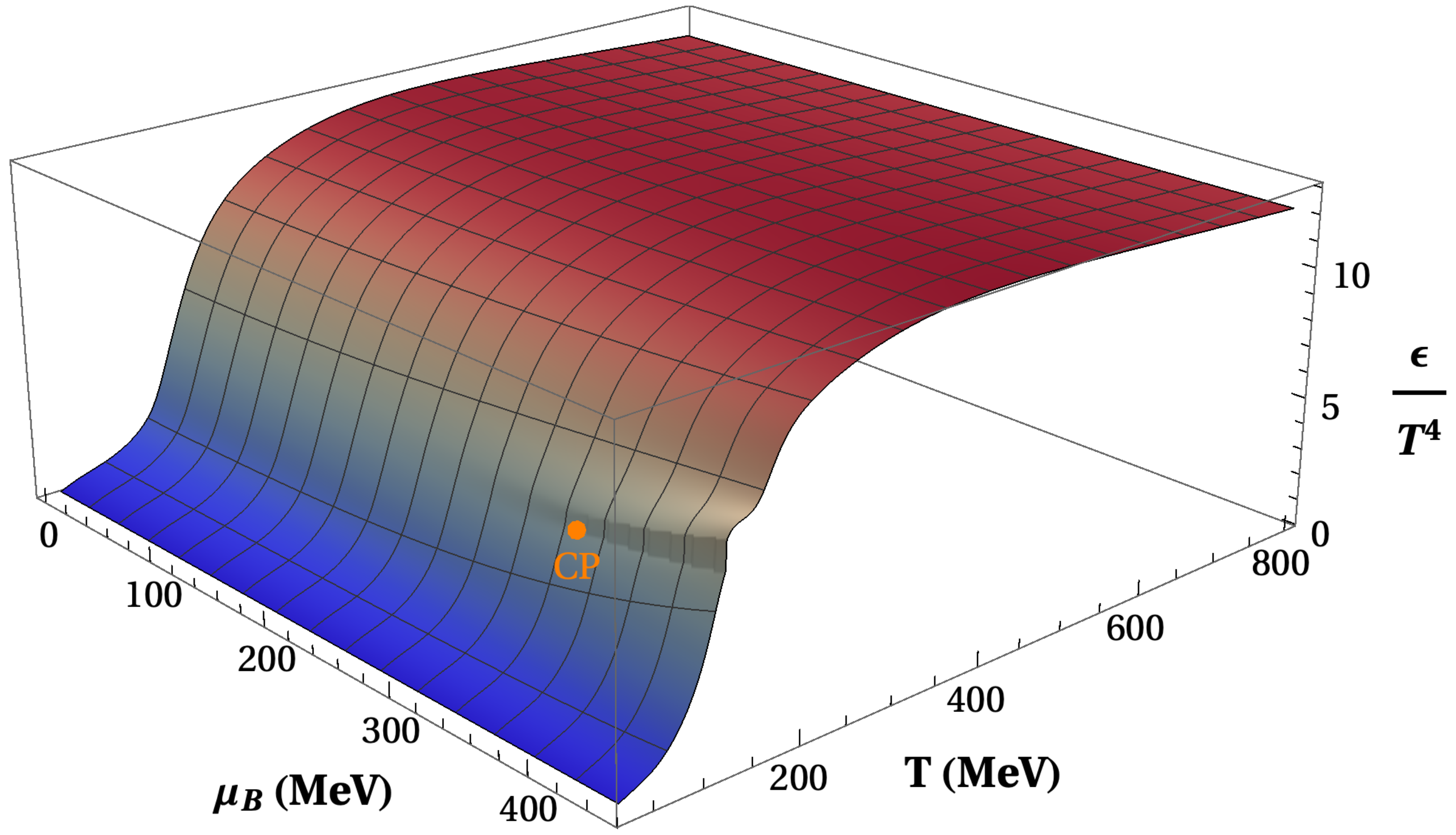
M. Stephanov, Prog.Theor.Phys.Suppl. (2004)
O. Scavenius et al, PRC (2001)
V. Vovchenko and V. Koch, J.Subatomic Part.Cosmol. (2025)

How to use EoS to connect to experiment?

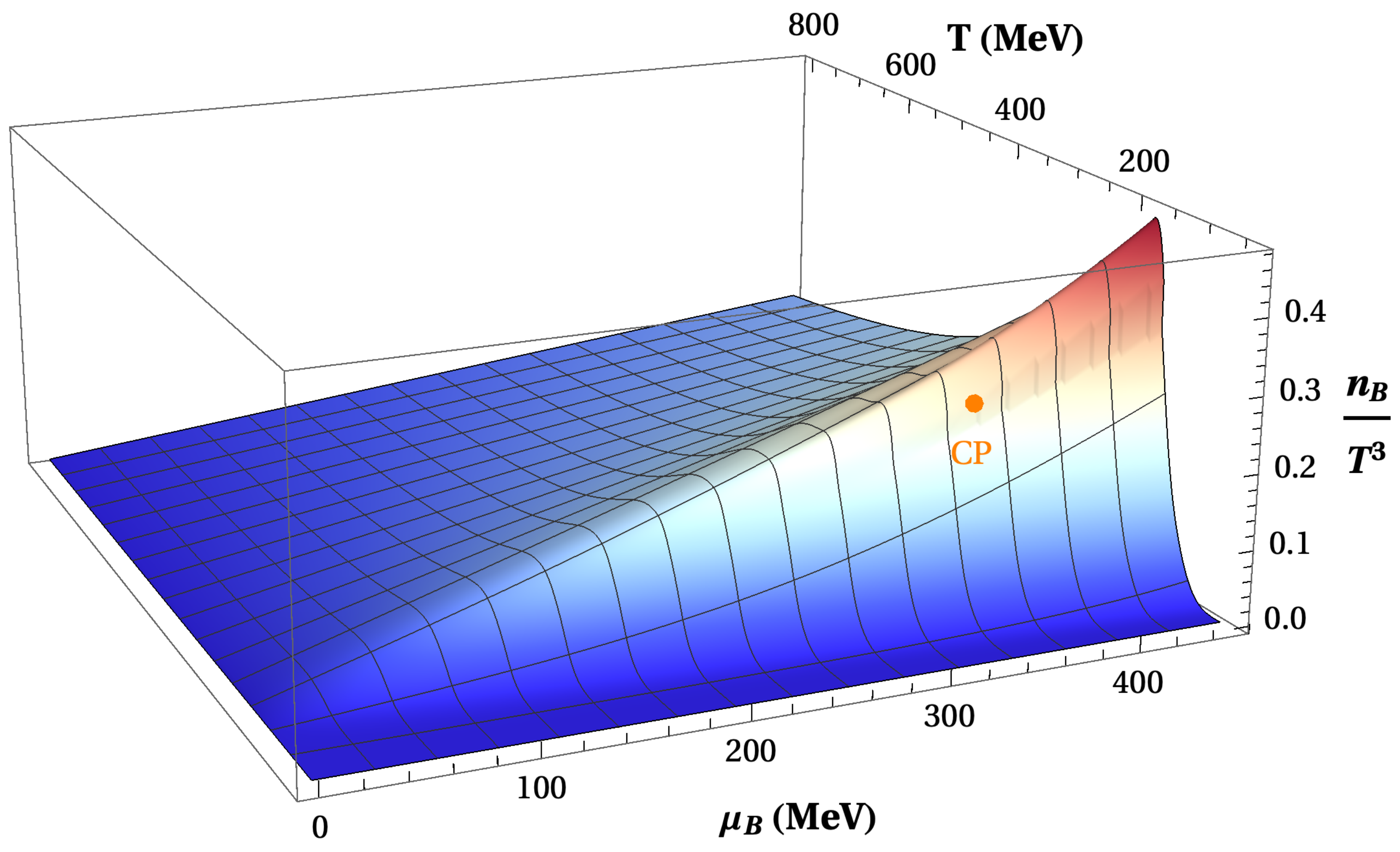
… in a model independent way



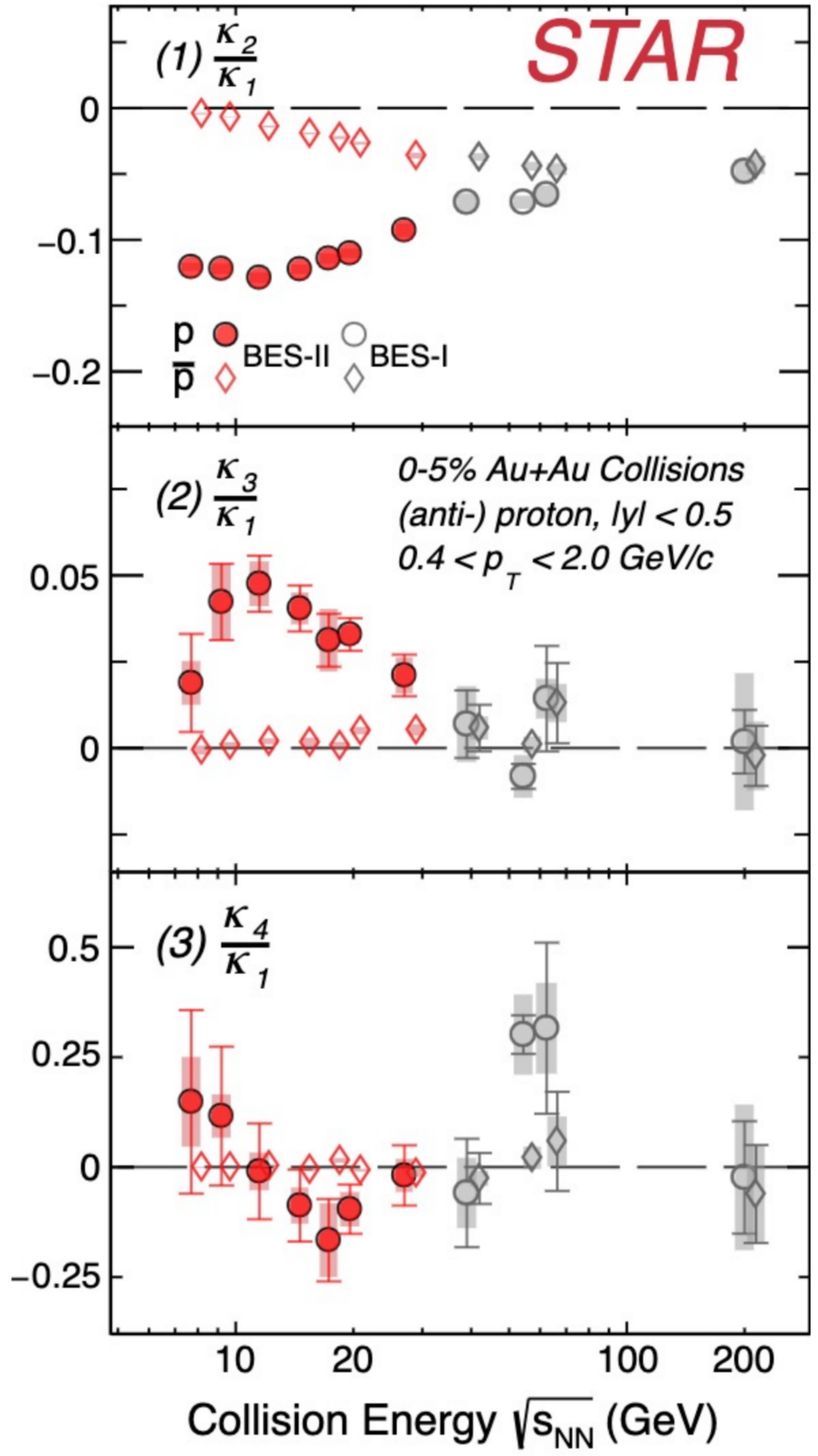
Pressure



Energy density



Baryon density



JMK et al, EPJ+ (2021)
A. Pandav (STAR collaboration), CPOD 2024

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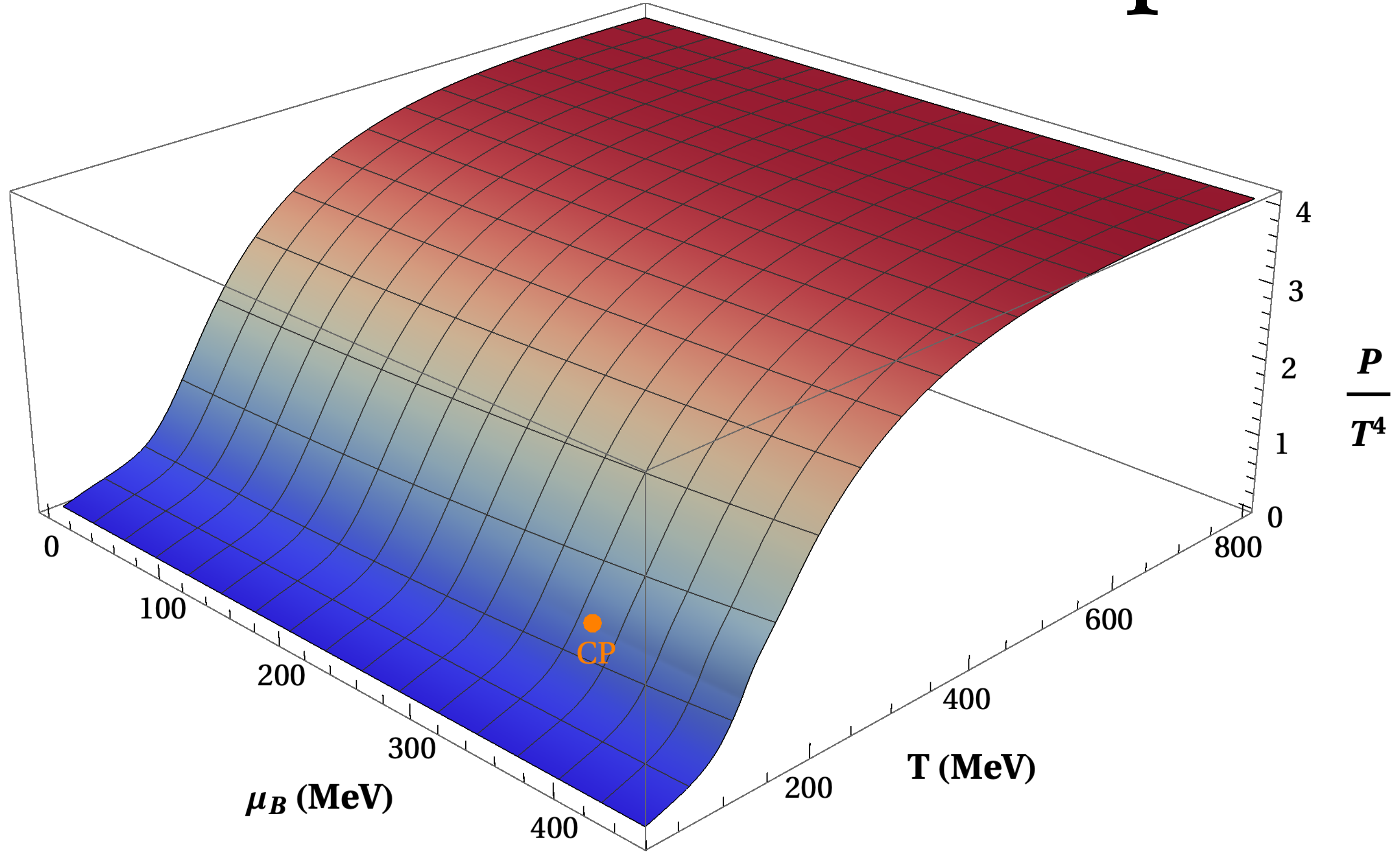

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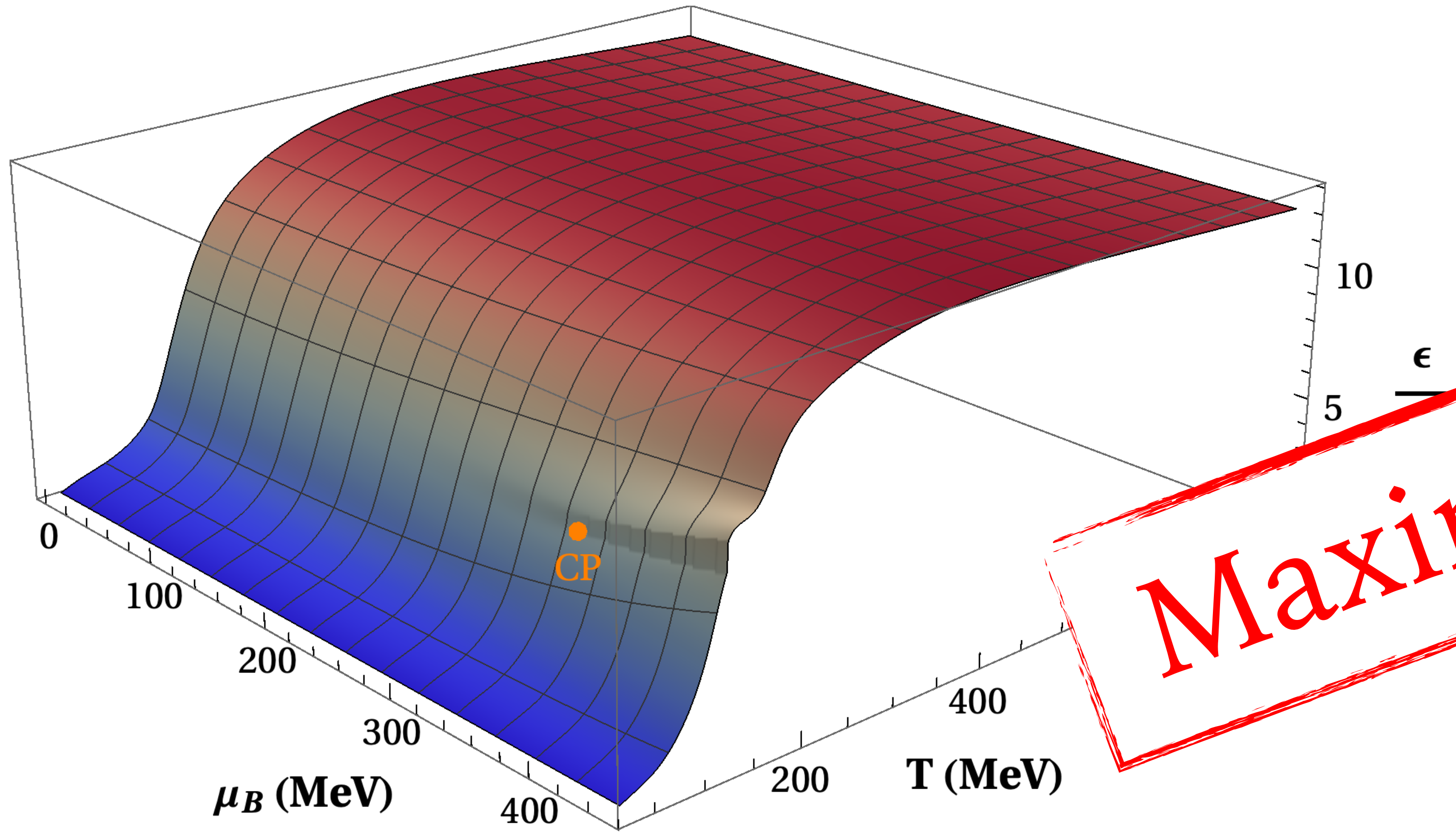

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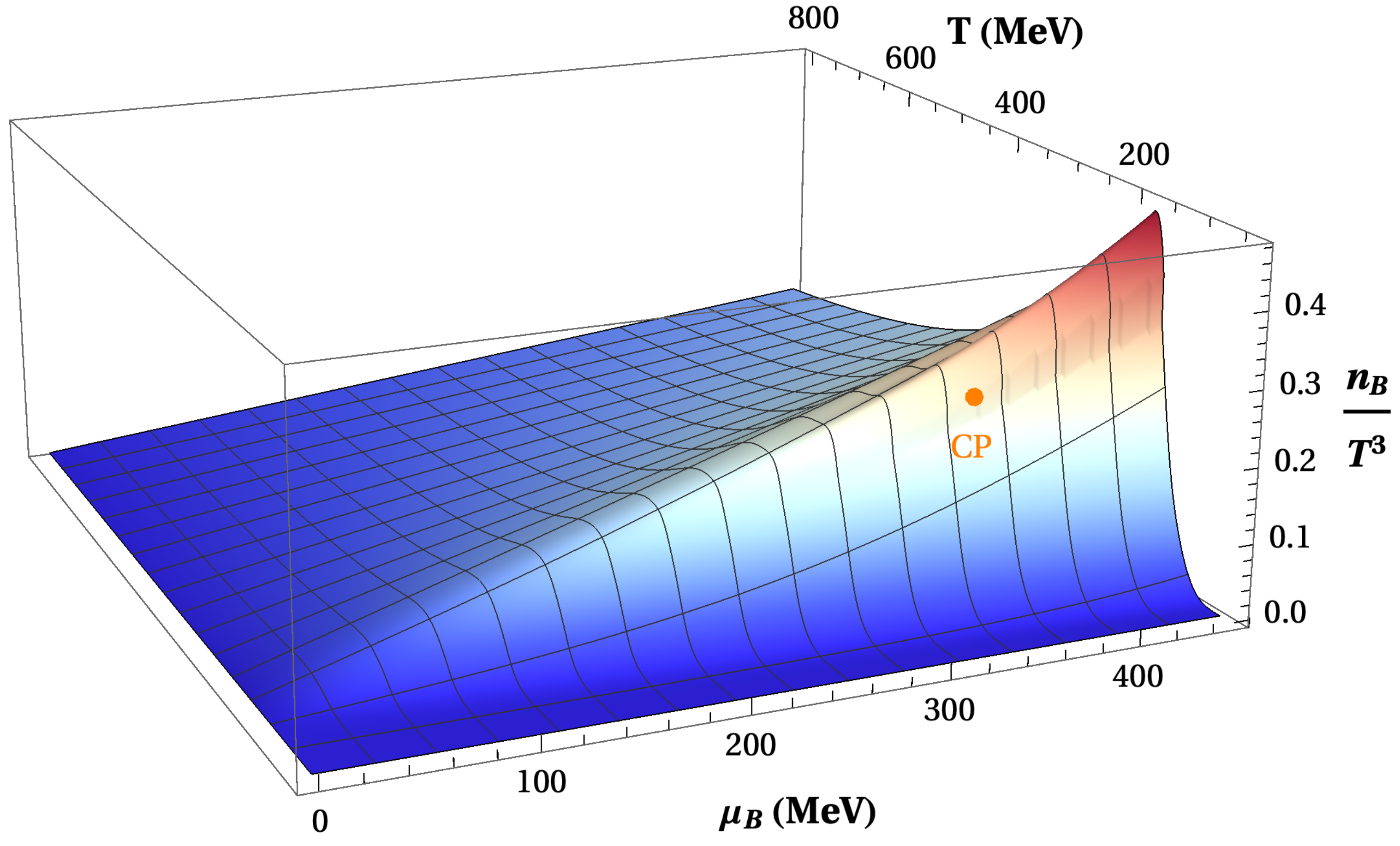

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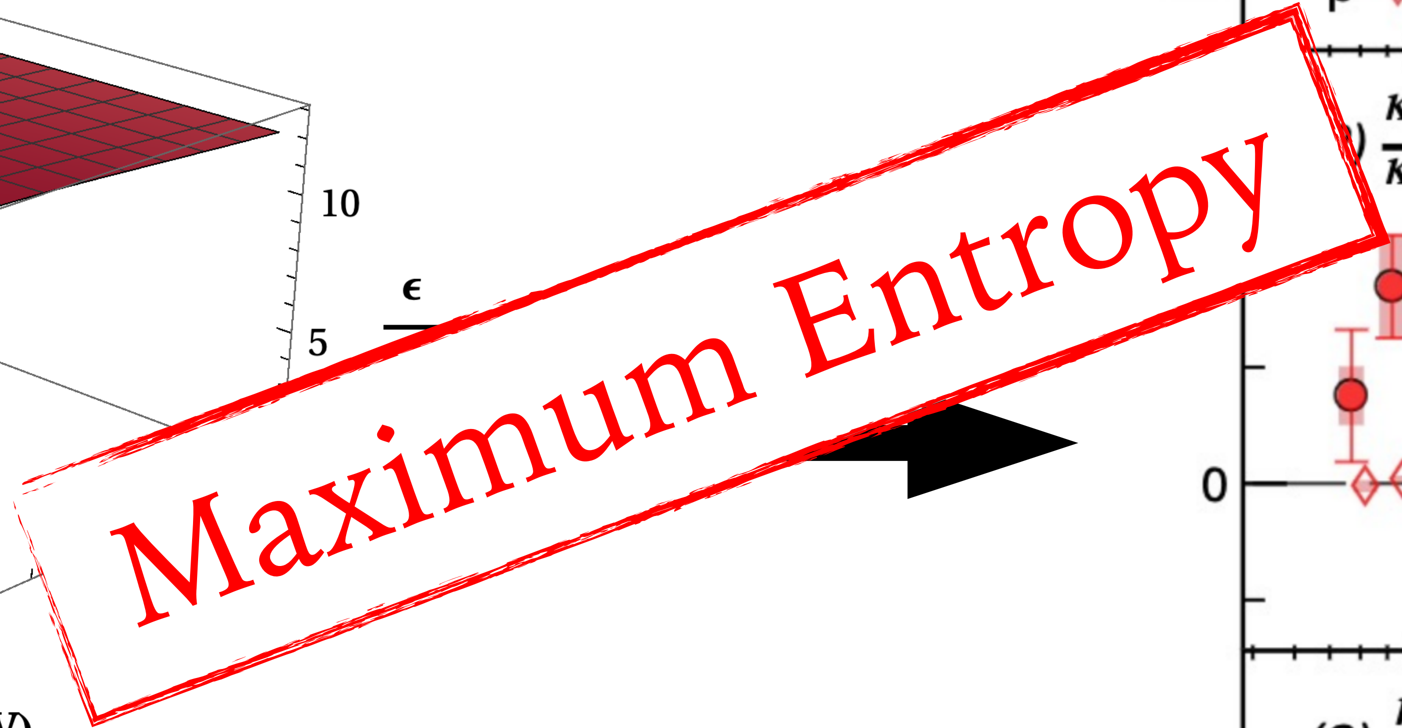

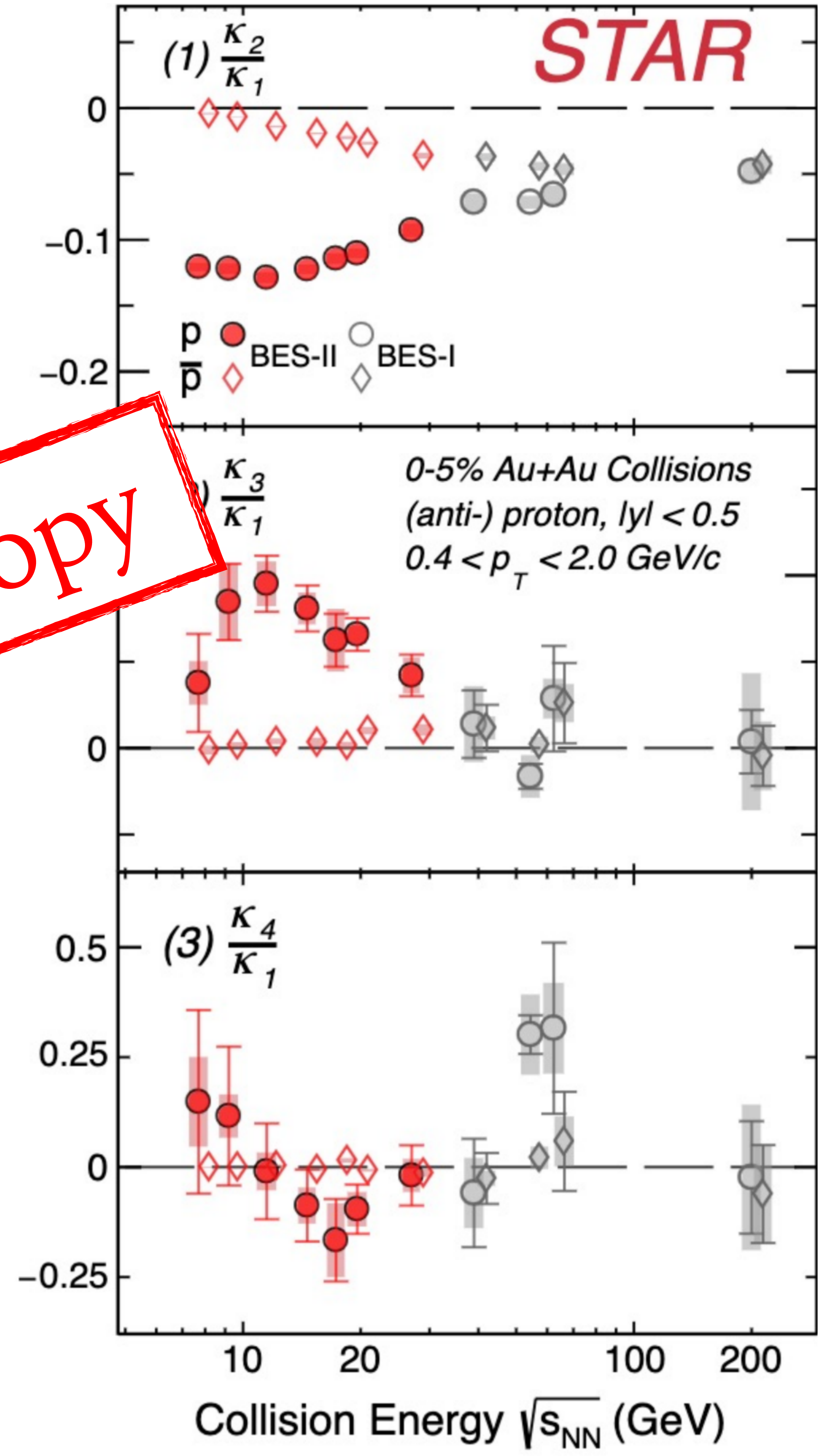

JMK et al, EPJ+ (2021)
A. Pandav (STAR collaboration), CPOD 2024

II. A first study of equilibrium proton factorial cumulants with MaxEnt

Connecting Hydro & Particle Fluctuations

Maximum entropy freezeout of hydrodynamic fluctuations

Maneesha Sushama Pradeep[1] and Mikhail Stephanov[1]
[1] *Department of Physics, University of Illinois, Chicago, IL 60607, USA*
(Dated: May 17, 2023)

We propose a general approach to freezing out *fluctuations* in heavy-ion collisions using the principle of maximum entropy. We find the results naturally expressed as a direct relationship between the *irreducible relative correlators* quantifying the deviations of hydrodynamic as well as hadron gas fluctuations from the ideal hadron gas baseline. The method also allows us to determine heretofore unknown parameters crucial for the freezeout of fluctuations near the QCD critical point in terms of the QCD equation of state.

$$\hat{\Delta} G_{A_1 \ldots A_k} = \hat{\Delta} \mathcal{H}_{a_1 \ldots a_k} \prod_{i=1}^{k} P^{a_i}_{A_i}$$

Irreducible relative cumulant (IRC)

(Particle) Gas

Hydro (fields)

Matching conditions projectors

Least-biased determination of particle multiplicity fluctuations consistent with hydrodynamic correlations

M. Pradeep and M. Stephanov, PRL (2023)

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translate → into MaxEnt

$$\Delta G_{AB} = \frac{g_A g_B}{ZT^2}\frac{m_A}{E_A}\frac{m_B}{E_B}\Delta H^{mm} f'_A f'_B$$

M. Pradeep and M. Stephanov, PRL (2023)

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translate → into MaxEnt

$$\Delta G_{AB} = \frac{g_A g_B}{ZT^2}\frac{m_A}{E_A}\frac{m_B}{E_B}\Delta H^{mm} f'_A f'_B$$

Extract unknown coupling in terms of EoS properties:

$$g_A = \sqrt{Z}\frac{E_A}{m_A}\left(\bar{H}^{-1}\right)_{mm}\frac{w_c}{n_c}\left(\frac{E_A}{w_c} - \frac{q_A}{n_c}\right), \quad \sqrt{Z} \propto \frac{\sin(\alpha_1)}{w\sin(\alpha_1-\alpha_2)}$$

M. Pradeep and M. Stephanov, PRL (2023)

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what about correlation length? $(\xi = 1/m_\sigma)$

translate into MaxEnt →

$$\Delta G_{AB} = \frac{g_A g_B}{Z T^2} \frac{m_A}{E_A} \frac{m_B}{E_B} \Delta H^{mm} f_A' f_B'$$

Extract unknown coupling in terms of EoS properties:

$$g_A = \sqrt{Z} \frac{E_A}{m_A} \left(\bar{H}^{-1} \right)_{mm} \frac{w_c}{n_c} \left(\frac{E_A}{w_c} - \frac{q_A}{n_c} \right), \quad \sqrt{Z} \propto \frac{\sin(\alpha_1)}{w \sin(\alpha_1 - \alpha_2)}$$

M. Pradeep and M. Stephanov, PRL (2023)

- Derive correlation length dependence on mapping parameters using universal scaling form with approximate rational critical exponents

plot ξ^2/w^2 to relate to p fluctuations:

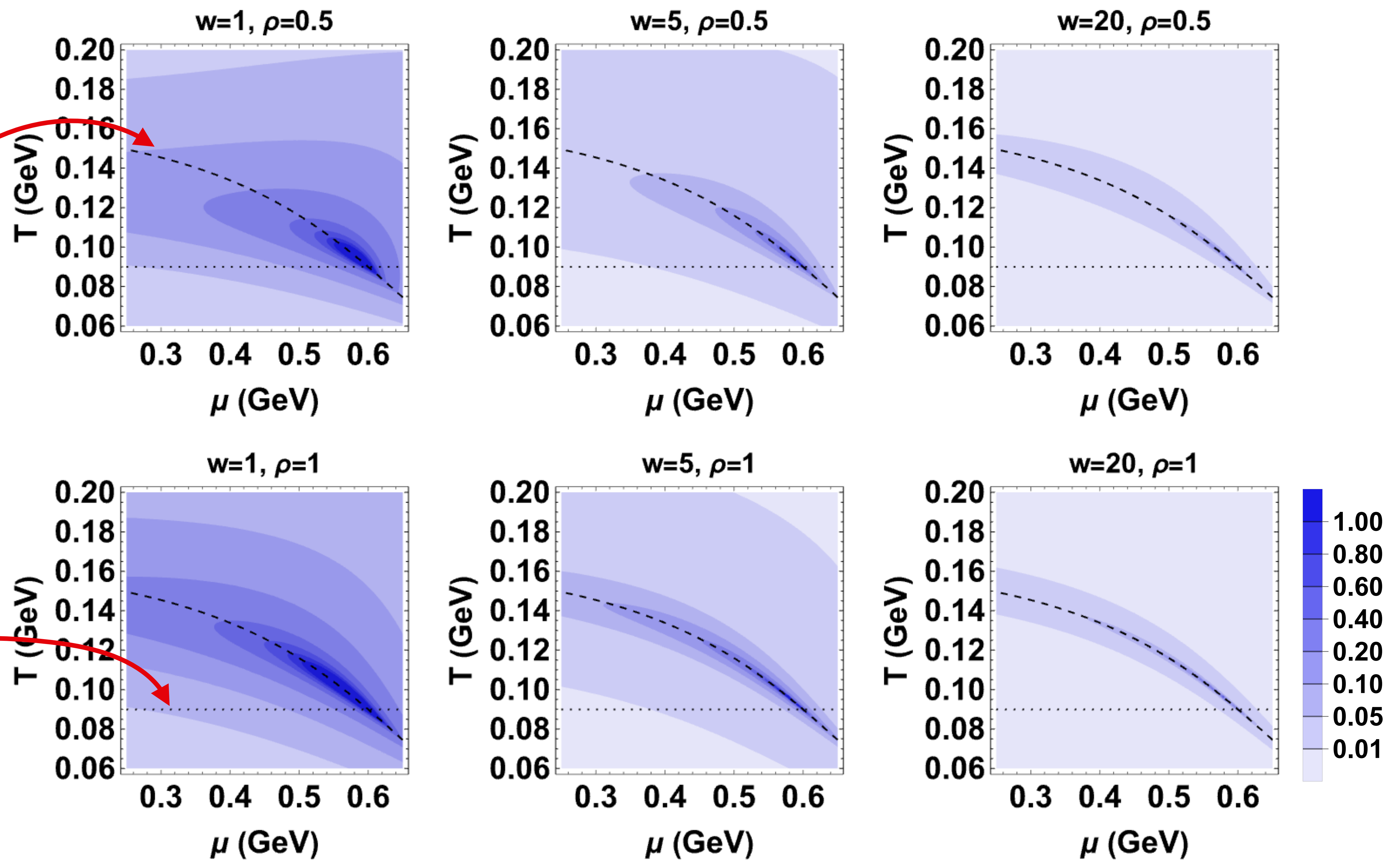
$$\langle \delta n_p \delta n_p \rangle = \langle \delta n_p^2 \rangle \propto g_p^2 \xi^{2-\eta}$$

Along r-axis:

$$\frac{\xi^2_{\rm QCD}}{w^2} \propto \frac{1}{w^{2/3}} \left(\frac{\rho}{\mu_c^2 - \mu^2} \right)^{4/3}$$

Along h-axis:

$$\frac{\xi^2_{\rm QCD}}{w^2} \propto \frac{1}{w^{6/5}(\mu_c^2 - \mu^2)^{4/5}}$$



$\mu_c = 600\,\text{MeV}, \alpha_2 = 0^{\text{o}}\,(\alpha_1 = 16.6^{\text{o}}, \text{T}_\text{c} = 90\,\text{MeV})$

JMK, K. Rajagopal, M. Pradeep, M. Stephanov, Y. Yin, PRD (2026, Editors' Suggestion)

Required Input for MaxEnt: EoS

- Maximum entropy does not require unknown parameters - takes only EoS as input
  - Because the EoS is not known yet, it has its own free parameters, e.g. mapping parameters from BEST EoS, for which we make a choice

$$\mu_c = 600\,\mathrm{MeV}, \alpha_2 = 0^o$$
$$(\alpha_1 = 16.6^o, T_c = 90\,\mathrm{MeV})$$
$$w = \{1,5,20\}\ \rho = \{0.5,1,2\}$$

__JMK,__ K. Rajagopal, M. Pradeep, M. Stephanov, Y. Yin, PRD (2026, Editors' Suggestion)

Required Input for MaxEnt: EoS

- Maximum entropy doe[...] ıly EoS as input
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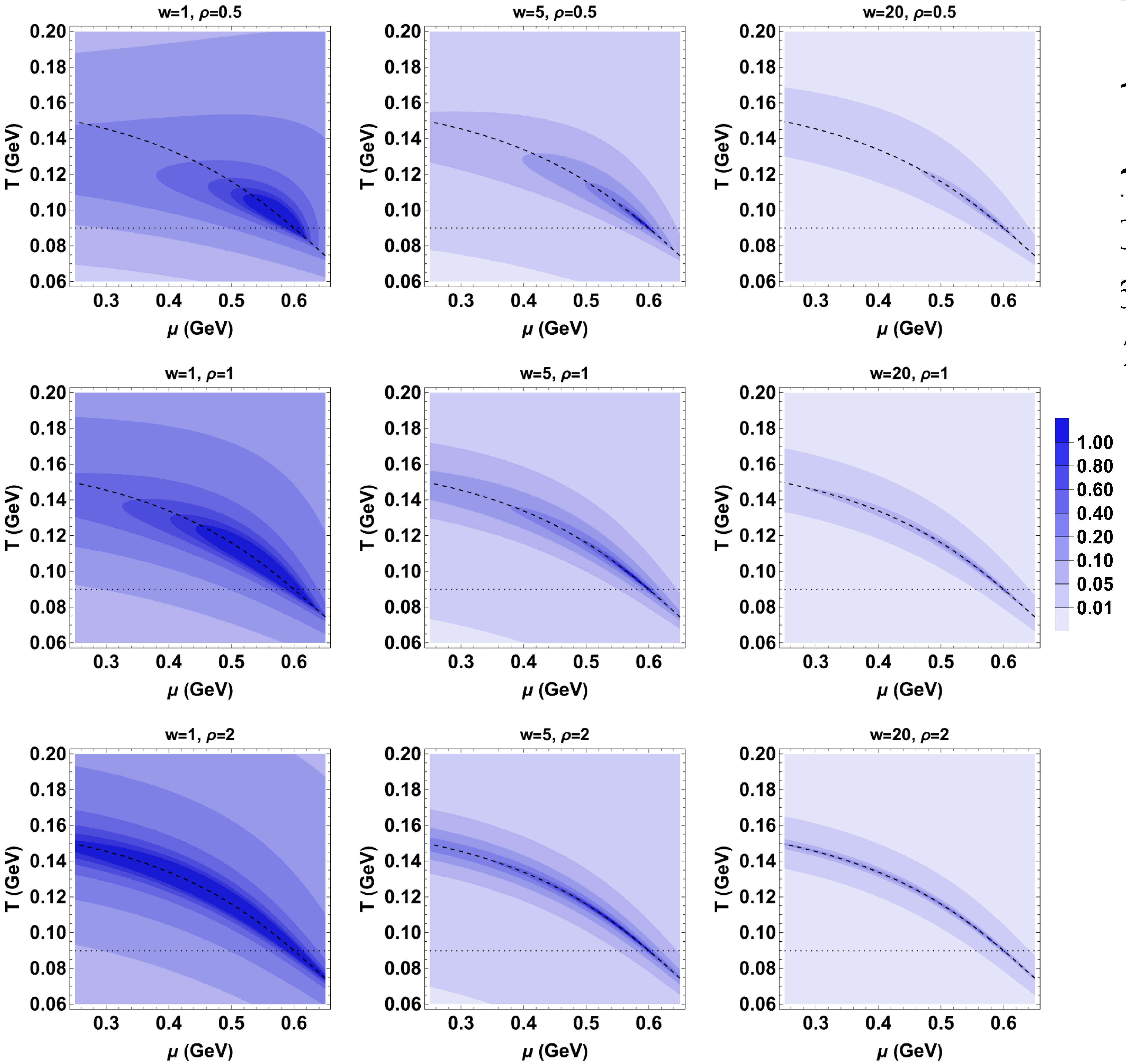
$$\hat{\Delta} G_{A_1\ldots A_k} = \hat{\Delta}\mathcal{H}_{a_1\ldots a_k} \prod_{i=1}^{k} P_{A_i}^{a_i}$$

matches fluctuations at all orders

$$\Delta H_{kn} \equiv \langle \delta n^k \rangle$$

$$k = 2$$



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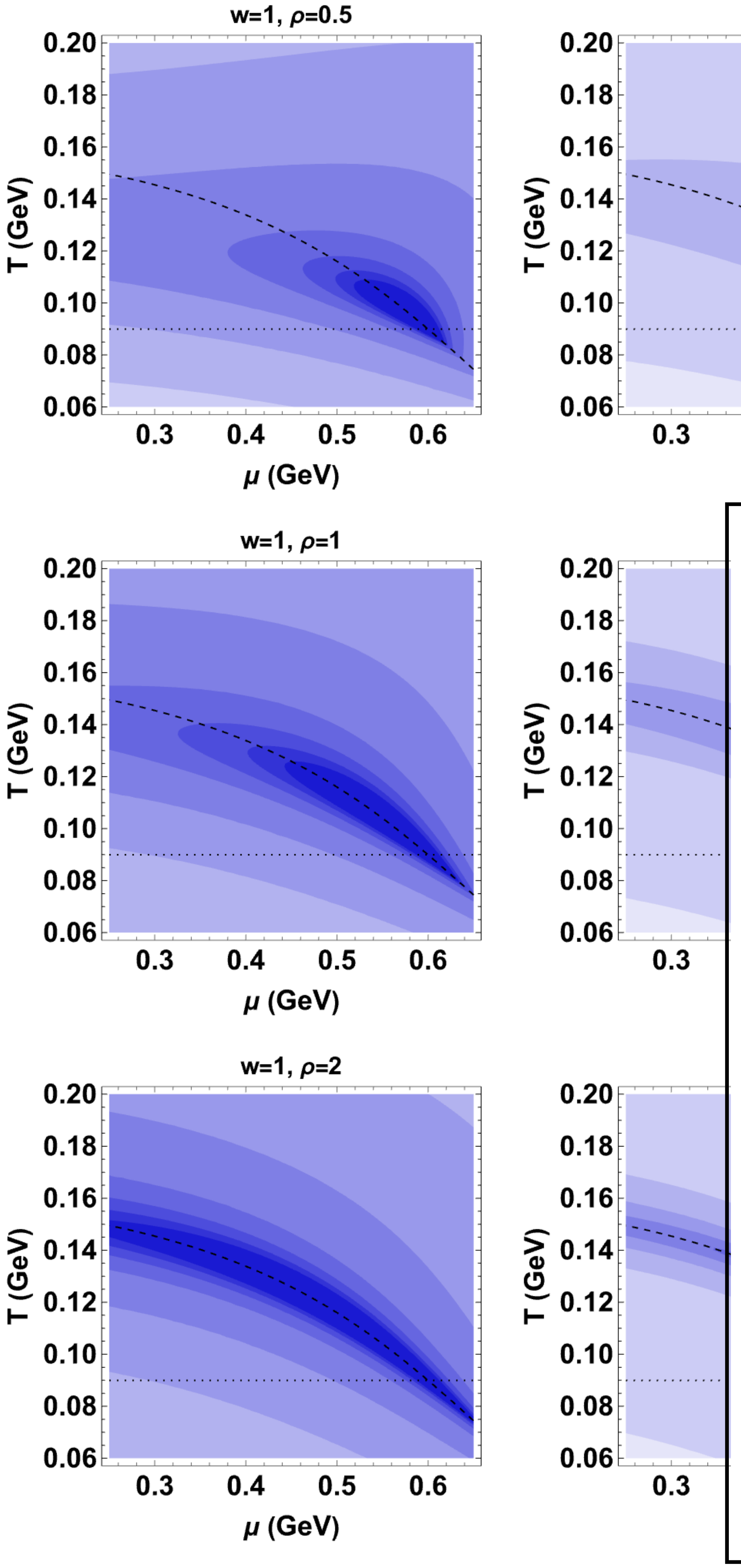

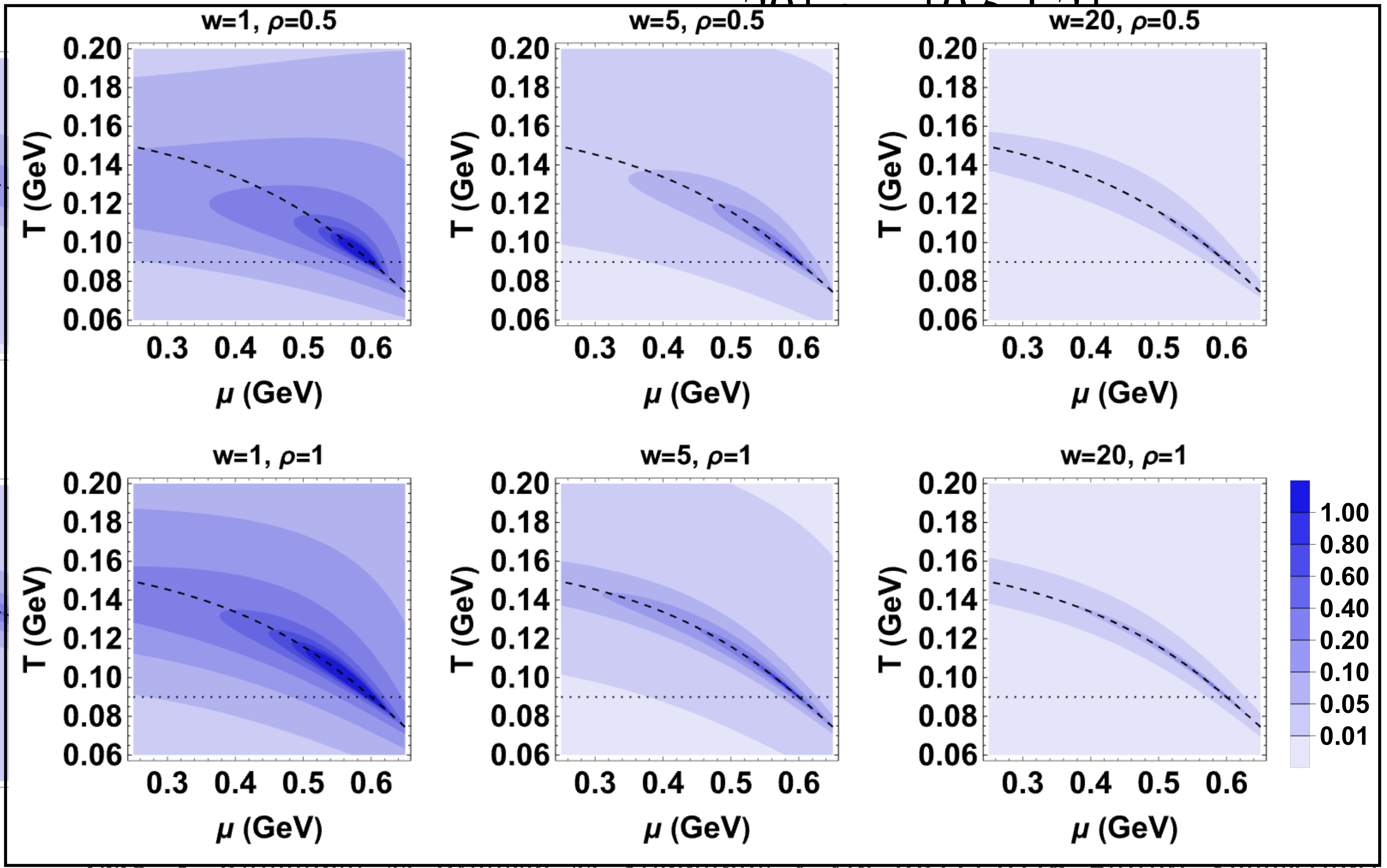

JMK, K. Rajagopal, M. Pradeep, M. Stephanov, Y. Yin, PRD (2026, Editors' Suggestion)

Required Input for MaxEnt: EoS

- Maximum entropy does … ly EoS as input
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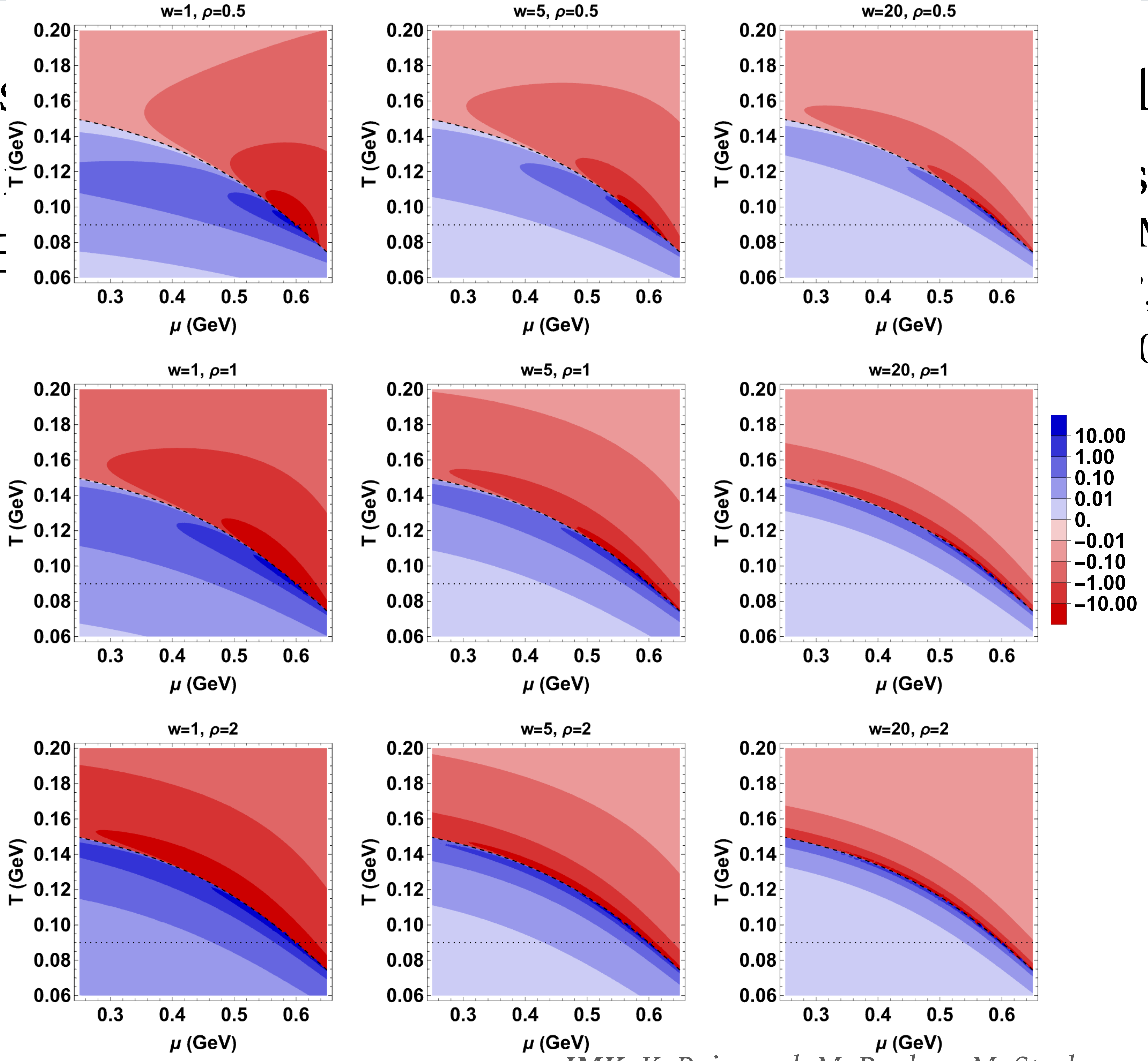

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matches fluctuations at all orders

$$\Delta H_{kn} \equiv \langle \delta n^k \rangle$$

$$k = 3$$

***JMK,** K. Rajagopal, M. Pradeep, M. Stephanov, Y. Yin, PRD (2026, Editors' Suggestion)*

Required Input for MaxEnt: EoS

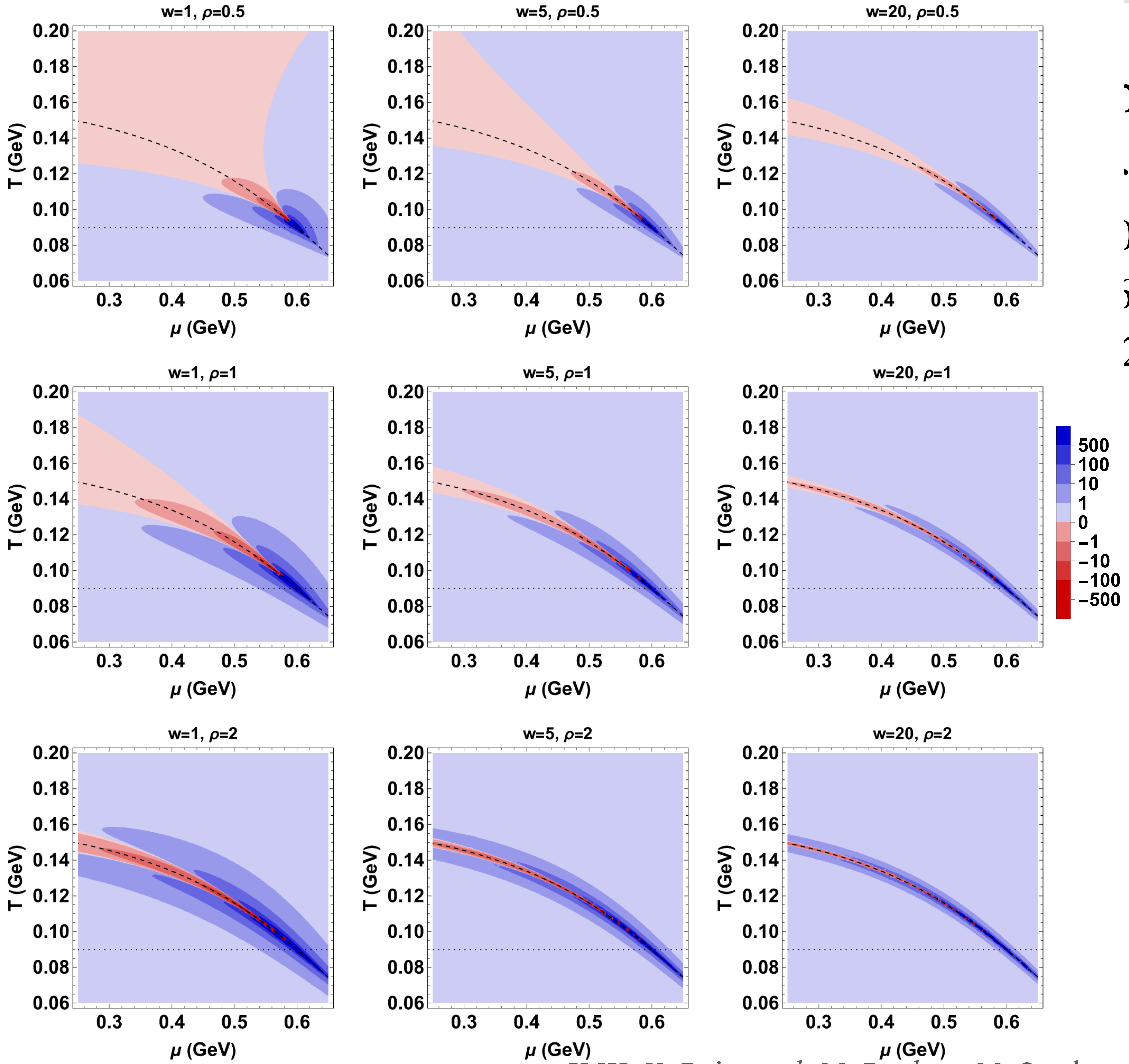
- ➤ Maximum entropy doe… …ly EoS as input
  - ➤ Because the EoS is … …s, e.g. mapping parameters from BI…
    …) MeV, $\alpha_2 = 0^o$
    …^{o}, $T_c = 90\,\mathrm{MeV}$)
    …20} $\rho = \{0.5,1,2\}$

$$\hat{\Delta}G_{A_1\ldots A_k} = \hat{\Delta}\mathcal{H}_{a_1\ldots a_k}\prod_{i=1}^{k} P_{A_i}^{a_i}$$

matches fluctuations at all orders

$$\Delta H_{kn} \equiv \langle \delta n^k \rangle$$

$$k = 4$$



JMK, K. Rajagopal, M. Pradeep, M. Stephanov, Y. Yin, PRD (2026, Editors' Suggestion)

Quantifying Critical Fluctuation Signatures

- Extract critical contribution to proton multiplicity fluctuations from an Ising-like critical point & study influence of unknown EoS parameters
  - Focus on deviations from Hadron Resonance Gas background of EoS close to critical point where singular part dominates

$$P(\mu, T) = P^{\text{reg}}(\mu, T) + P^{\text{sing}}(\mu, T)$$

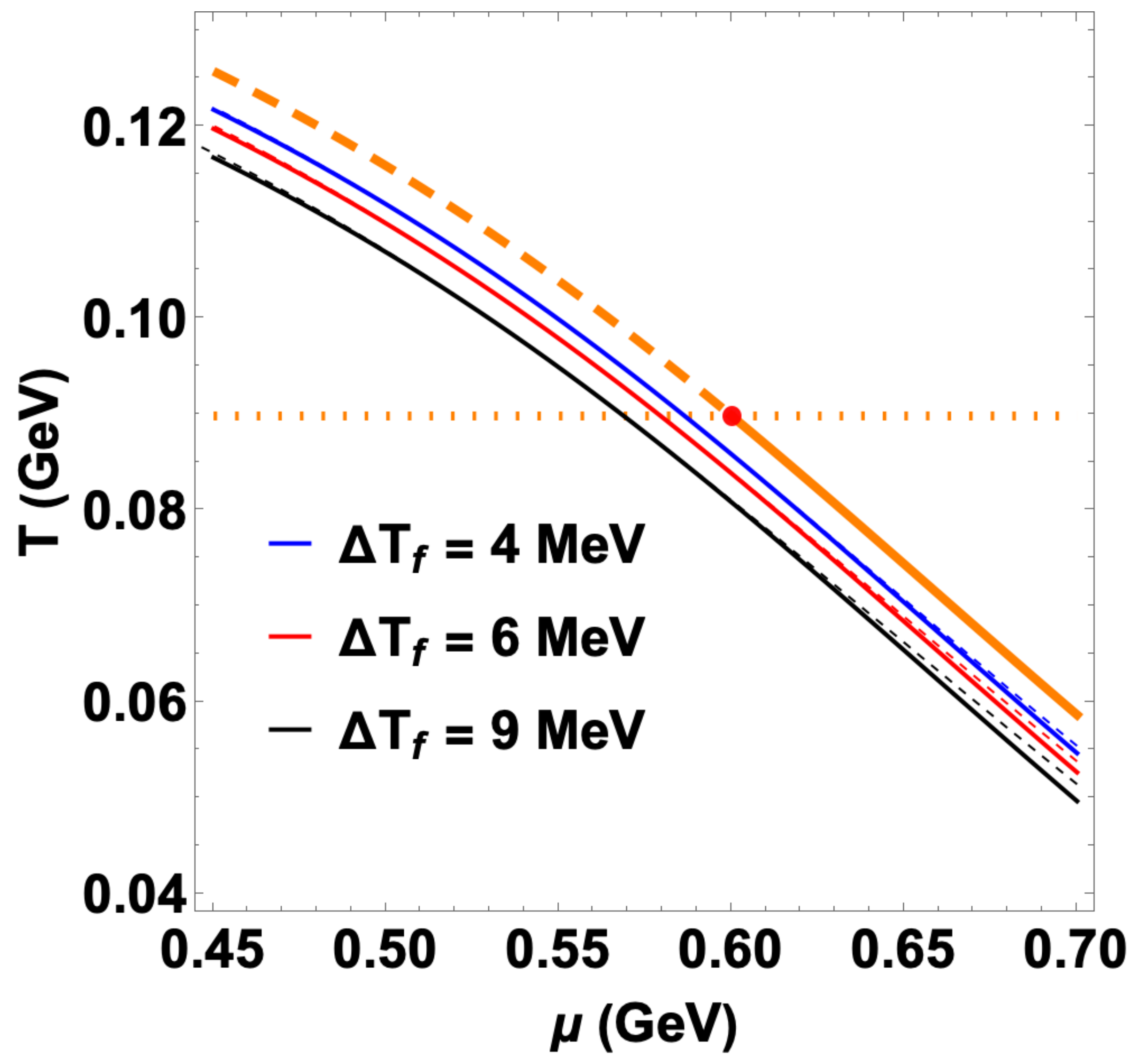
HRG   3D Ising model

  - Use quadratic map for Ising transition line

M. Kahangirwe et al, PRD (2024)

  - Calculate multiplicity fluctuations along freeze-out lines

$$T_f(\mu_B) = T_{\text{crossover}}(\mu_B) - \Delta T_f$$



***JMK,** K. Rajagopal, M. Pradeep, M. Stephanov, Y. Yin, PRD (2026, Editors' Suggestion)*

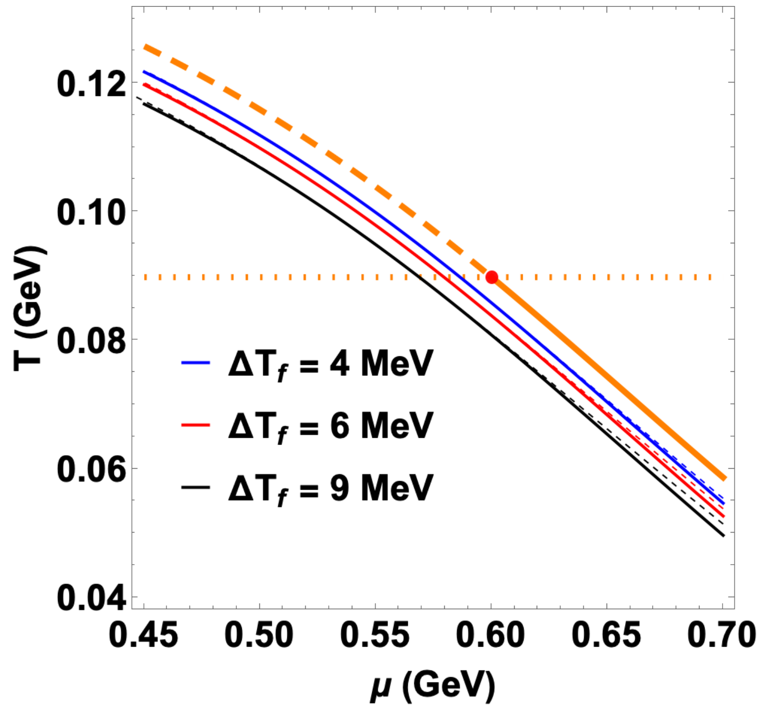
Proton Factorial Cumulants from MaxEnt

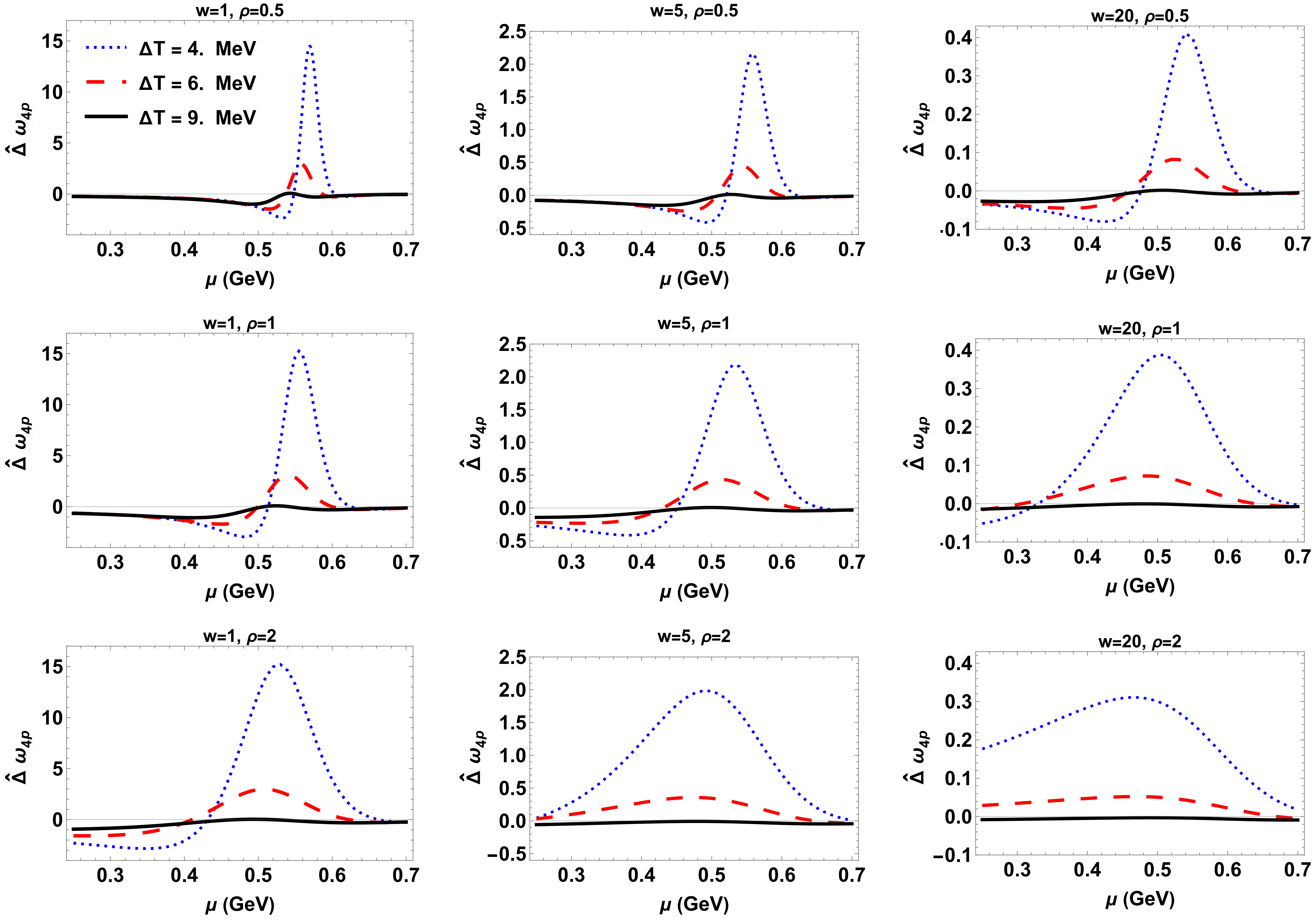
➤ Critical contribution to fourth order proton factorial cumulants in equilibrium for range of EoS parameters and freeze-out scenarios

Normalized proton factorial cumulants:

$$\widehat{\Delta}\omega_{kp} = \frac{\widehat{\Delta}H_{a_1\ldots a_k}P_p^{a_1}\ldots P_p^{a_k}}{\langle N_p\rangle} = \frac{\kappa_{4p}}{\kappa_{1p}}$$

Along freeze-out curves:





JMK, K. Rajagopal, M. Pradeep, M. Stephanov, Y. Yin, PRD (2026, Editors' Suggestion)

Proton Factorial Cumulants from MaxEnt

➤ Critical contribution to fourth order proton factorial cumulants in equilibrium for range of EoS parameters and freeze-out scenarios

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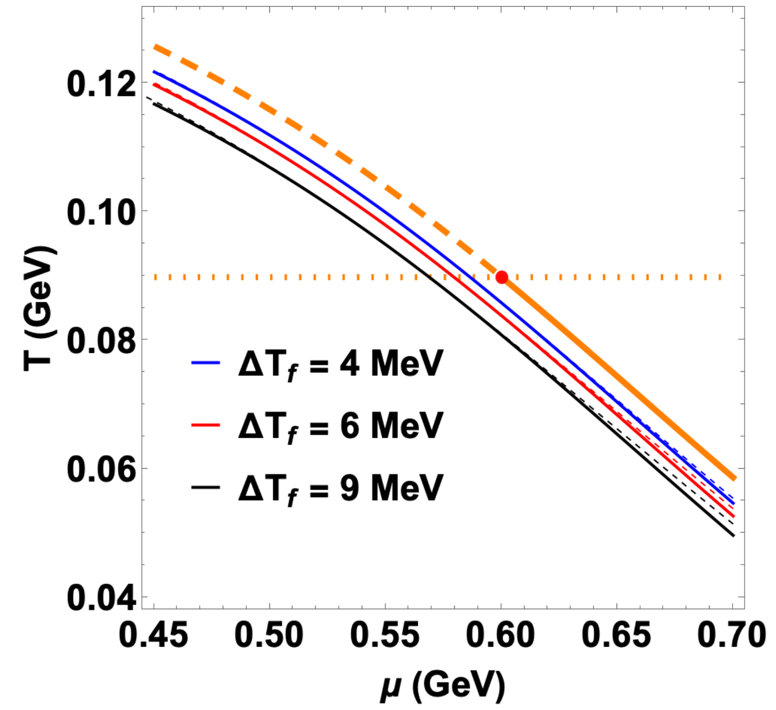

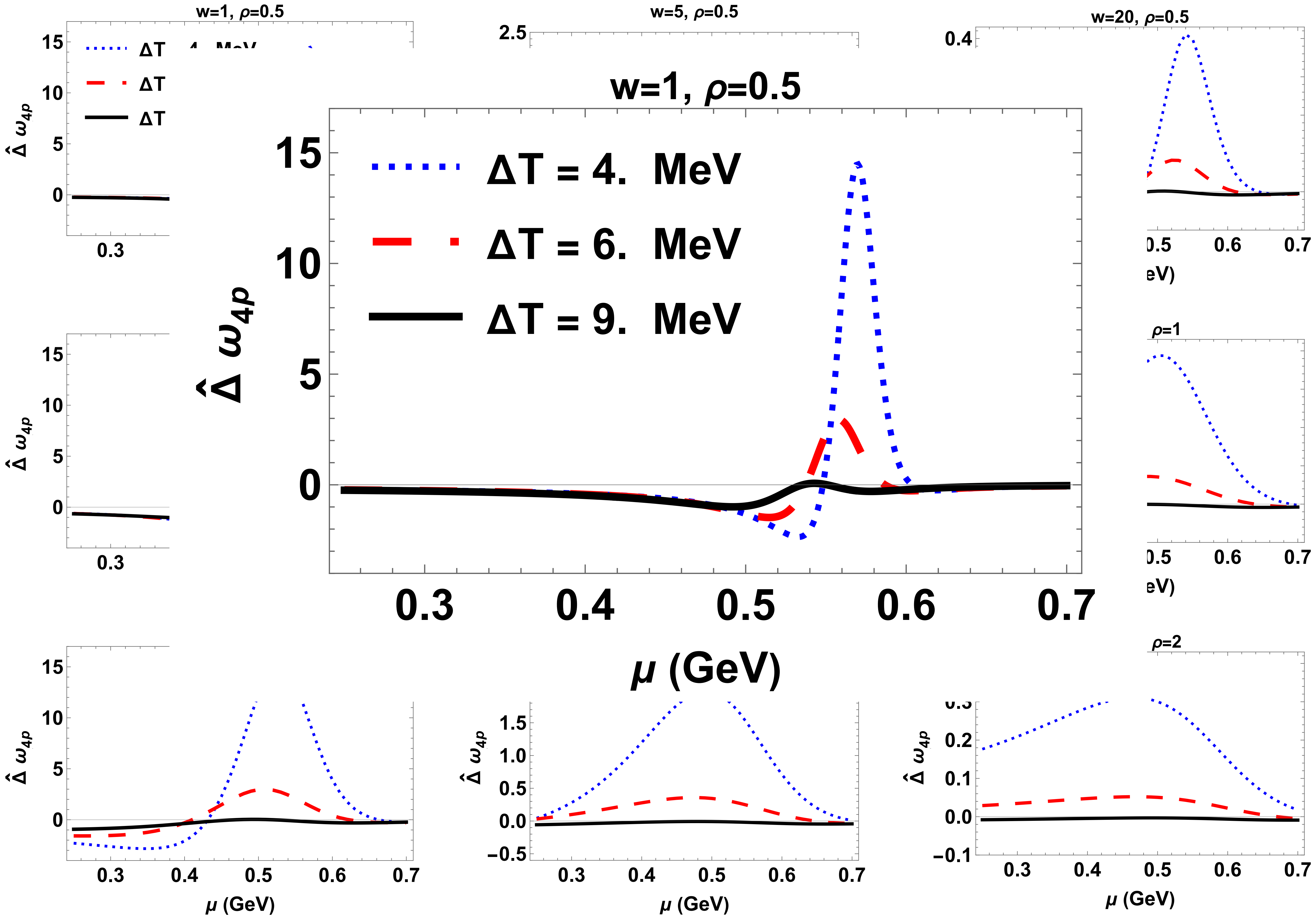

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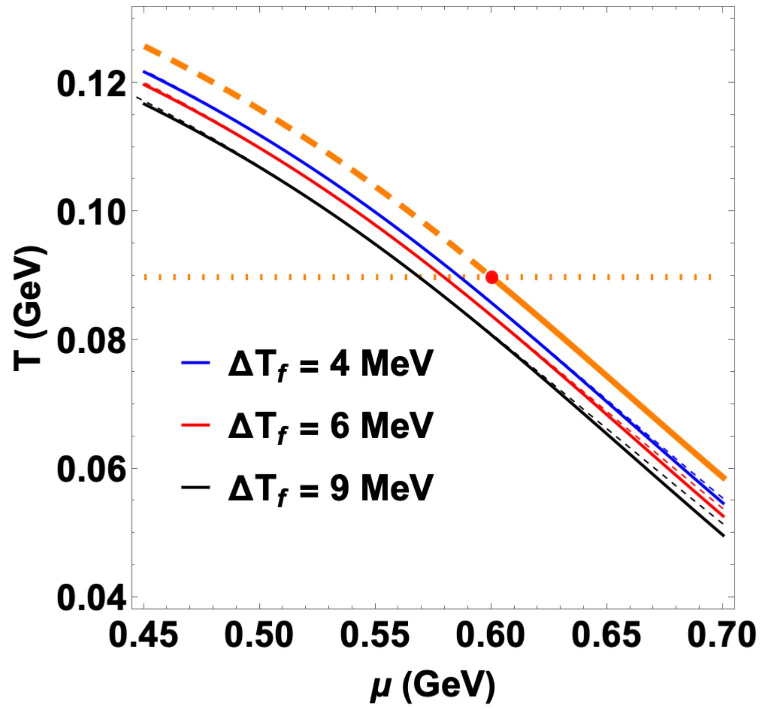
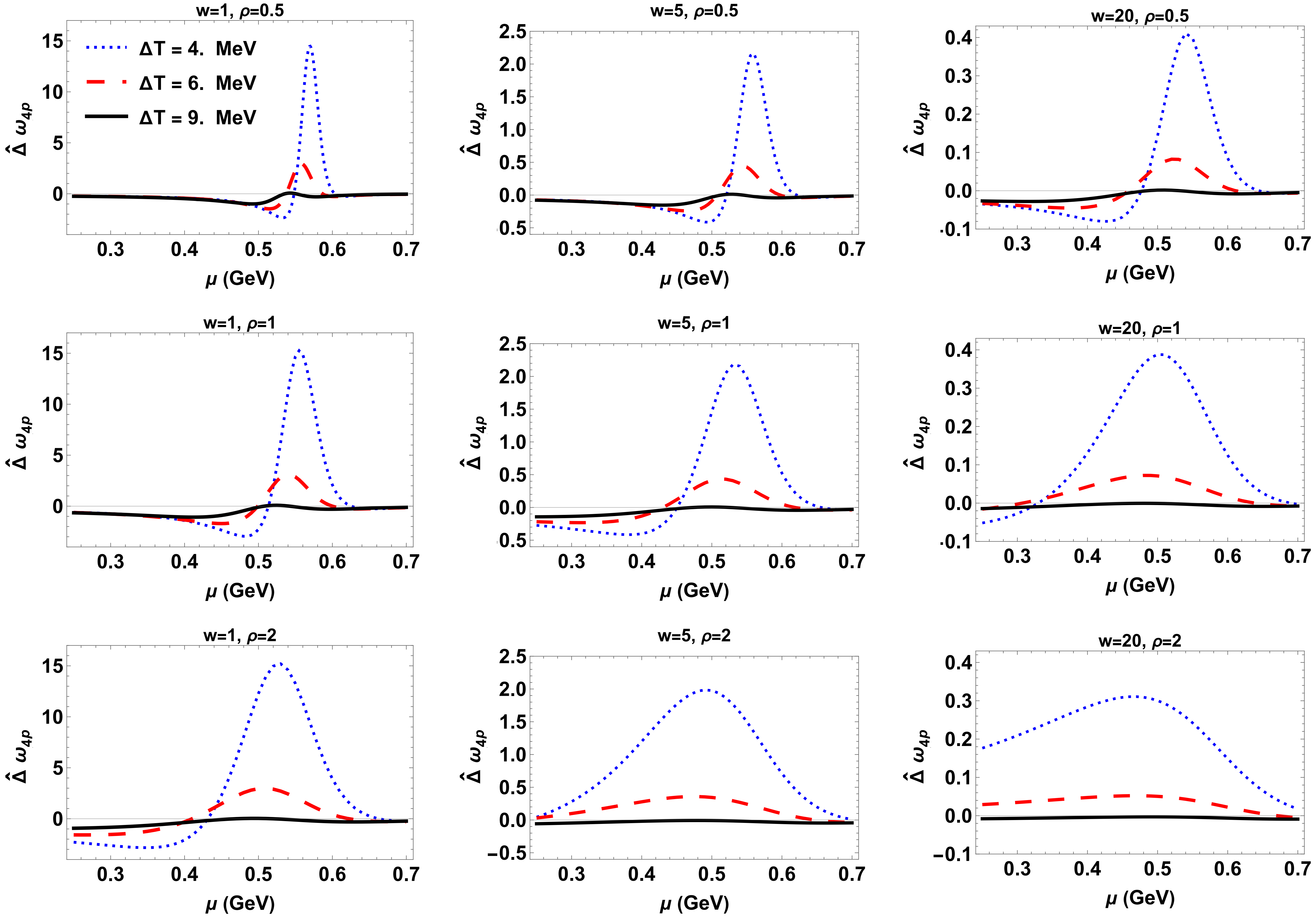
Along freeze-out curves:





Increasing ρ increases peak width

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Proton Factorial Cumulants from MaxEnt

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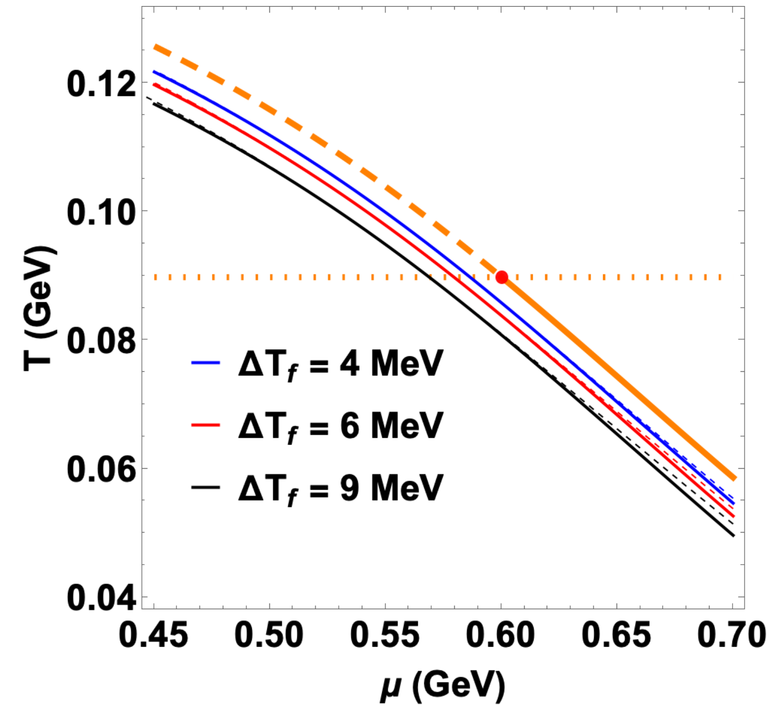

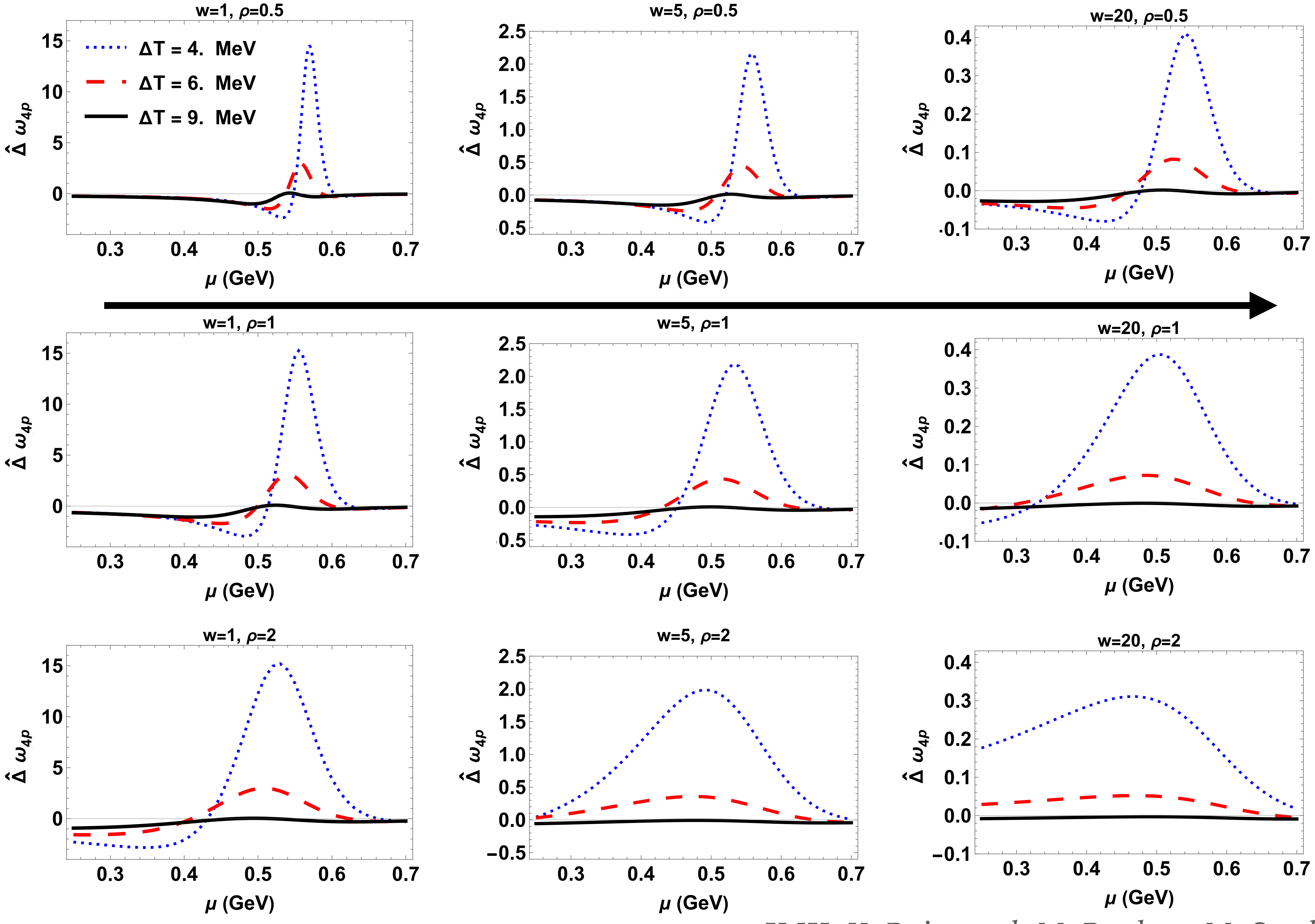

Increasing w reduces peak height

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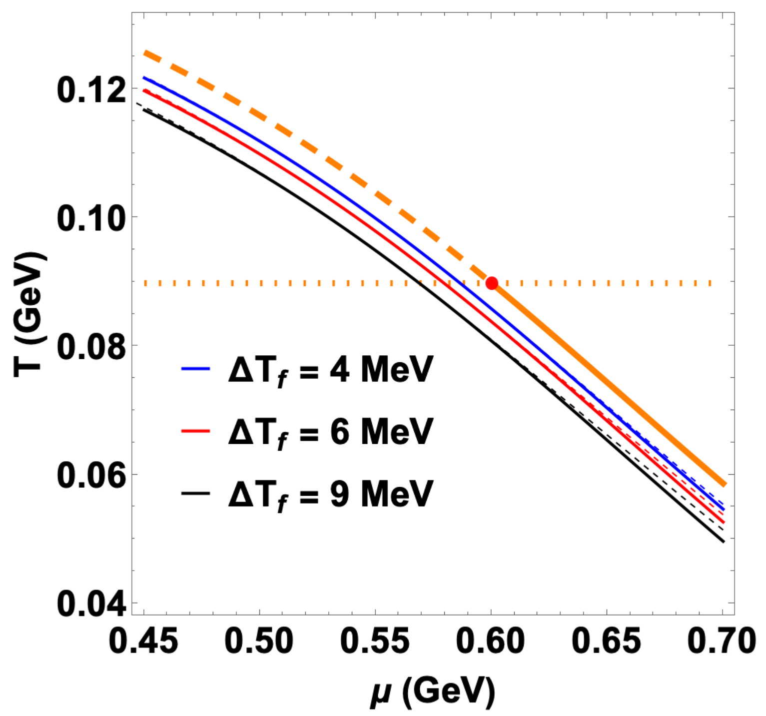
Proton Factorial Cumulants from MaxEnt

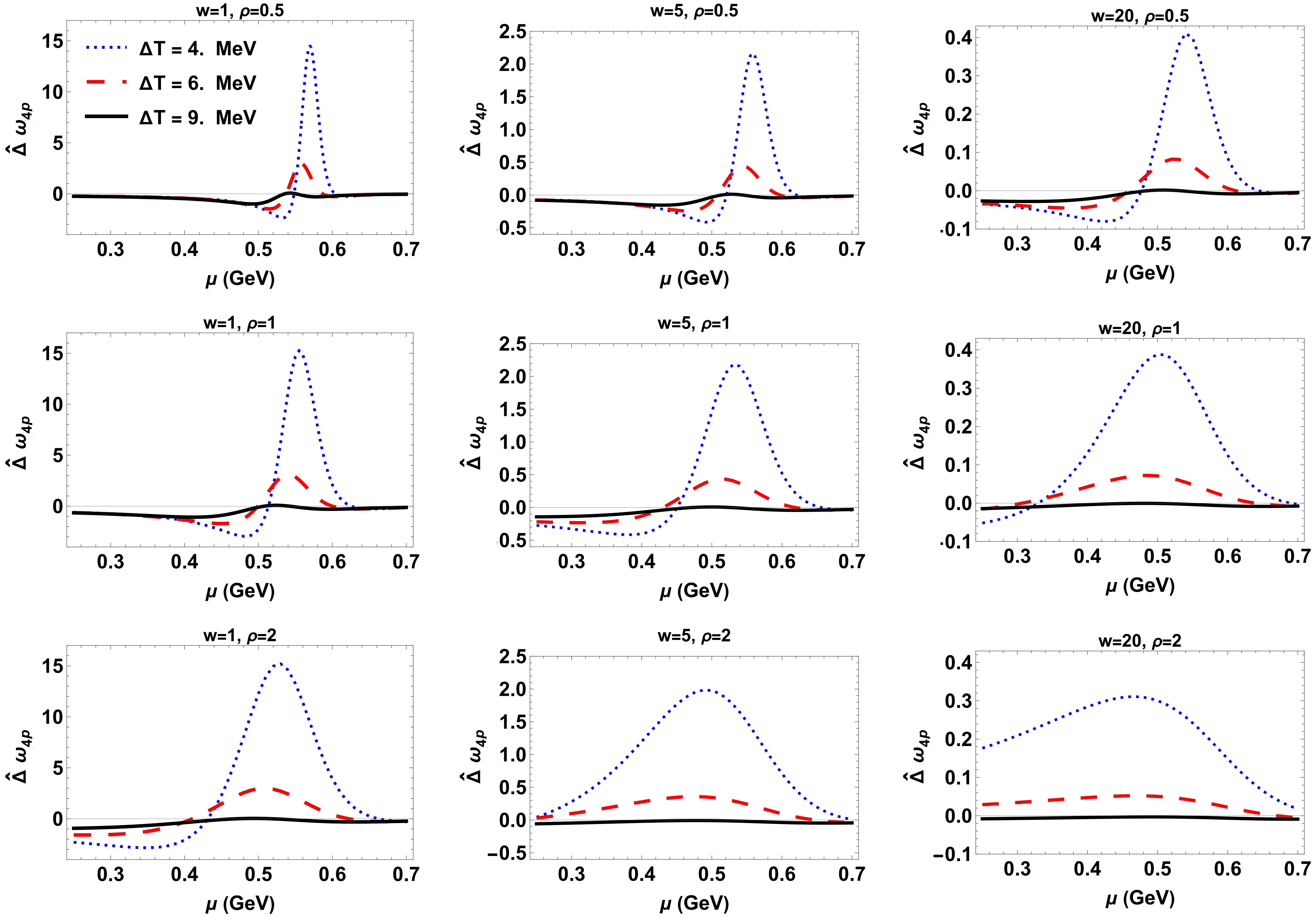
- Critical contribution to fourth order proton factorial cumulants in equilibrium for range of EoS parameters and freeze-out scenarios

Normalized proton factorial cumulants:

$$\widehat{\Delta}\omega_{kp} = \frac{\widehat{\Delta}H_{a_1\ldots a_k}P_p^{a_1}\ldots P_p^{a_k}}{\langle N_p\rangle}$$
$$= \frac{\kappa_{4p}}{\kappa_{1p}}$$

Along freeze-out curves:





Scaling of magnitude:
$\hat{\Delta}\omega_p^k \sim w^{-1.2}\Delta T_f^{1.2-k} \sim \xi^{k(5-\eta)/2-3}$

Uncertainty in ΔT_f: factorial cumulants can vary by order of magnitude

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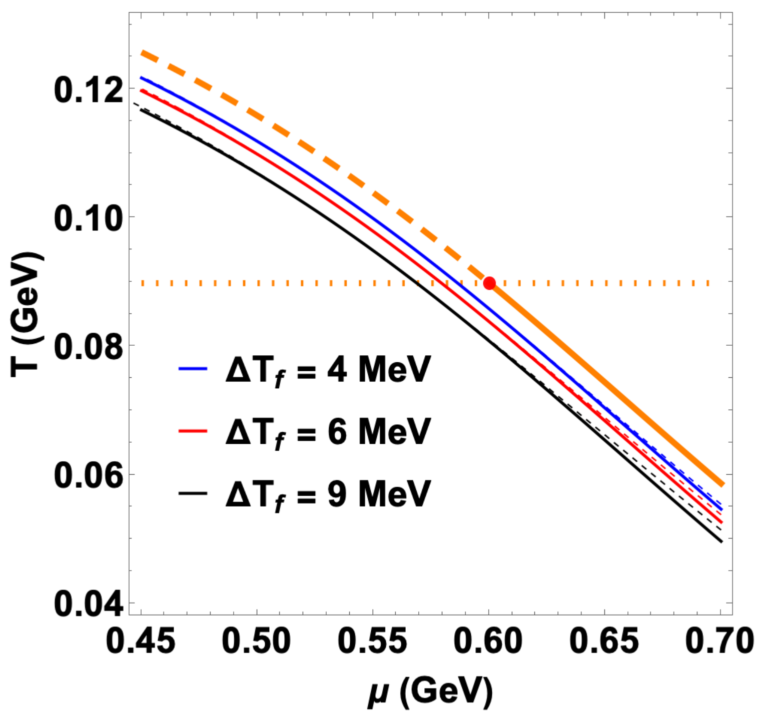
Proton Factorial Cumulants from MaxEnt

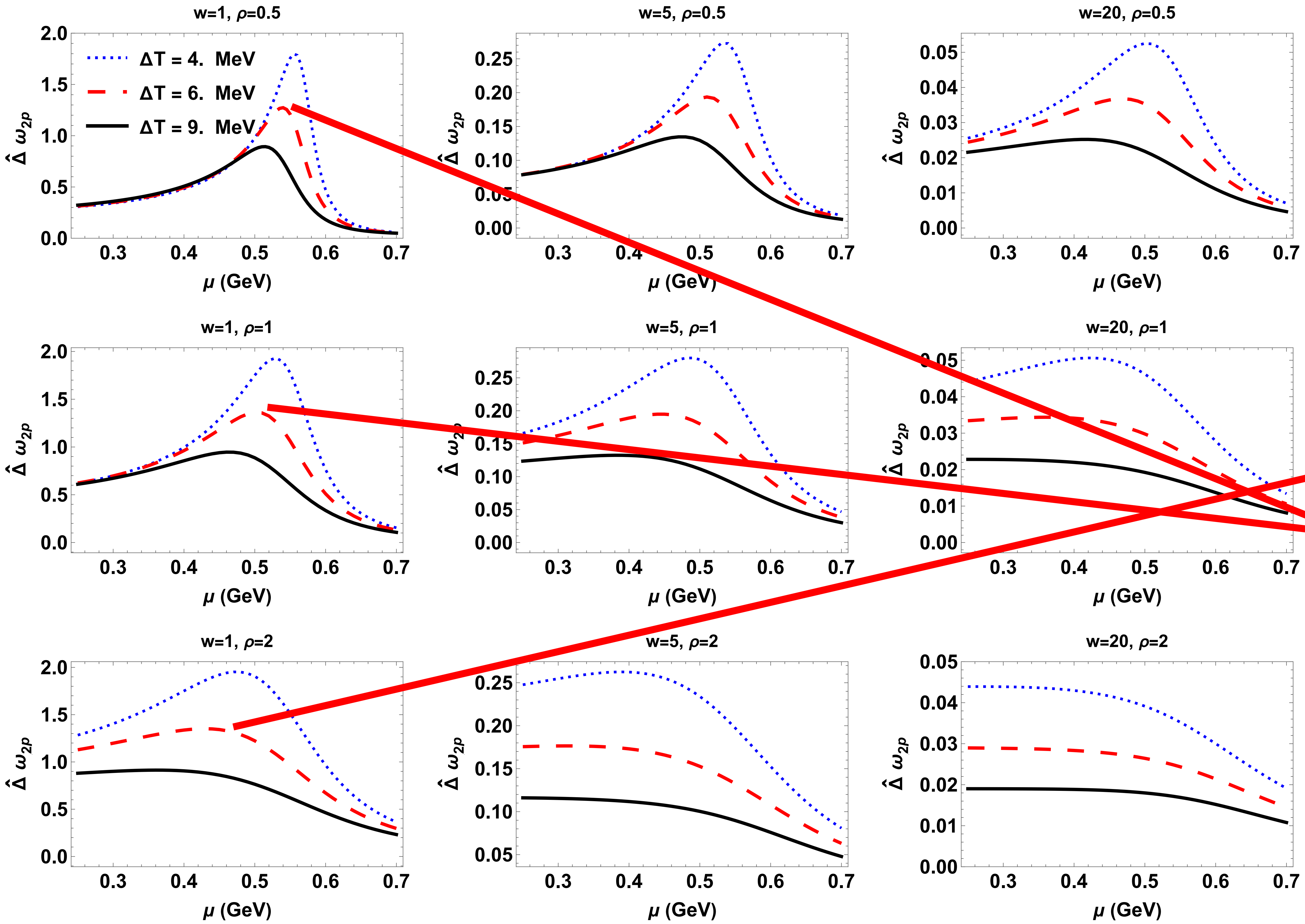
➤ Critical contribution to second order proton factorial cumulants in equilibrium for range of EoS parameters and freeze-out scenarios

Normalized proton factorial cumulants:

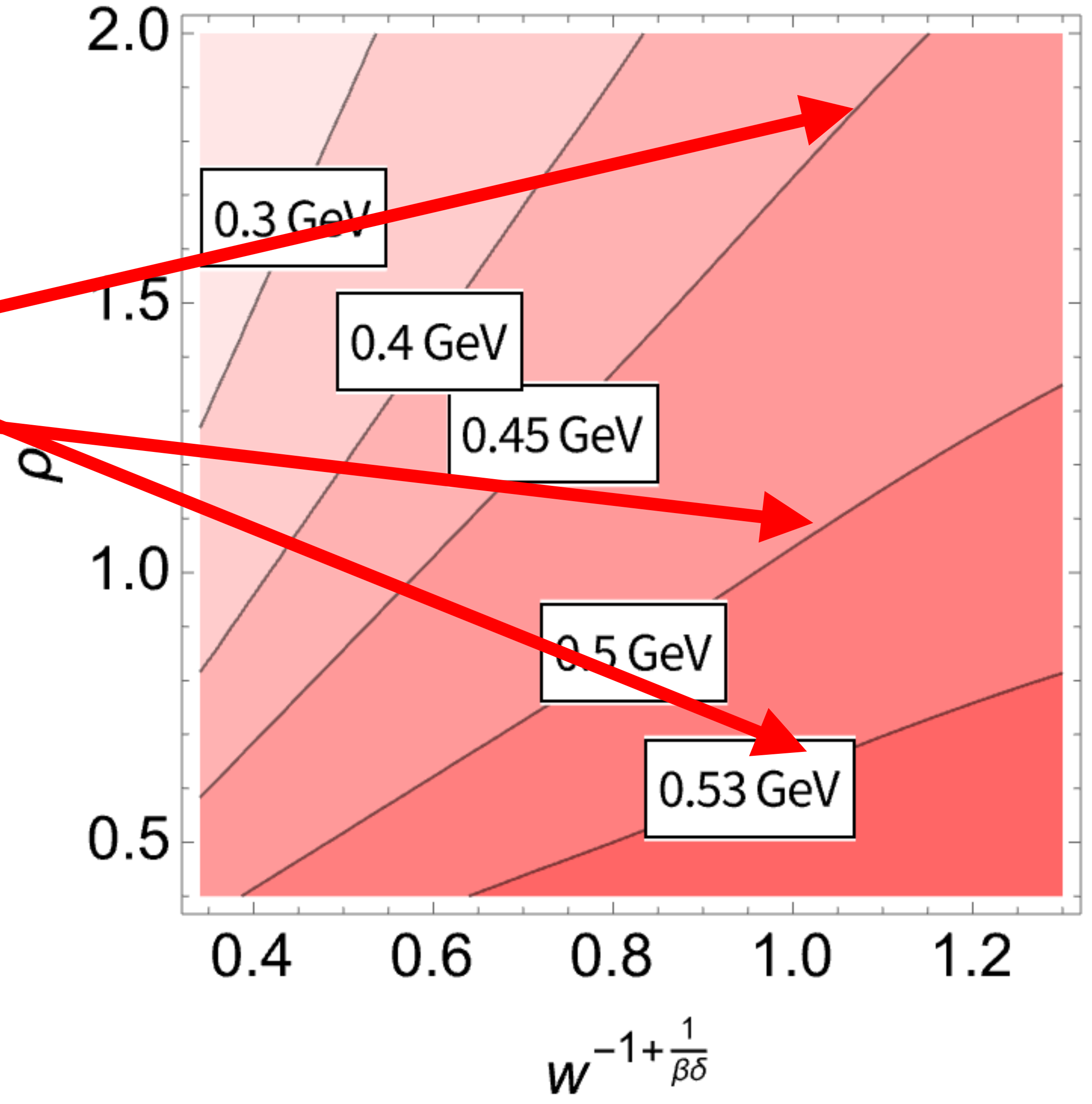
$$\widehat{\Delta}\omega_{kp} = \frac{\widehat{\Delta}H_{a_1\ldots a_k} P_p^{a_1} \ldots P_p^{a_k}}{\langle N_p \rangle} = \frac{\kappa_{4p}}{\kappa_{1p}}$$

Along freeze-out curves:





Scaling of location: $\rho w^{1-1/\beta\delta}$

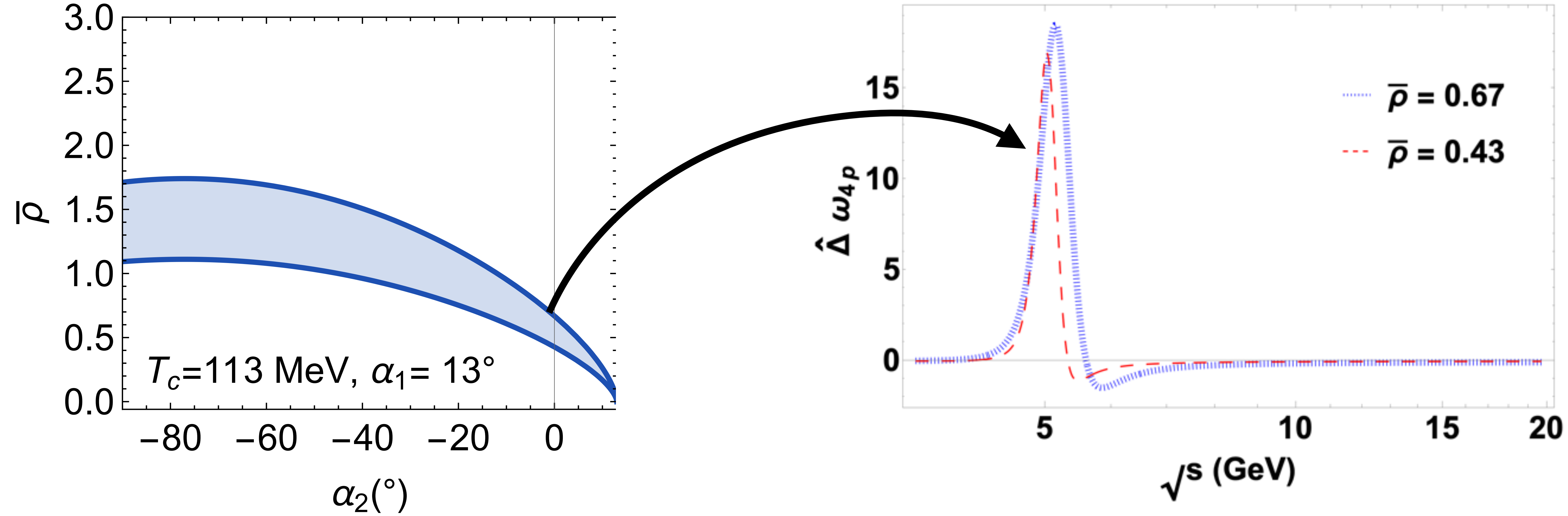


JMK, K. Rajagopal, M. Pradeep, M. Stephanov, Y. Yin, PRD (2026, Editors' Suggestion)

Other Constraints on EoS from $\hat{\omega}_{kp}$ with MaxEnt

- Padé estimates for CP can also constrain (via imaginary part, c_2) the combination of non-universal parameters → fix $T_c\,, \alpha_1\,, \alpha_2$ to determine $\bar{\rho} = \rho w^{1-1/(\beta\delta)}$

$$\bar{\rho} = \left(\frac{1}{c_2 T_c^{\beta\delta-1}} \frac{|\sin\alpha_{12}|}{|\sin\alpha_1|^{\beta\delta+1}} \right)^{1/\beta\delta}$$



Uncertainty on Lattice Padé estimates translates to spread of $\bar{\rho}$
with location of critical signal stable

G. Basar, M. Pradeep, M. Stephanov, arXiv:2603.23635

Future Work: Critical Effects Beyond Equilibrium

- Direct protons
  - ✓ Qualitative results persist with resonance feed-down (Appendix B)
- Current background EoS is HRG
  - ➡ Extend to more realistic EoS constrained by lattice
  - ➡ Compare to non-critical baseline

    P. Braun-Munzinger, B. Friman et al, NPA (2021)
    V. Vovchenko, V. Koch & C. Shen, PRC (2022)
- Assumption of thermal equilibrium
  - ➡ Full Hydro+ simulations

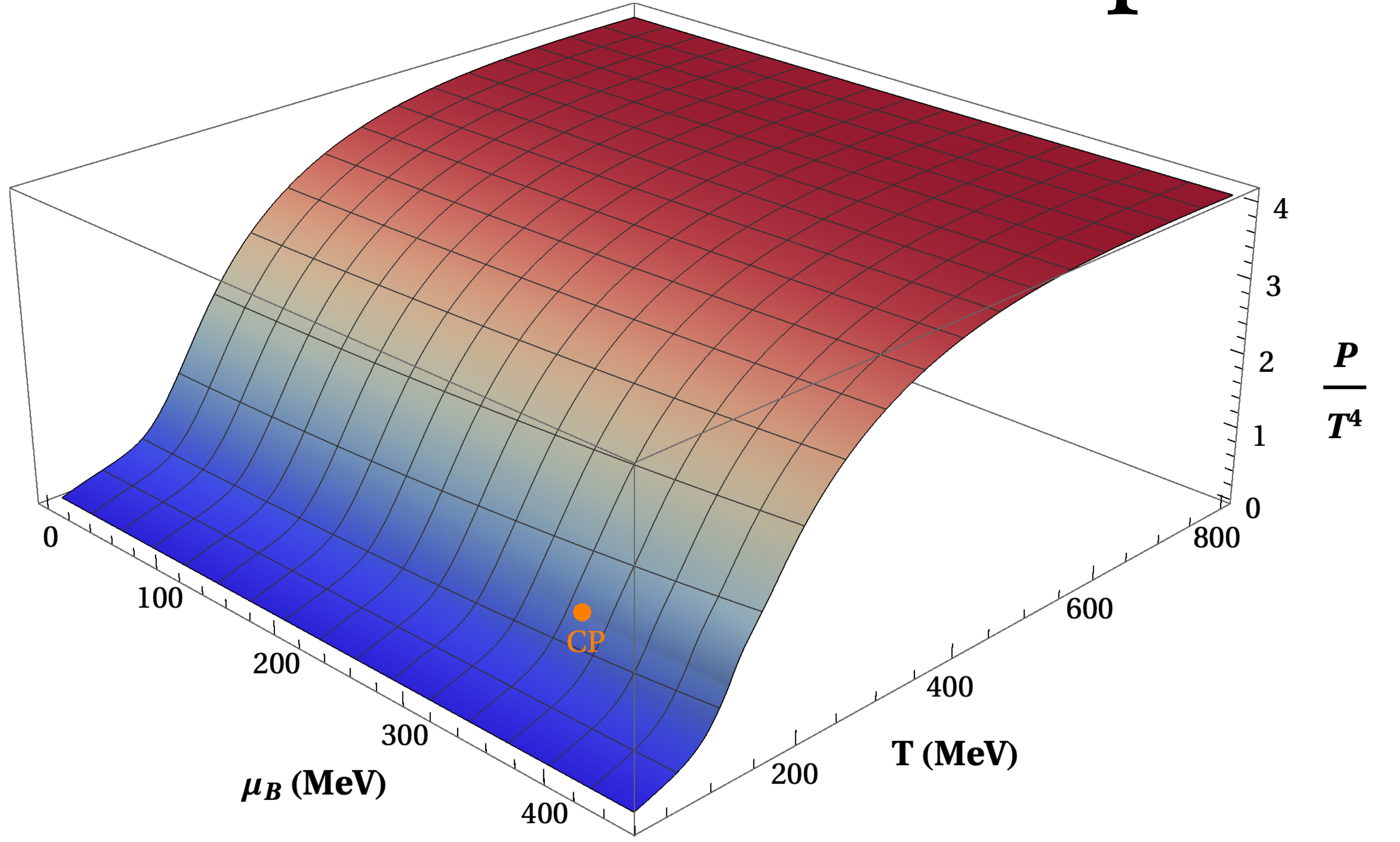
    M. Stephanov and Y. Yin, PRD (2018)
    K. Rajagopal, G. Ridgway et al, PRD (2020)
    M. Pradeep, K. Rajagopal et al, PRD (2022)
- Simultaneous particlization and chemical freeze-out
  - ➡ Couple to hadronic afterburner

    Critical effects seen to survive in SMASH:
    J. Hammelman, M. Bluhm et al, PRC (2024)

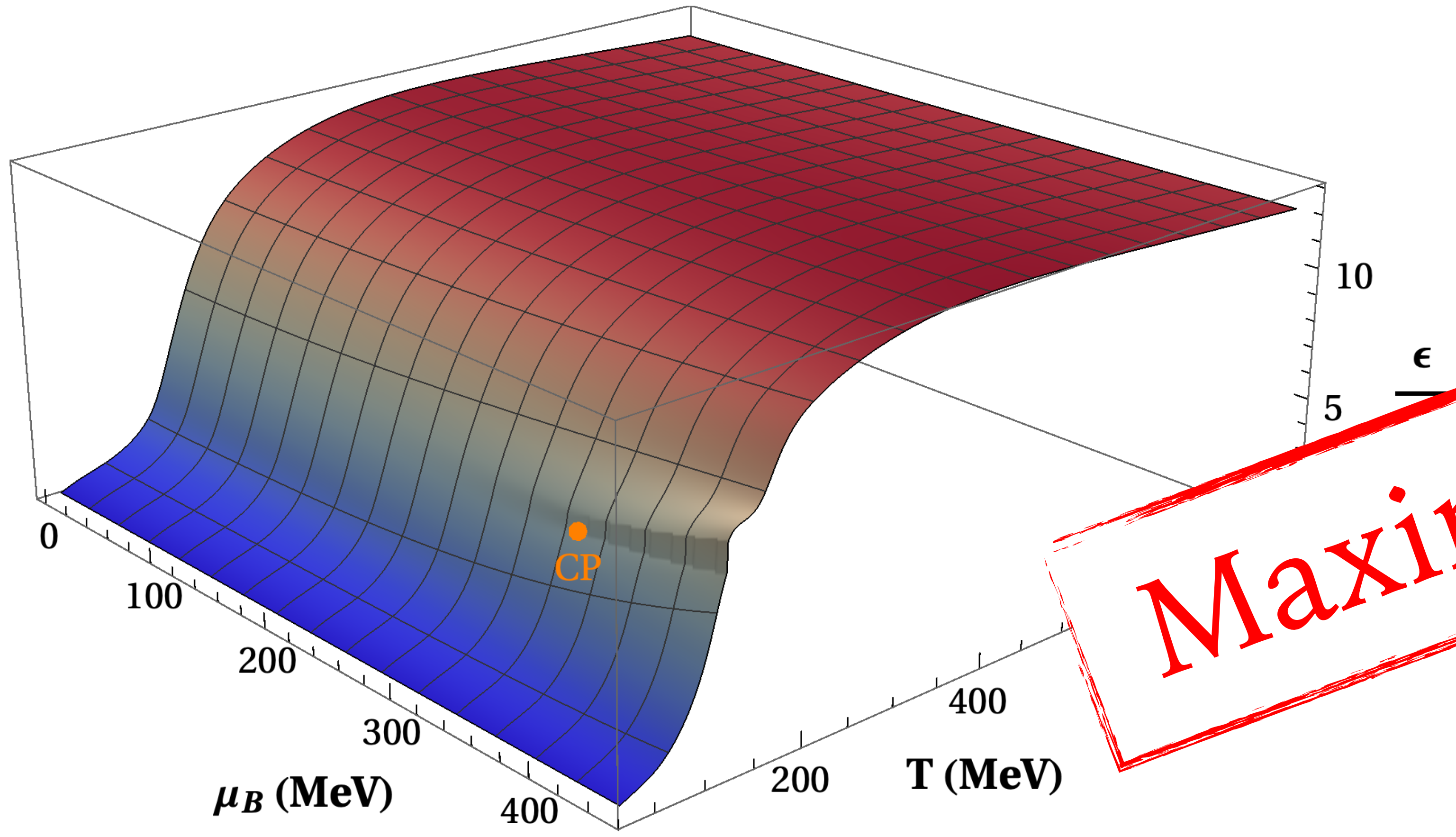
***JMK,** K. Rajagopal, M. Pradeep, M. Stephanov, Y. Yin, PRD (2026, Editors' Suggestion)*

How to use EoS to connect to experiment?

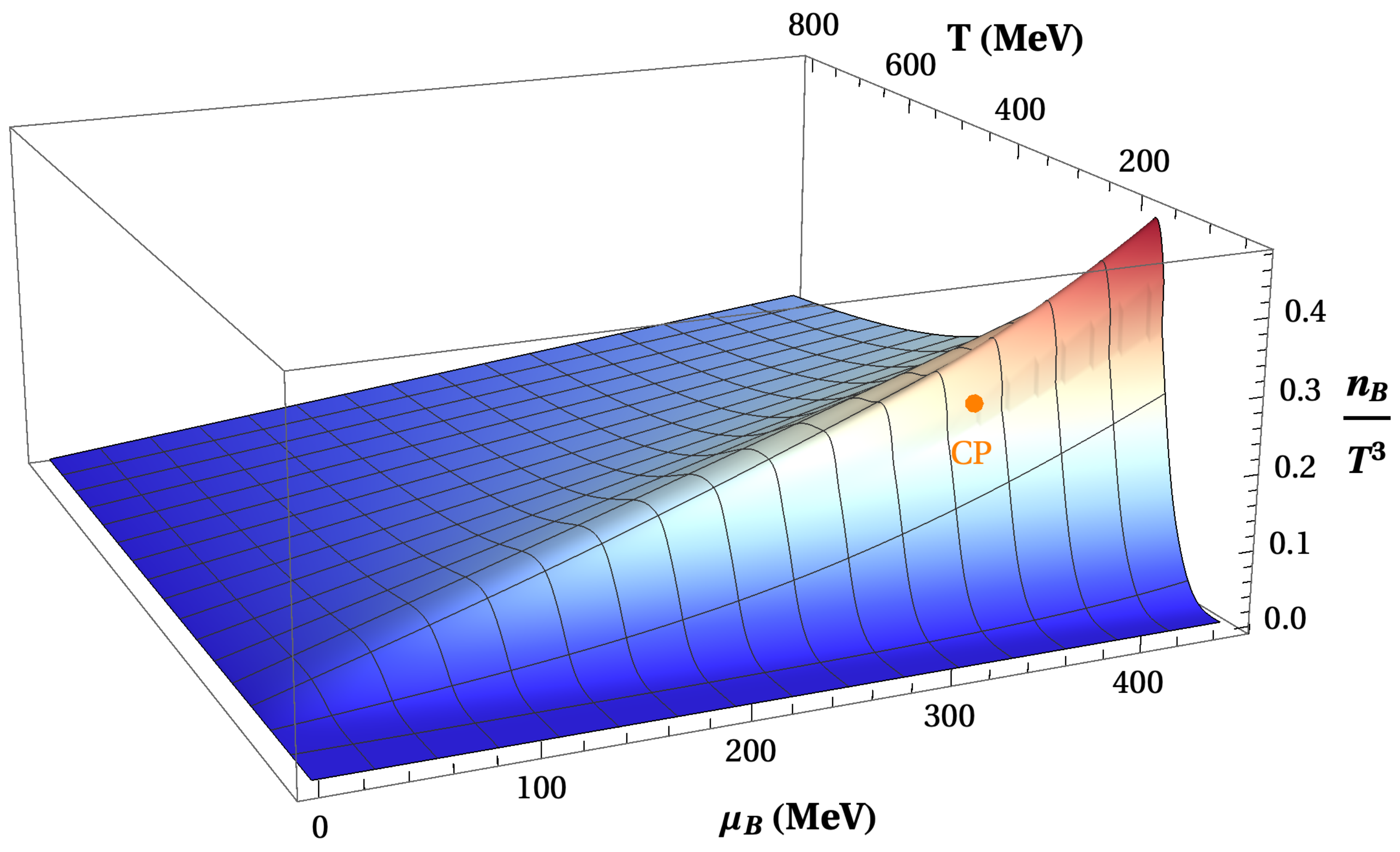
… in a model independent way



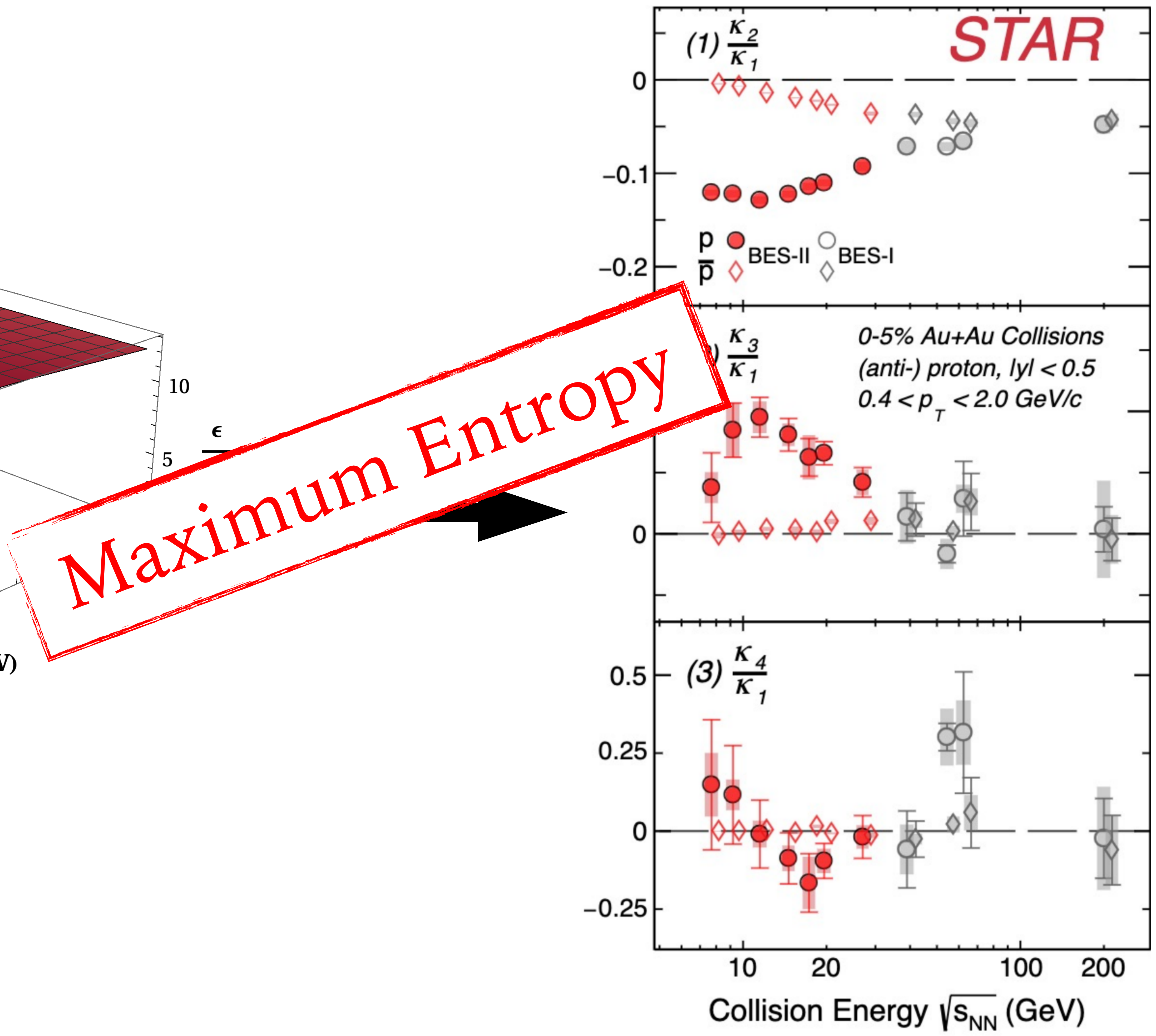
Pressure



Energy density



Baryon density



JMK et al, EPJ+ (2021)
A. Pandav (STAR collaboration), CPOD 2024

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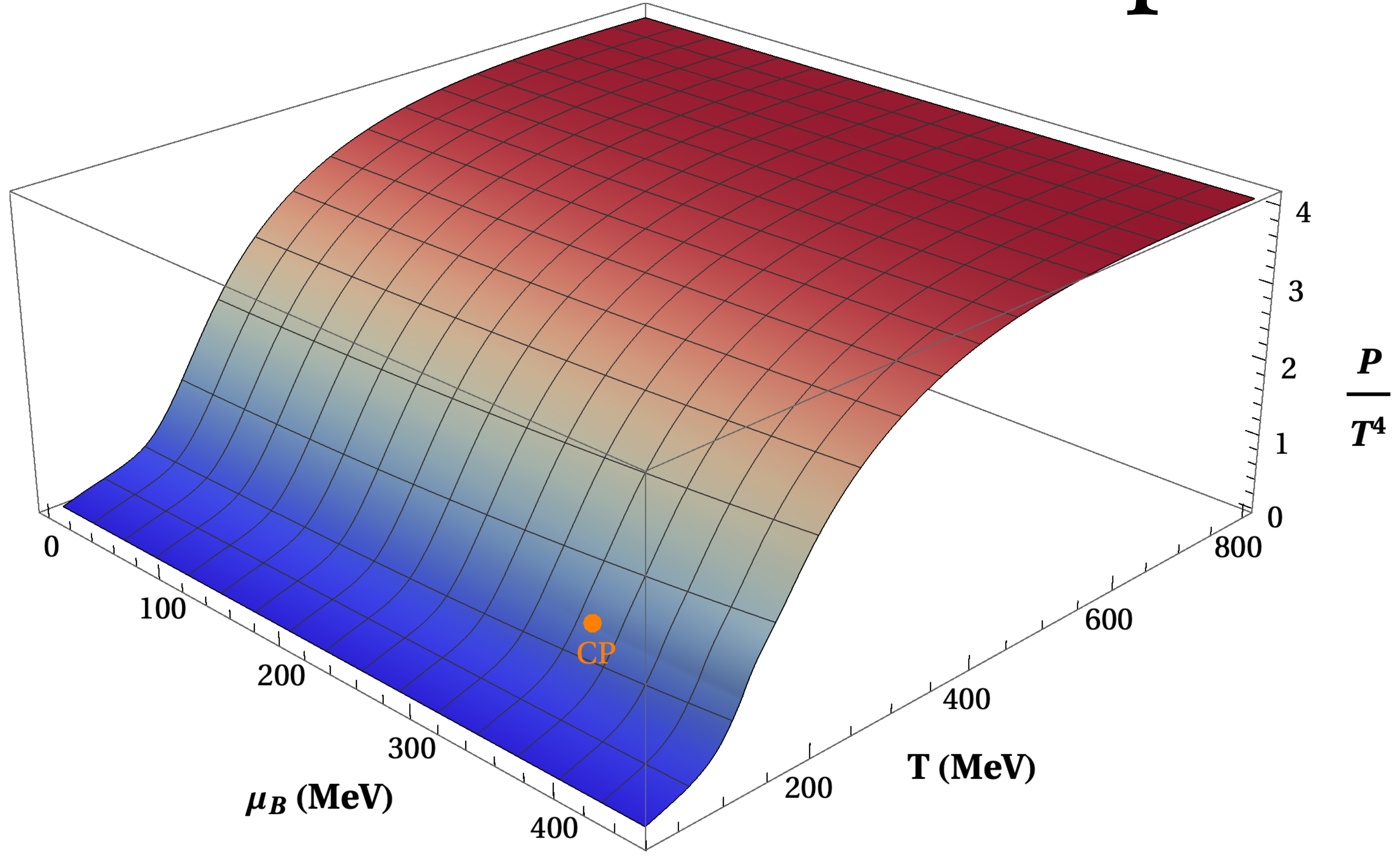
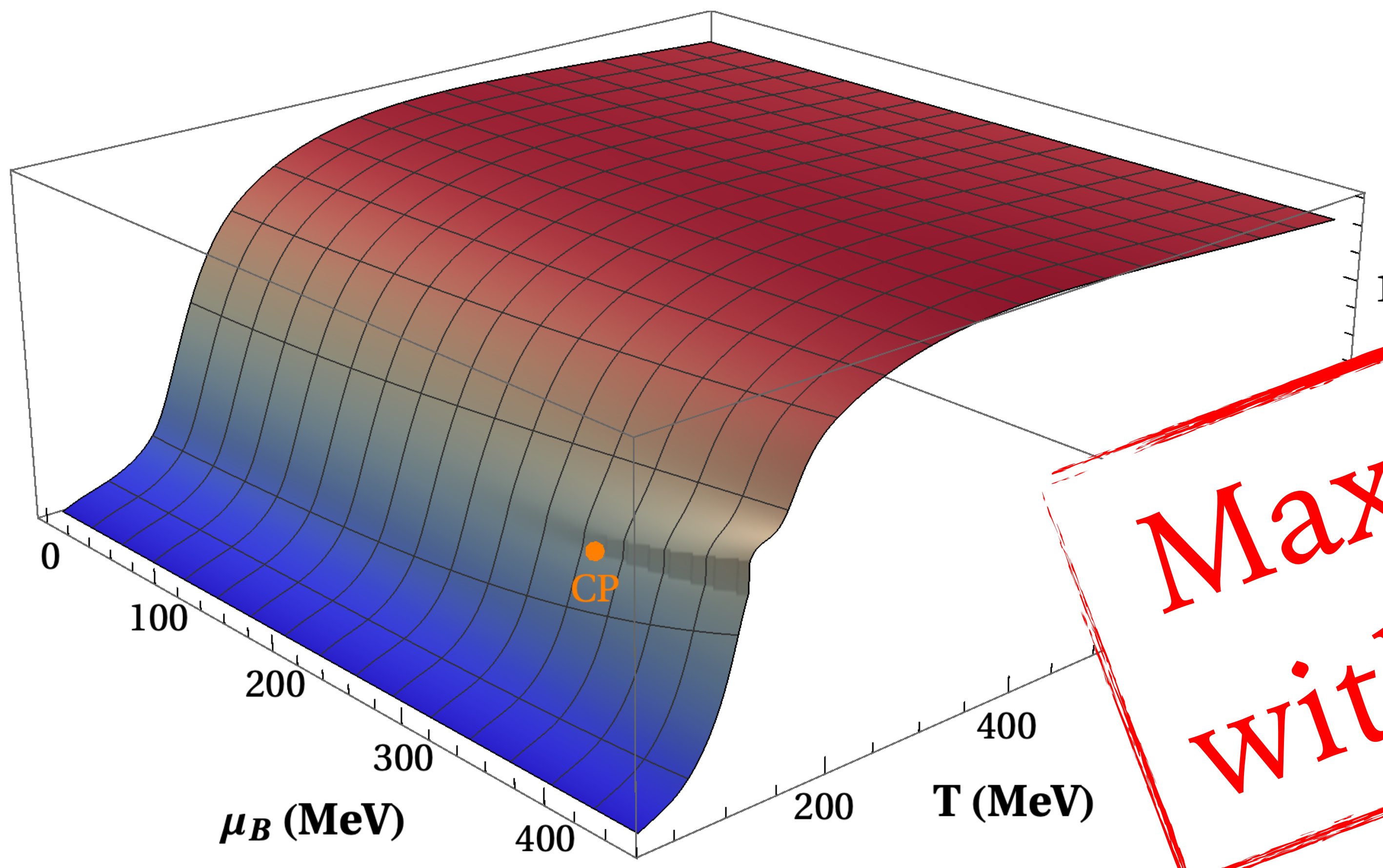
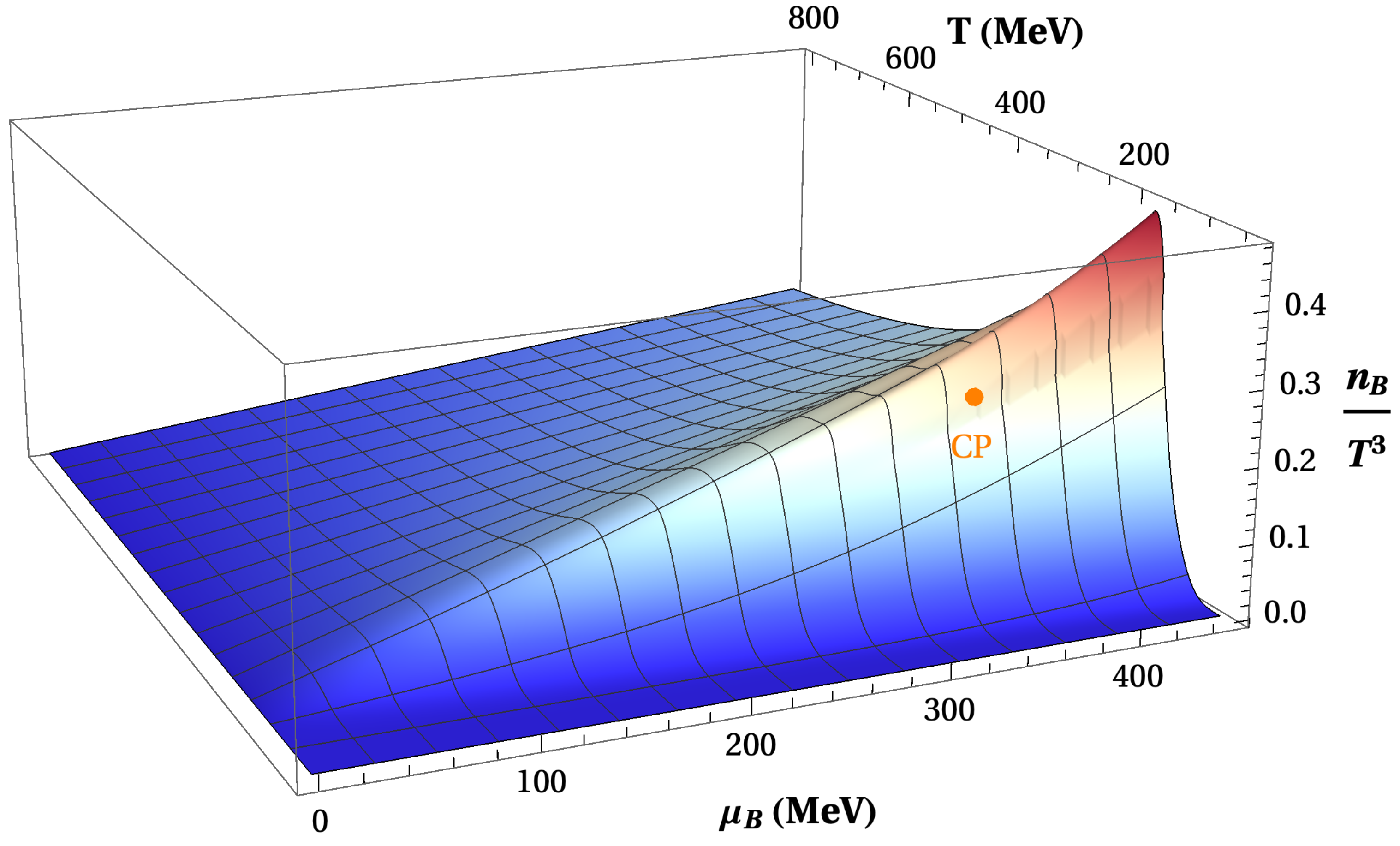


Pressure



Energy density



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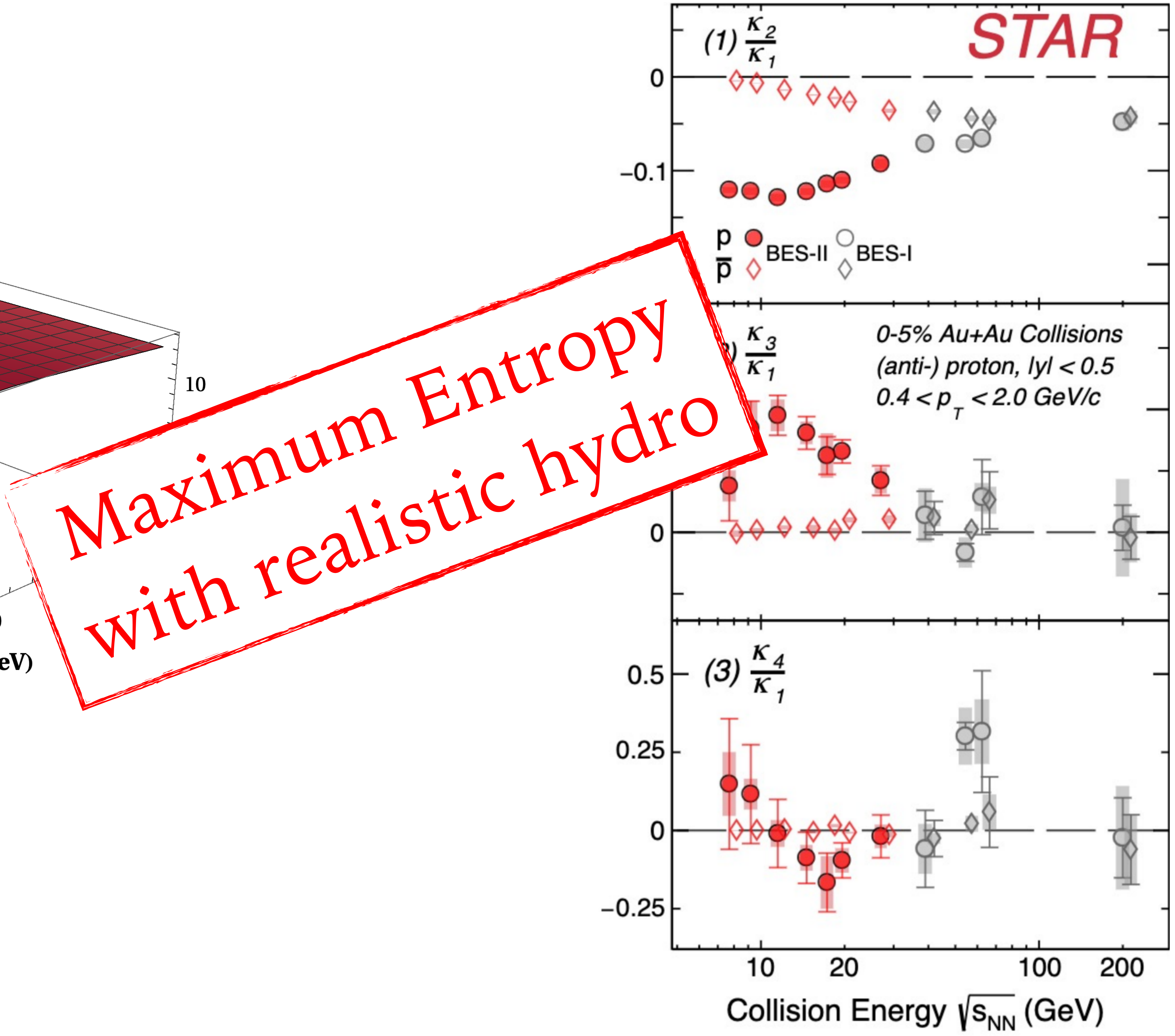

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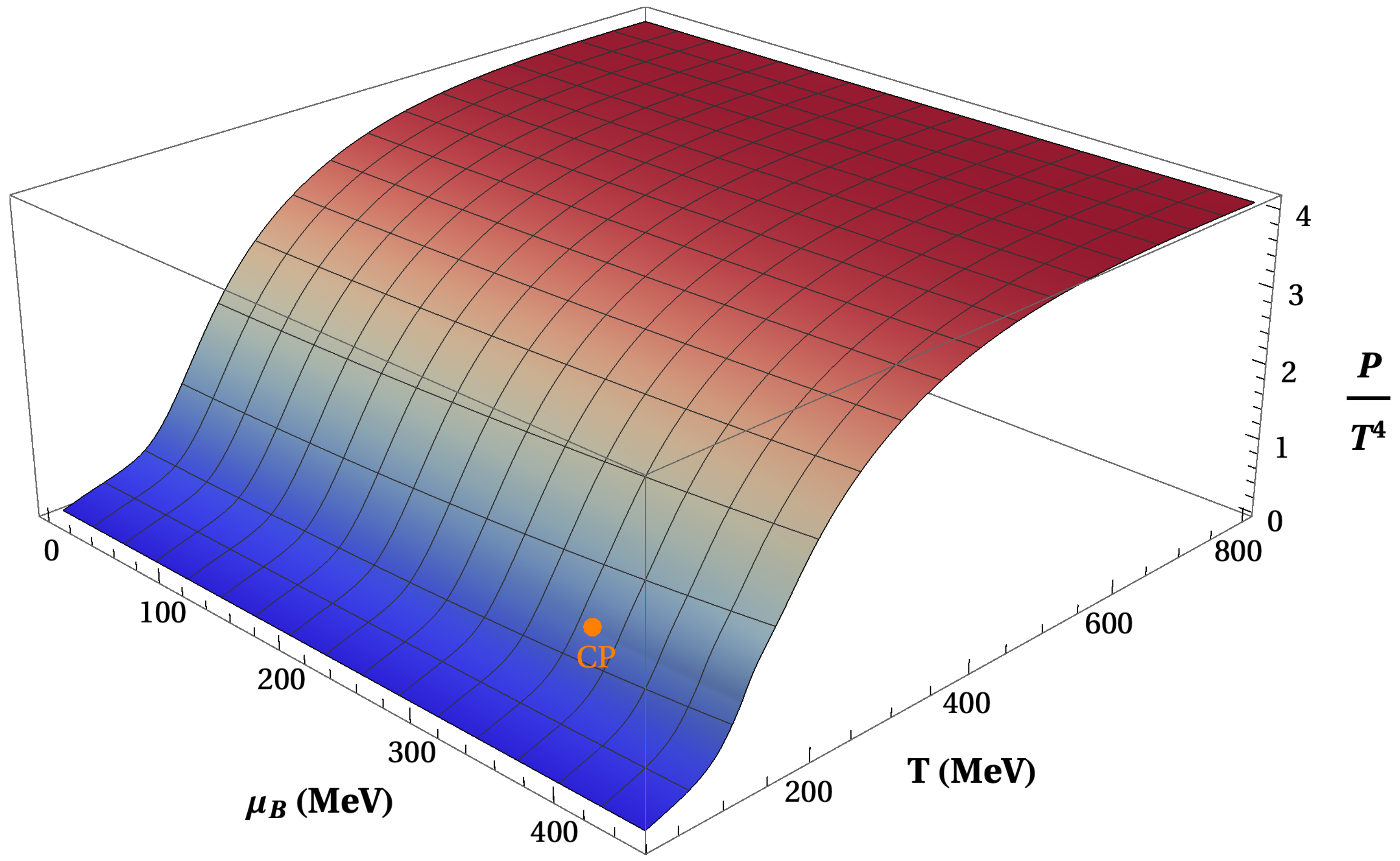
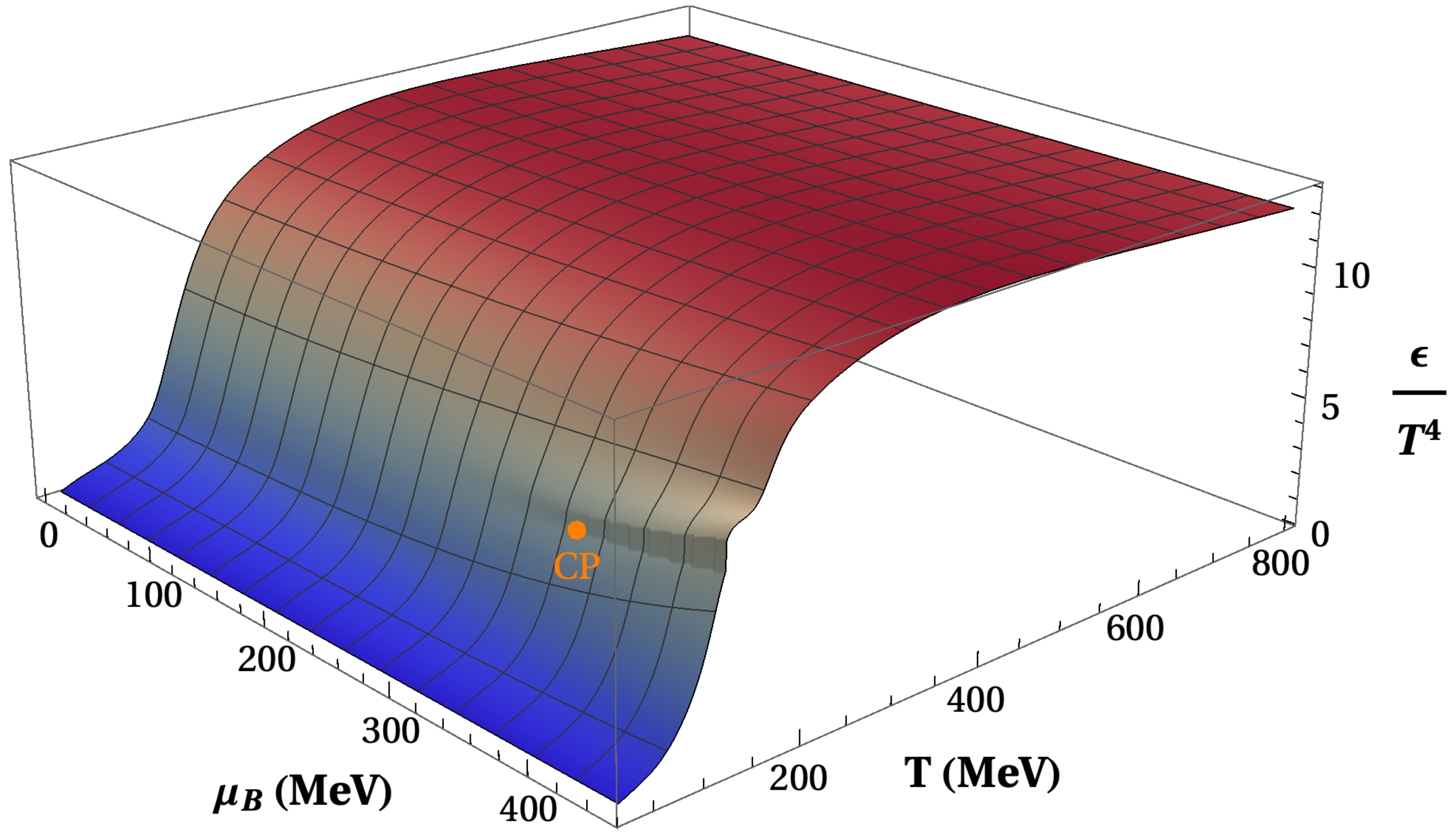
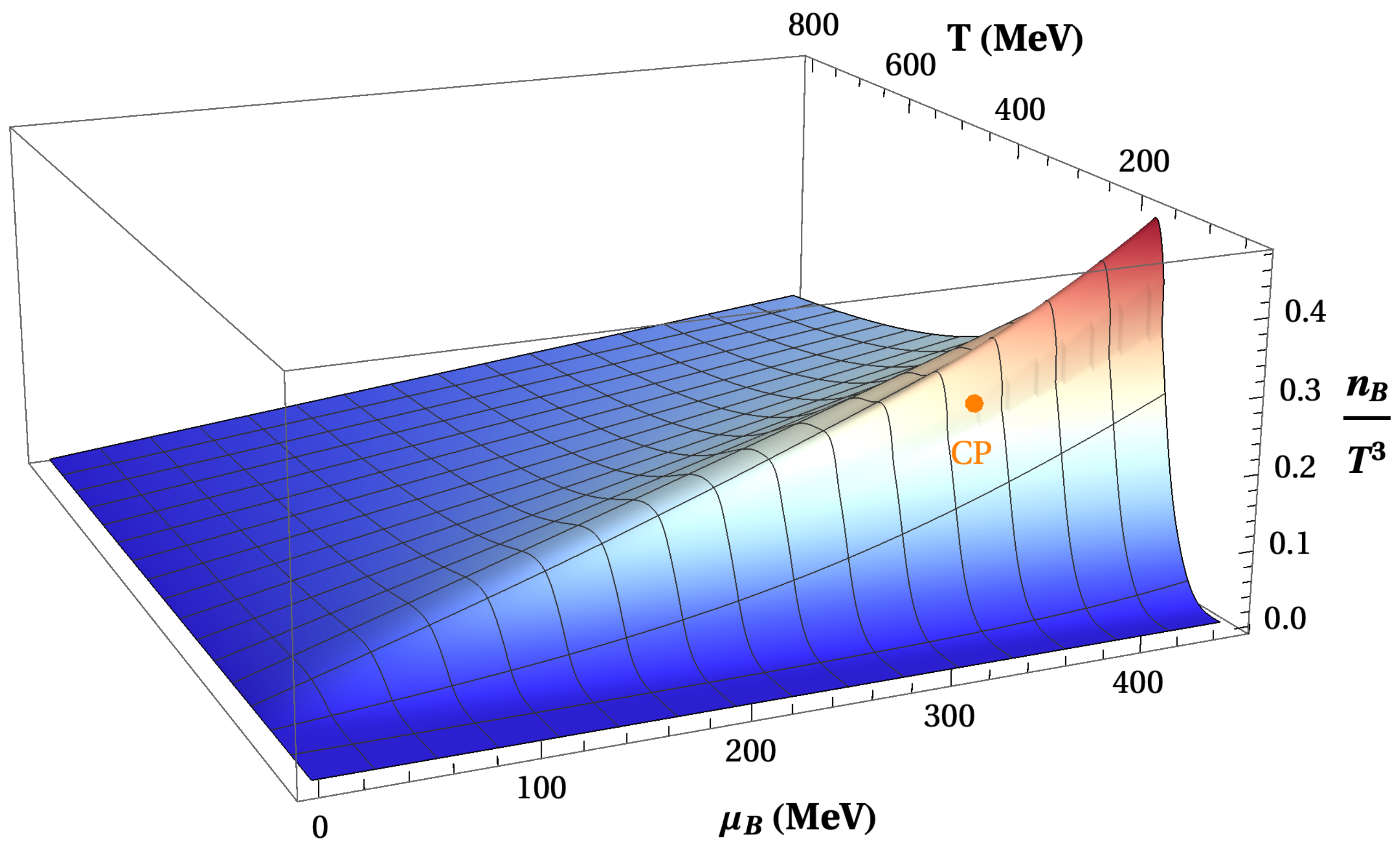
How to extract EoS from experiment?



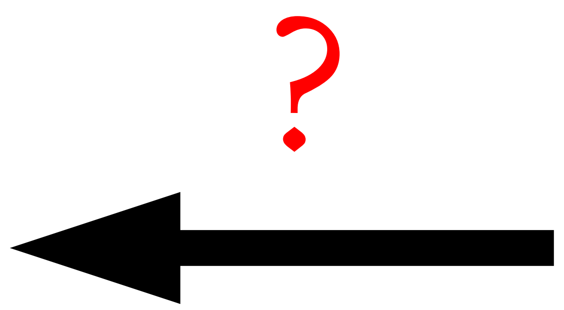
Pressure

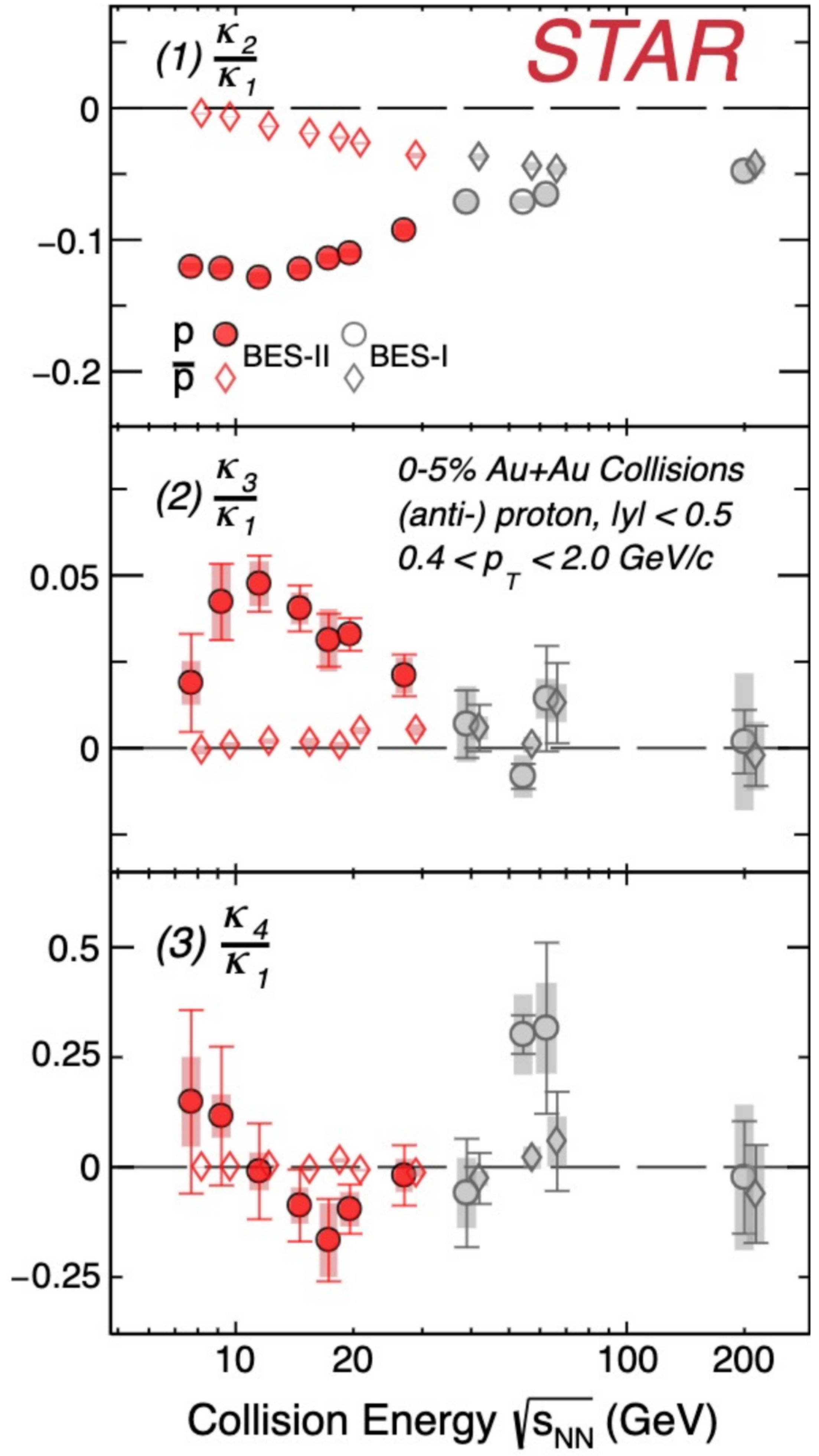


Energy density



Baryon density





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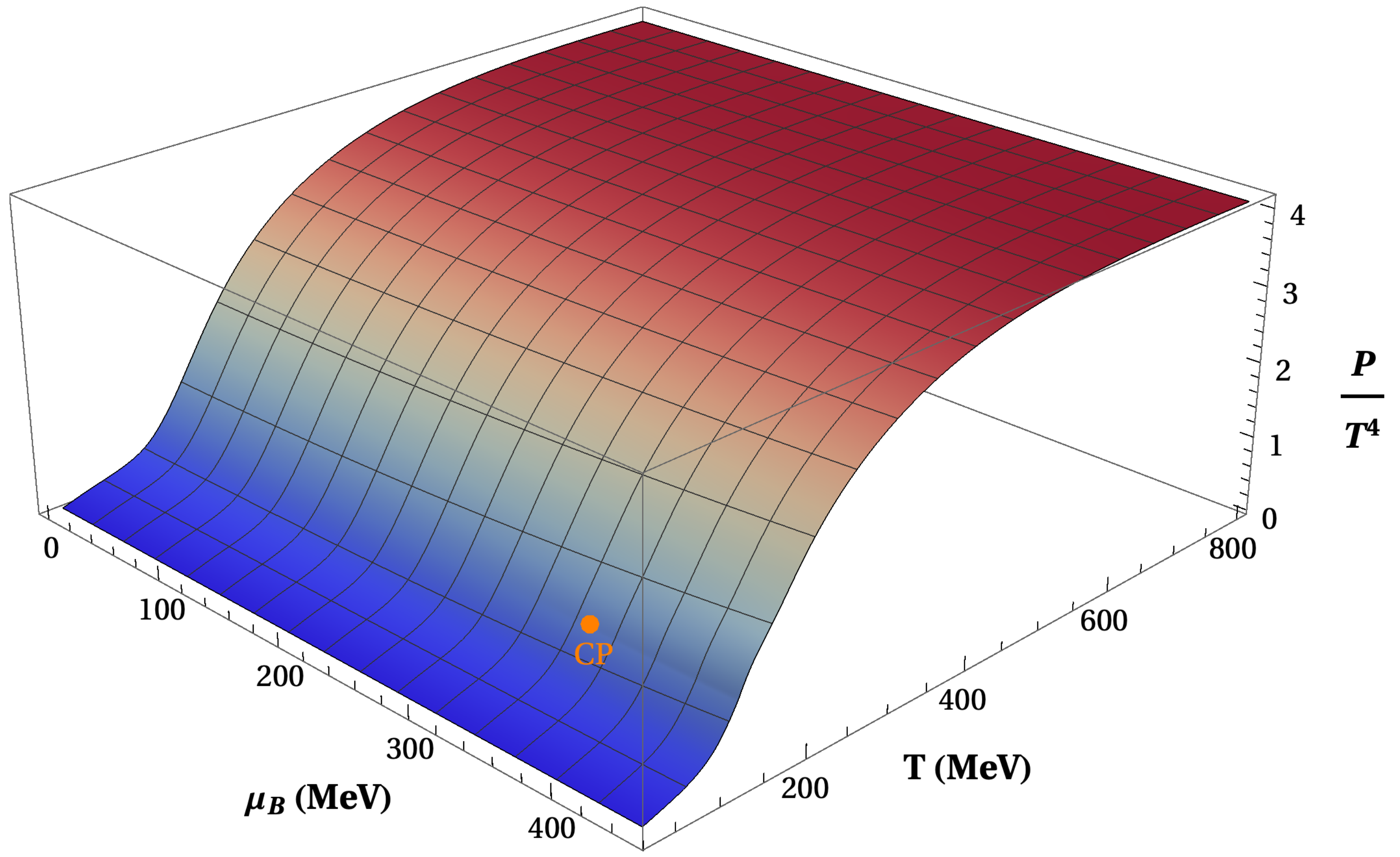
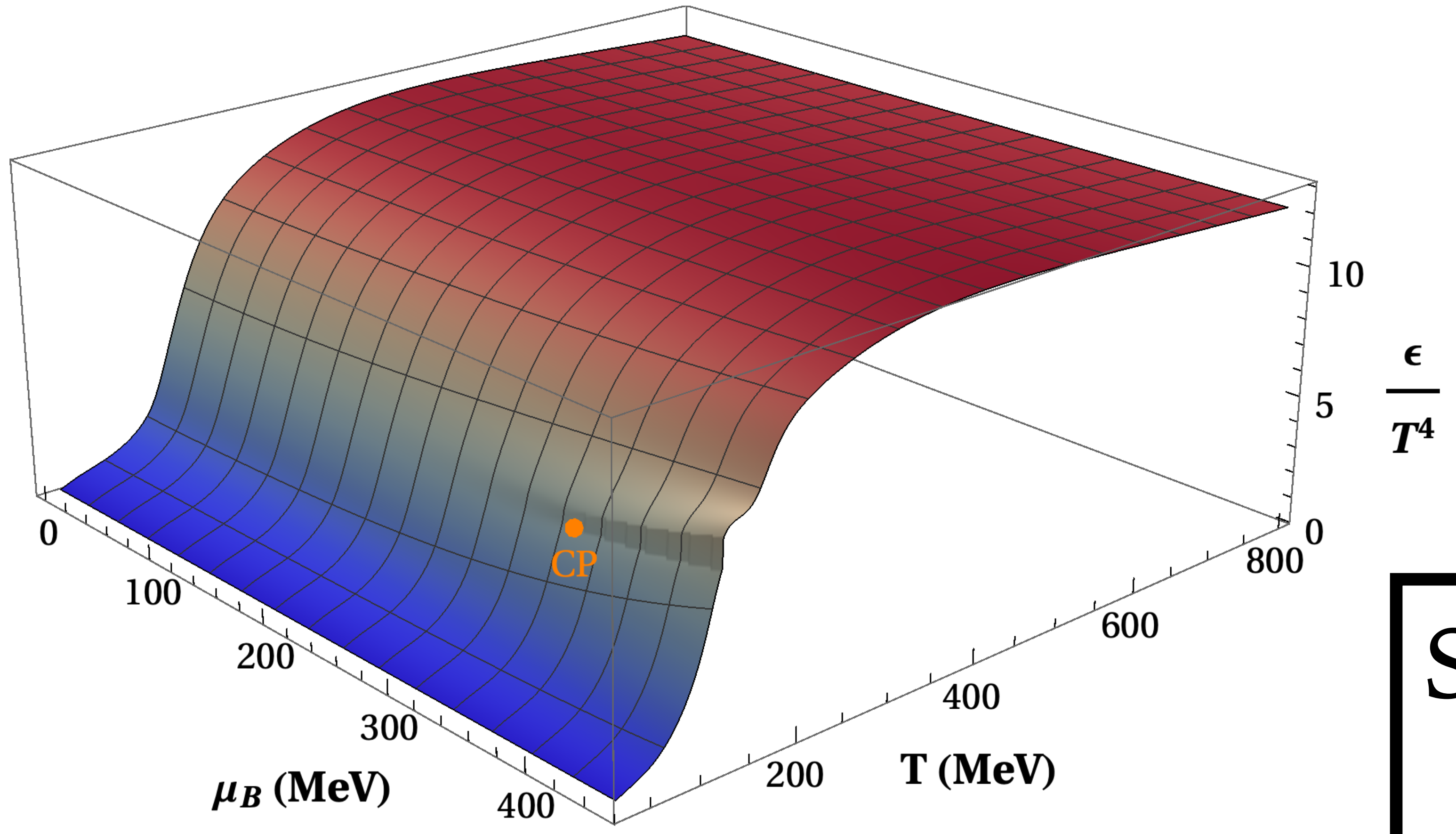
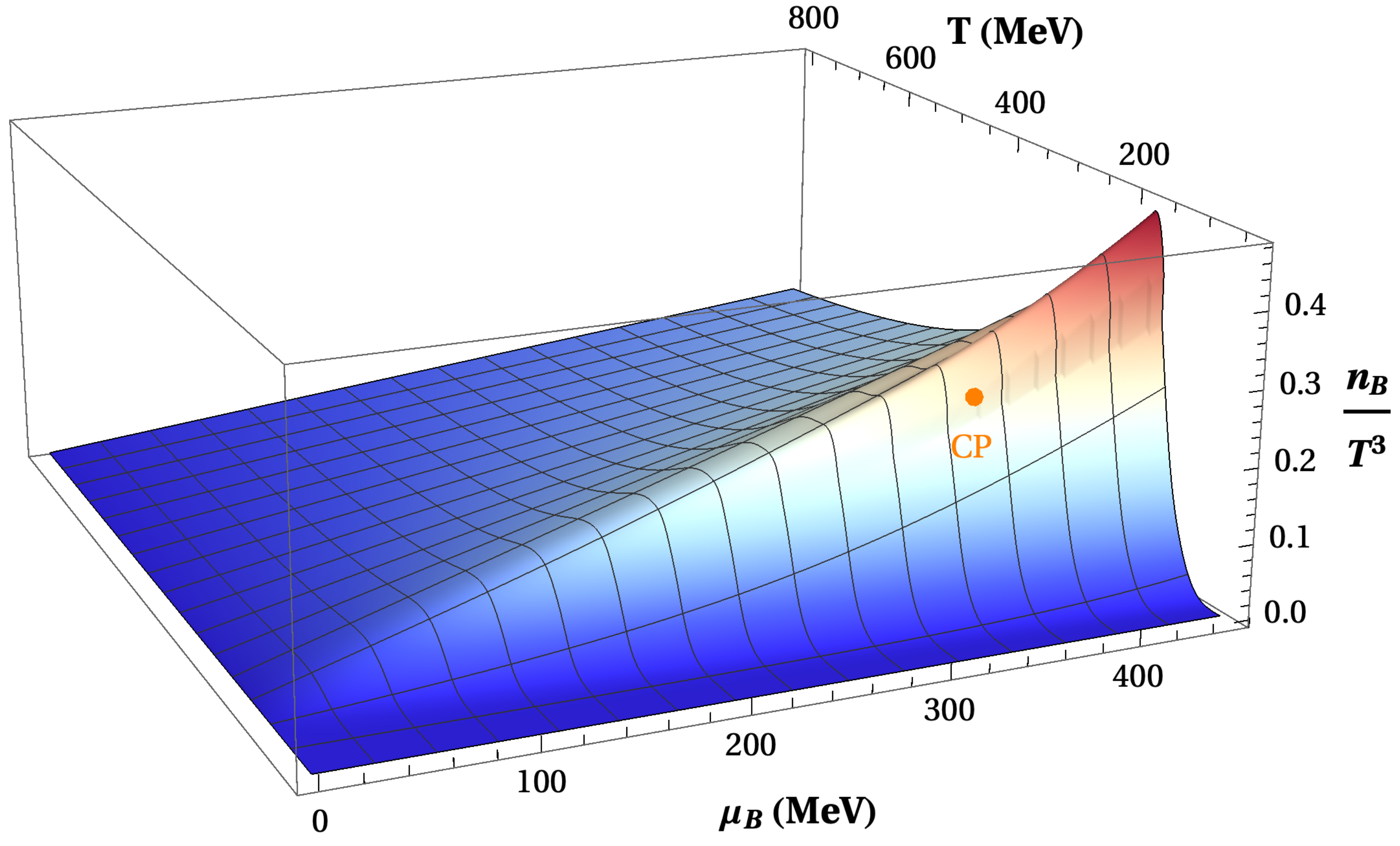


Pressure



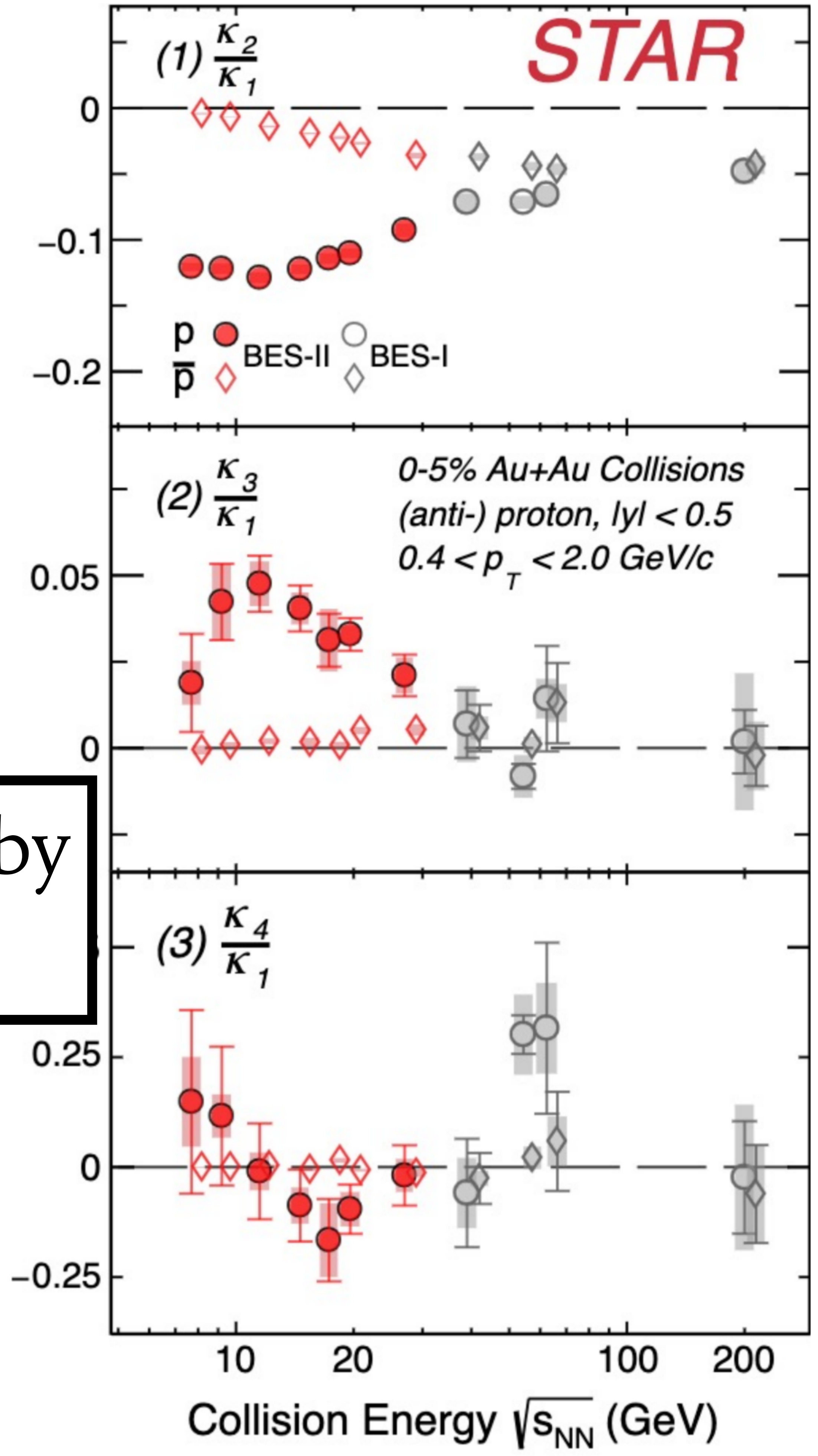
Energy density



Baryon density

?

See next talk by Grégoire!



JMK et al, EPJ+ (2021)
A. Pandav (STAR collaboration), CPOD 2024

Conclusions

- The search for the critical point/first order transition continues…
- Criticality incorporated within the original BEST EoS
  - Updated to strangeness neutrality, higher coverage of phase diagram, first order features + ongoing work: 4D EoS
- Max Ent provides a framework that takes the EoS as input and calculates the particle multiplicity fluctuations in the least biased way
  - Dependence of proton factorial cumulants on EoS parameters derived
  - Future work: implement these new updates into Hydro+/freeze-out
- Next talk: inverse problem - use experimental data to obtain EoS inputs