Theoretical Challenges in the Search for the QCD Critical Point and the Chiral Phase Transition

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QCD Critical Point and Hydrodynamic Evolution
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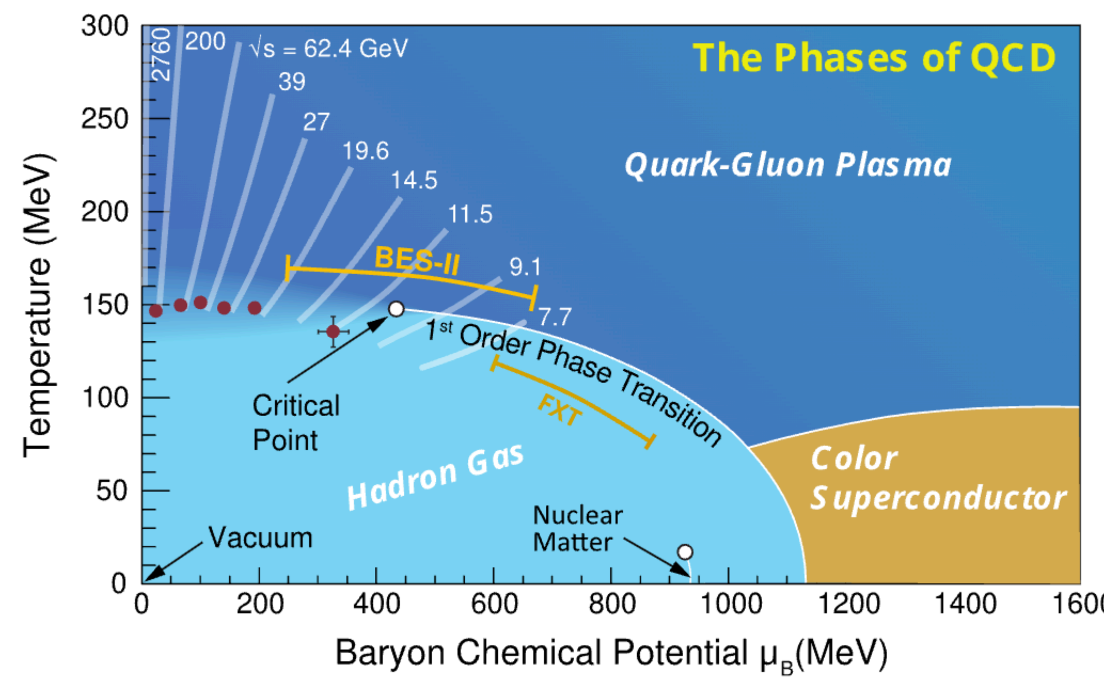
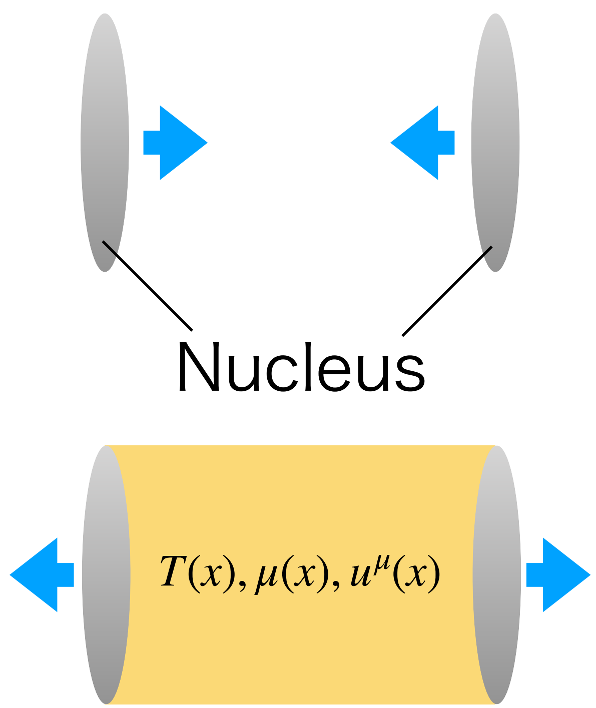
QCD phase diagram

- Many-Body Phenomena in QCD Matter
- QGP at low density and high temperature (Heavy-ion collision and lattice QCD)
- Exploration in the dense region
  - QCD critical point
  - Chiral phase transition
  - Nuclear/Quark superfluid
  - Color superconductivity

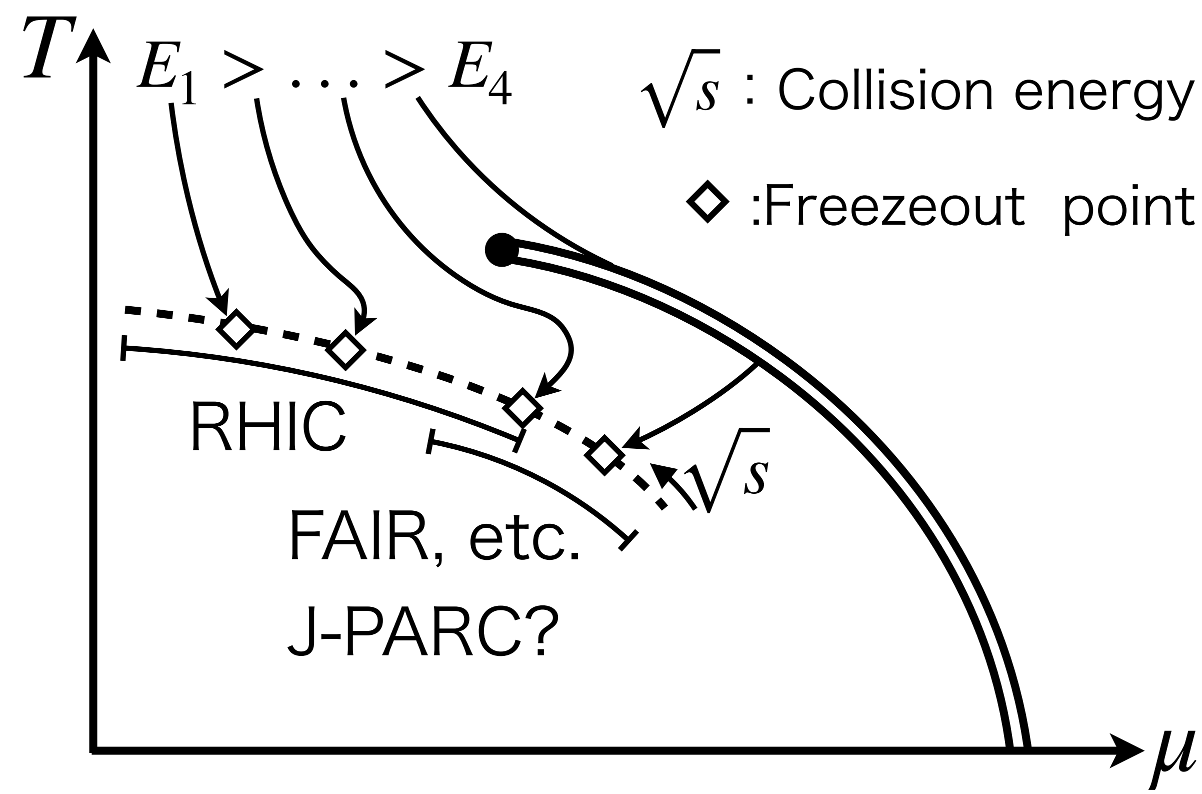


L. Du, Sorensen, Stephanov (2024)

Heavy-ion collisions



Relativistic hydrodynamics



- Relativistic Heavy Ion Collider (RHIC)、Beam-Energy-Scan (BES)

  (BES-I: 2010-2017, BES-II: 2019-2021)
- Future low-energy collisions: FAIR, HIAF, NICA, J-PARC

Recent results

Pandav (STAR), Critical Point and Onset of Deconfinement (CPOD) 2024

A review by Stephanov (2024) arXiv:2410.0286

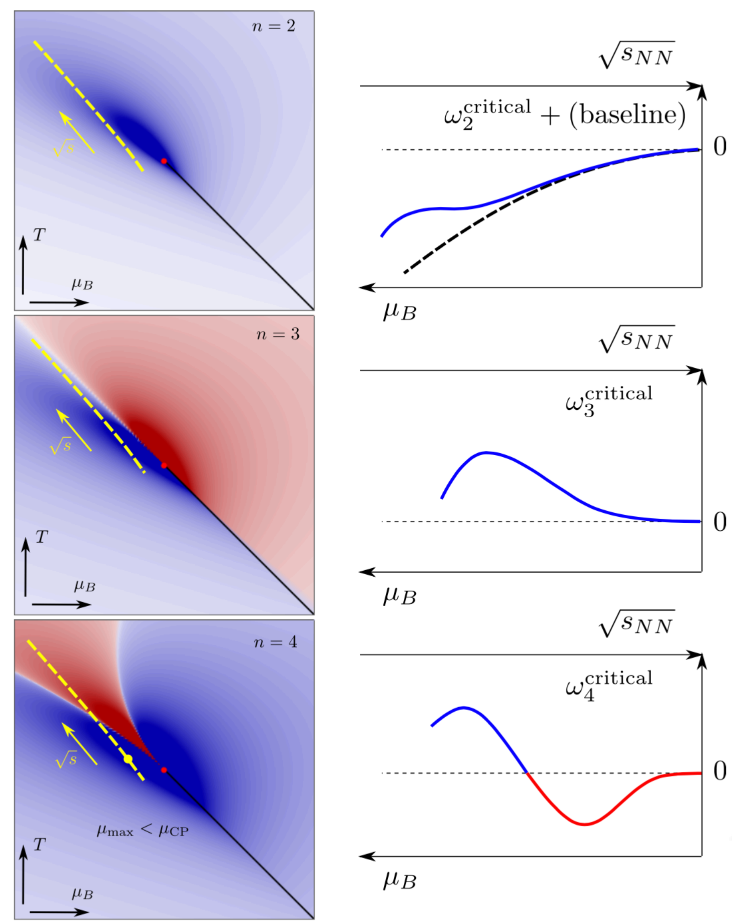
Baryon number fluctuation

$$\frac{\partial n(\mu, T)}{\partial \mu} \sim \langle \delta n \delta n \rangle_{\text{cum.}}$$

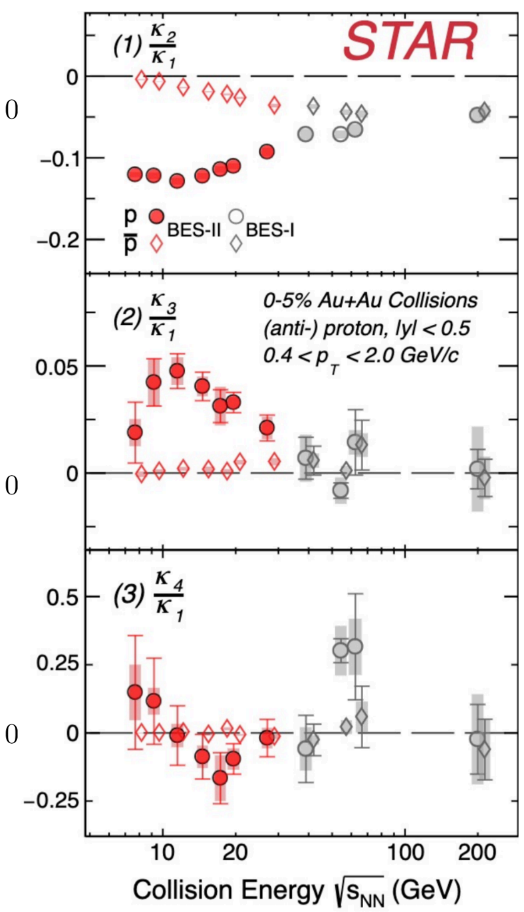
$$\frac{\partial^2 n}{\partial \mu^2} \sim \left\langle (\delta n)^3 \right\rangle_{\text{cum}}$$

$$\frac{\partial^3 n}{\partial \mu^3} \sim \left\langle (\delta n)^4 \right\rangle_{\text{cum}}$$

Theory



Experiment



Consistent with equilibrium fluctuations expected near the critical point

Next: quantitative comparison with experimental results

Theoretical challenges

1. Non-equilibrium effect on fluctuations

   Equilibrium expectations are insufficient due to memory effects

   - Hydrodynamic evolution near the critical point

     M. Pradeep, NS, M. Stephanov, and H.-U. Yee, PRC **109**, 064905 (2024), arXiv:2402.09519

   - Fluctuation dynamics: an effective field theoretical perspective

     NS and Yi Yin, JHEP **2022**, 124 (2022), arXiv:2111.14667

2. Complementary probes beyond fluctuations

   - A nonfluctuational signature: freeze-out point jump

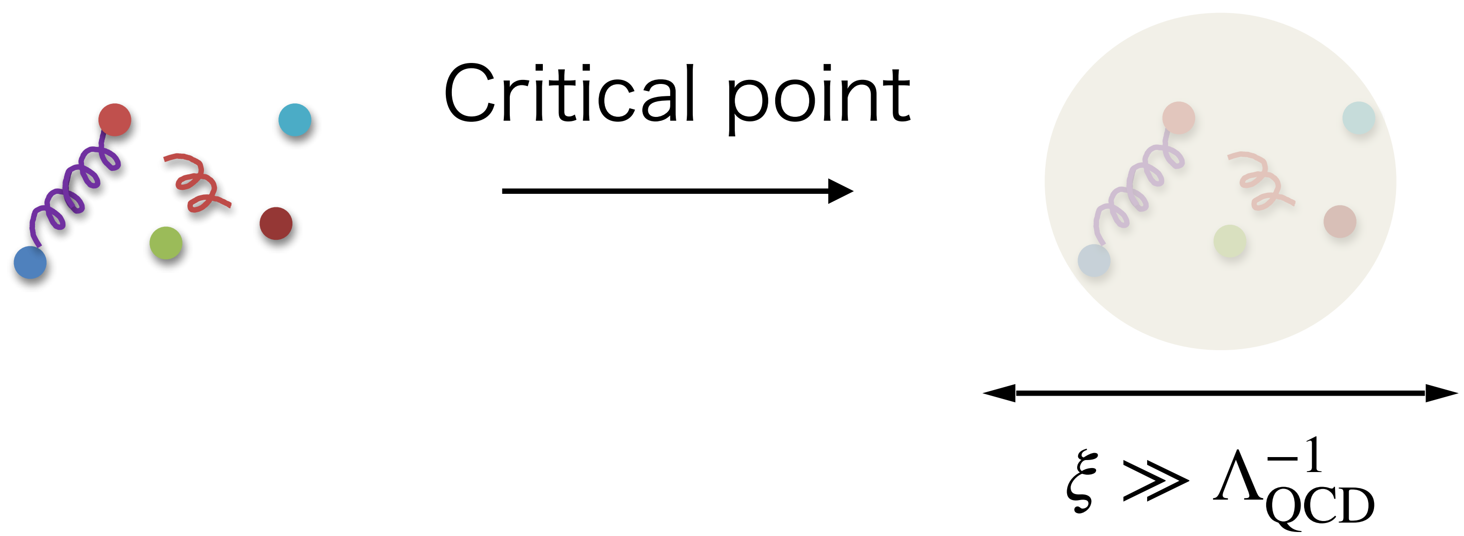
     M. Pradeep, NS, M. Stephanov, and H.-U. Yee, EPJ Web of Conferences **364**, 15012 (2026)

   - Nonequilibrium phase conversion during the first-order phase transition

     NS, M. Stephanov, and H.-U. Yee (in preparation)

Ideal hydrodynamics

- Entropy and baryon number conservation $\longrightarrow \hat{s} = \frac{s}{n} = \text{const}\,.$
- The universality of critical phenomena:

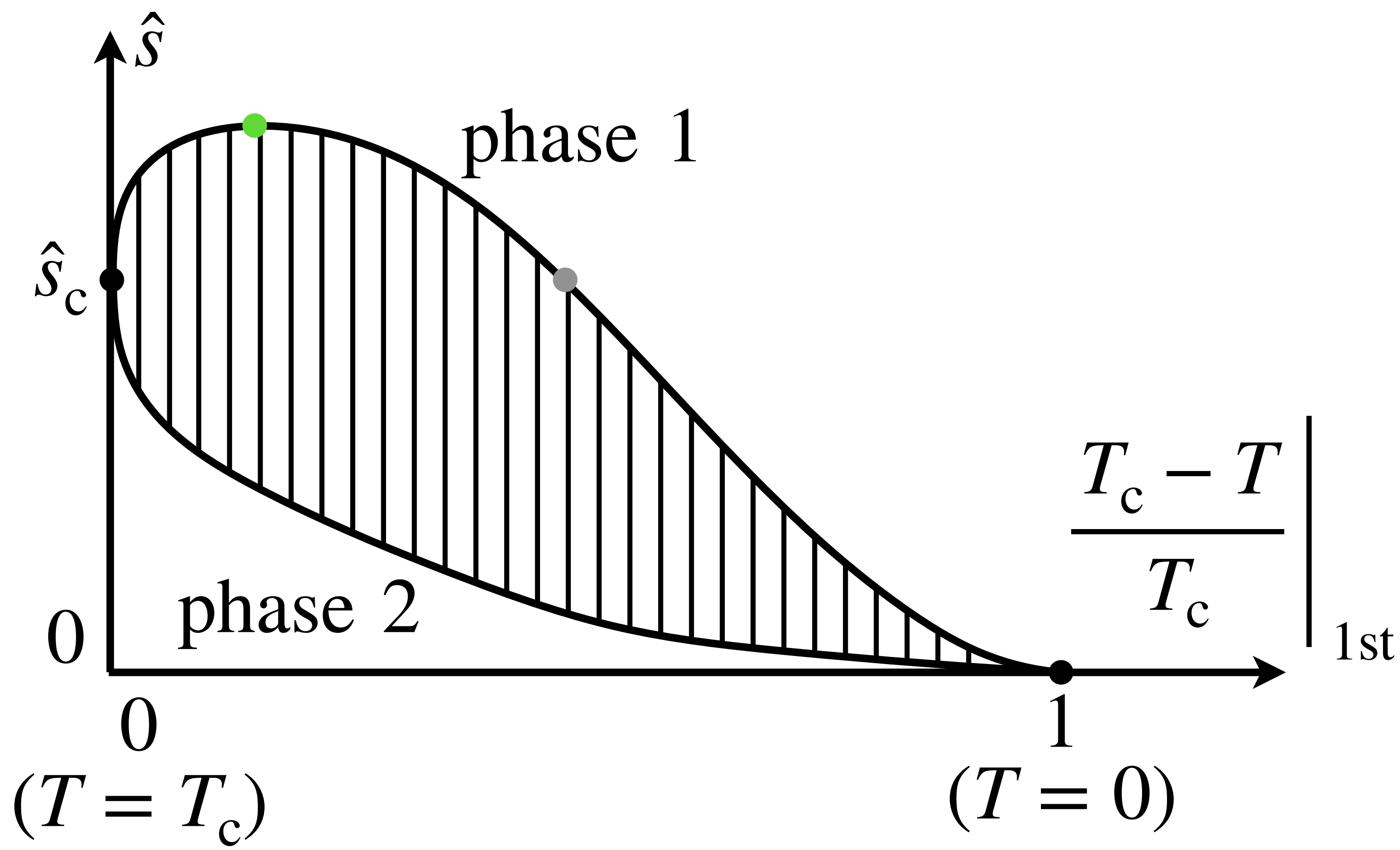


QCD thermodynamics ≃ 3D Ising model

$\longrightarrow$ Universal behaviors of isentropes

Non-monotonic specific entropy

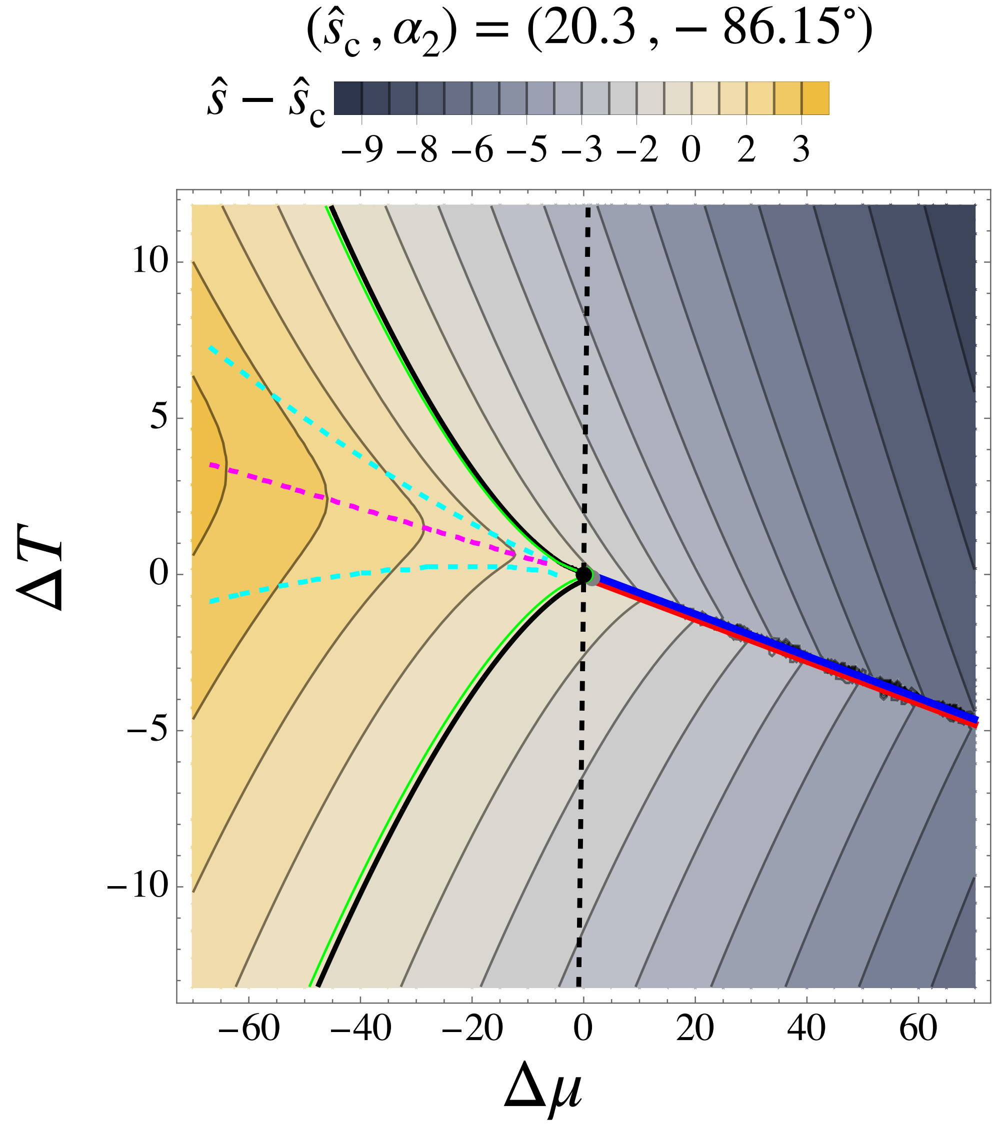
Pradeep, Sogabe, Stephanov, and Yee (2024)

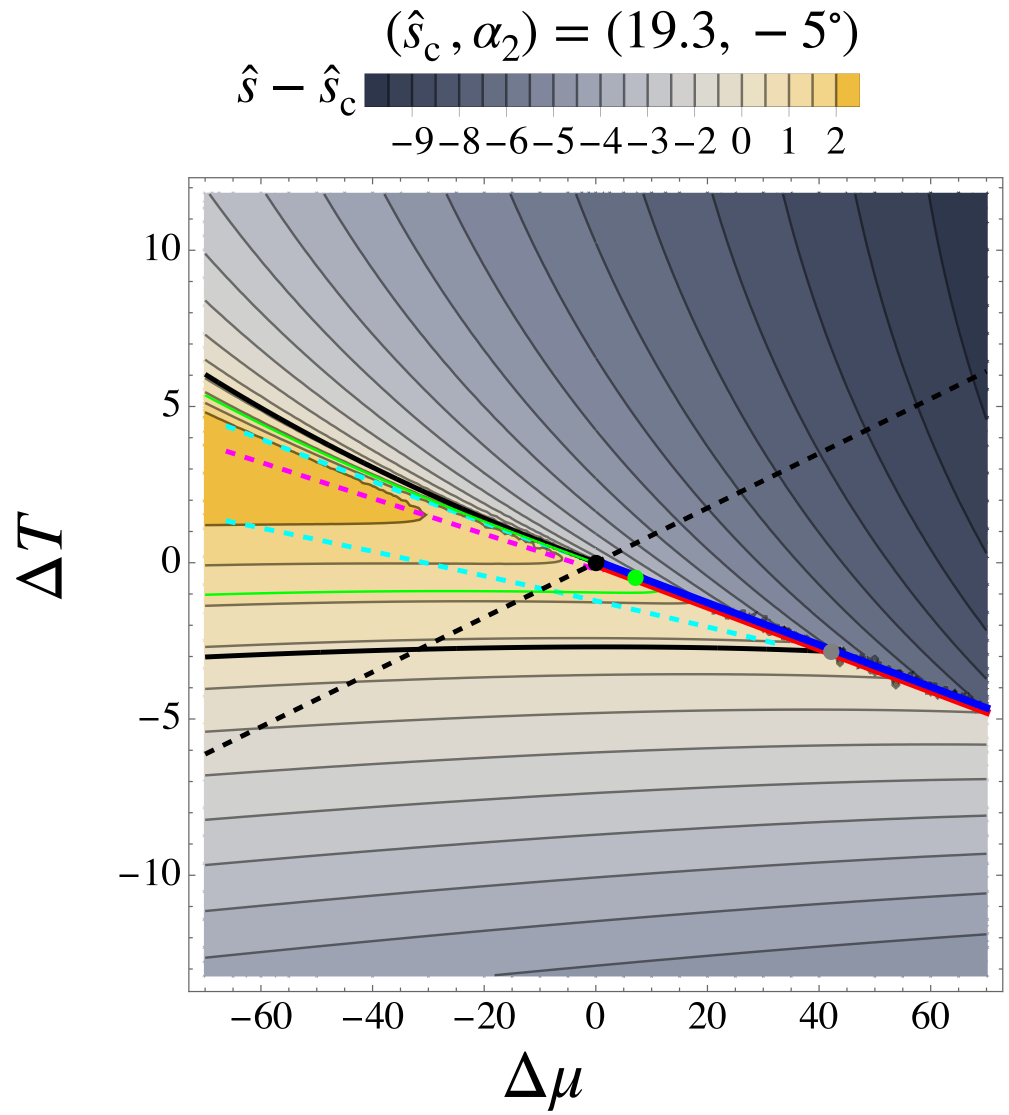


- Near T_c : $\hat{s} \sim$ (order parameter) $\sim \pm\,(T - T_c)^{\beta}$ $(\beta = 0.326)$
- The third law of thermodynamics: $\hat{s}(T=0) = 0$

On the (μ, T) plane

Pradeep, NS, Stephanov, and Yee (2024)

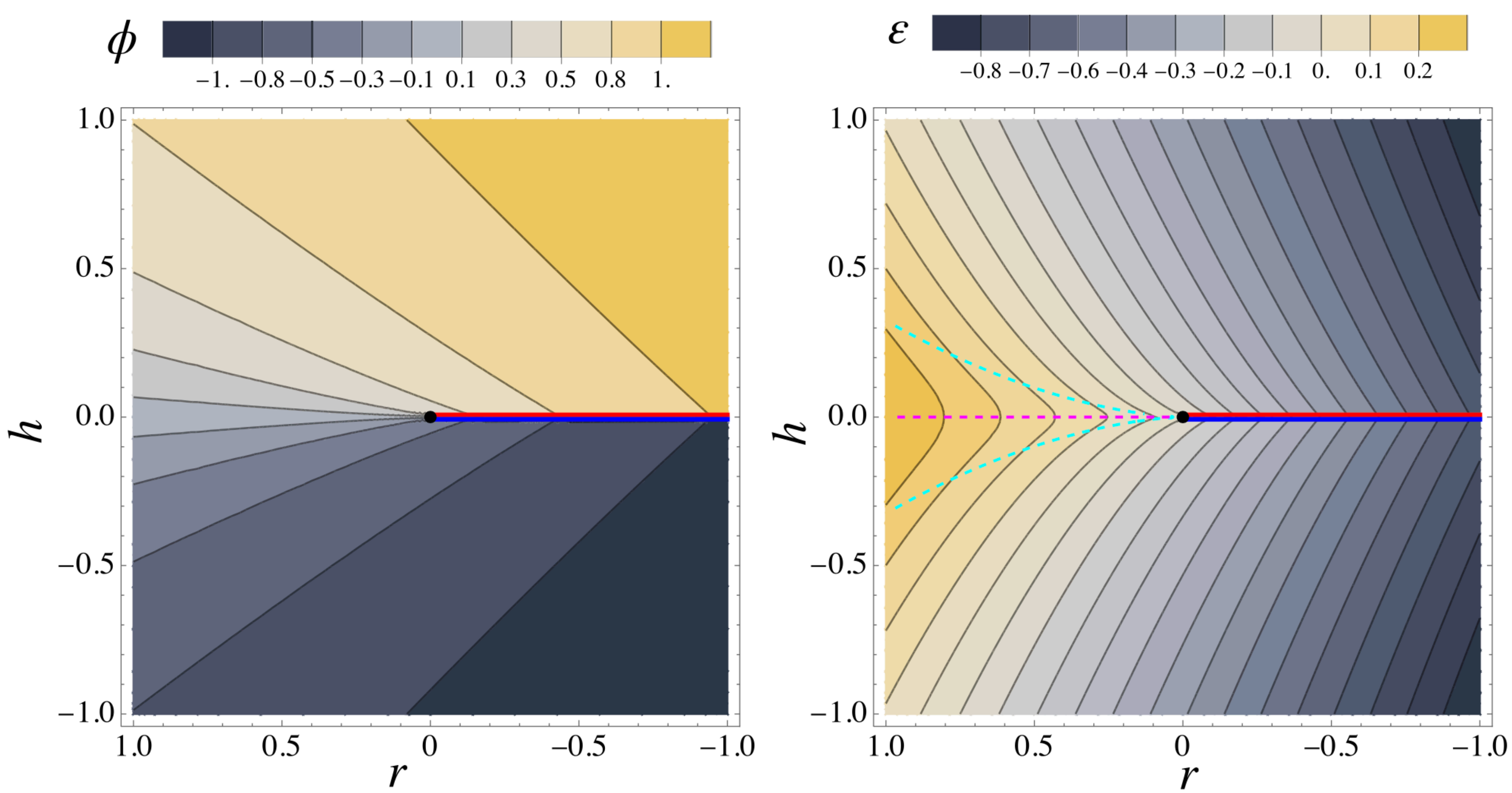




Intuitive picture

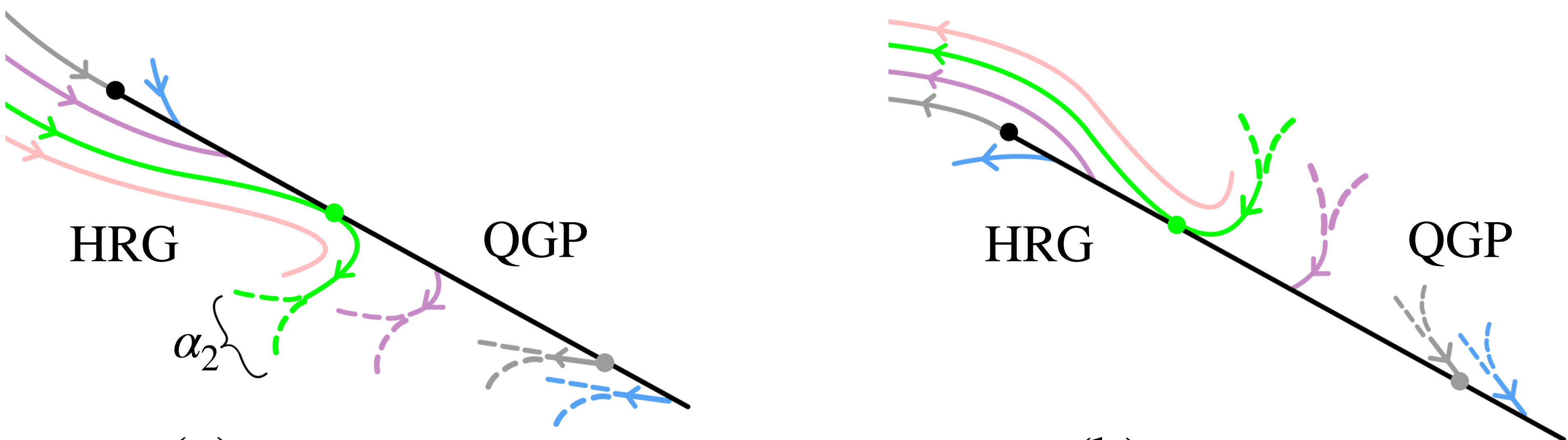
$$\hat{s} - \hat{s}_{\mathrm{c}} = \pm \left.\frac{\partial \hat{s}}{\partial \phi}\right|_{\varepsilon,\,\mathrm{c}} \phi + \left.\frac{\partial \hat{s}}{\partial \varepsilon}\right|_{\phi,\,\mathrm{c}} \varepsilon + \cdots$$

$\phi \equiv G_h \sim \pm(-r)^{\beta}\ (h=0) \qquad \varepsilon = G_r \sim (-r)^{1-\alpha}\ (h=0) \qquad (\alpha = 0.11, \quad \beta = 0.326)$

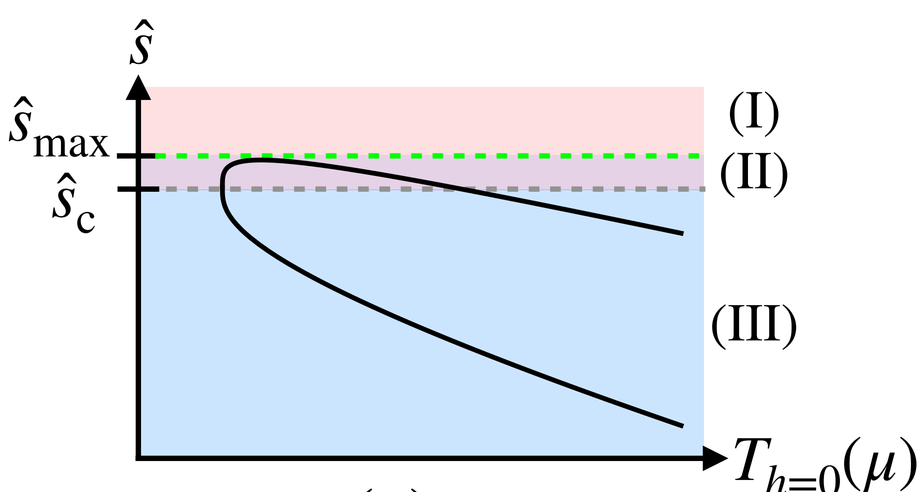


Classification of isentropes

Pradeep, Sogabe, Stephanov, and Yee (2024)



(a) $\hat{s}_c > \cot \alpha_1$ (b) $\hat{s}_c < \cot \alpha_1$



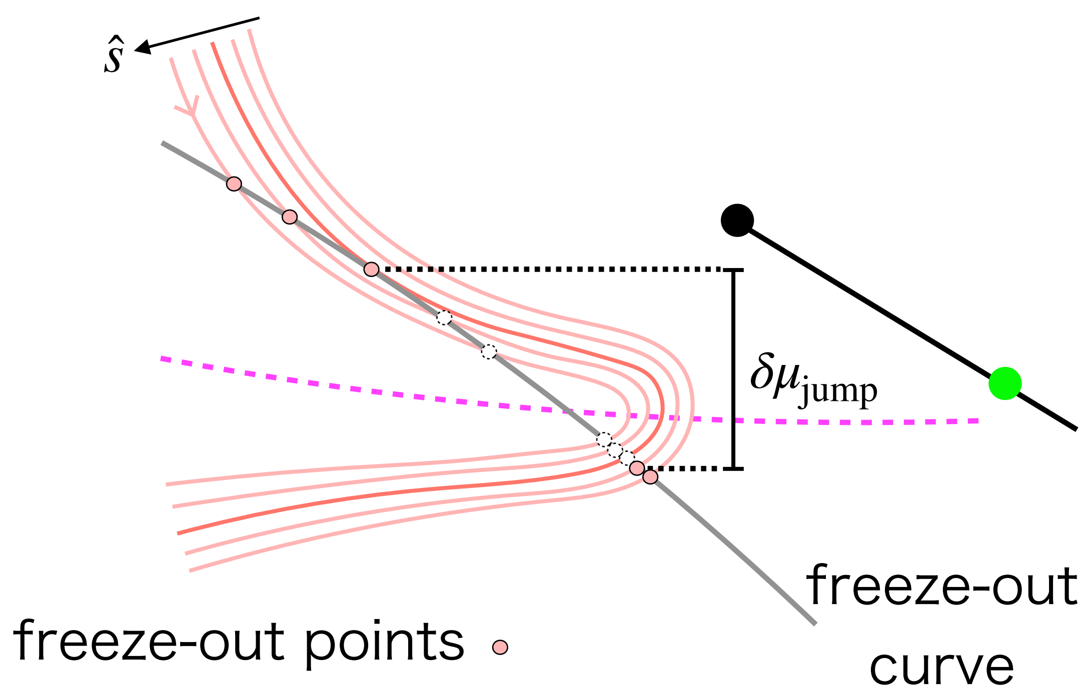
(c)

	(a)	(b)
(I)	Crossover	
(II)	HRG → HRG	QGP → QGP
(III)	QGP → HRG	

(d)

Implications for freeze-out

M. Pradeep, NS, M. Stephanov, and H.-U. Yee (QM 2026)



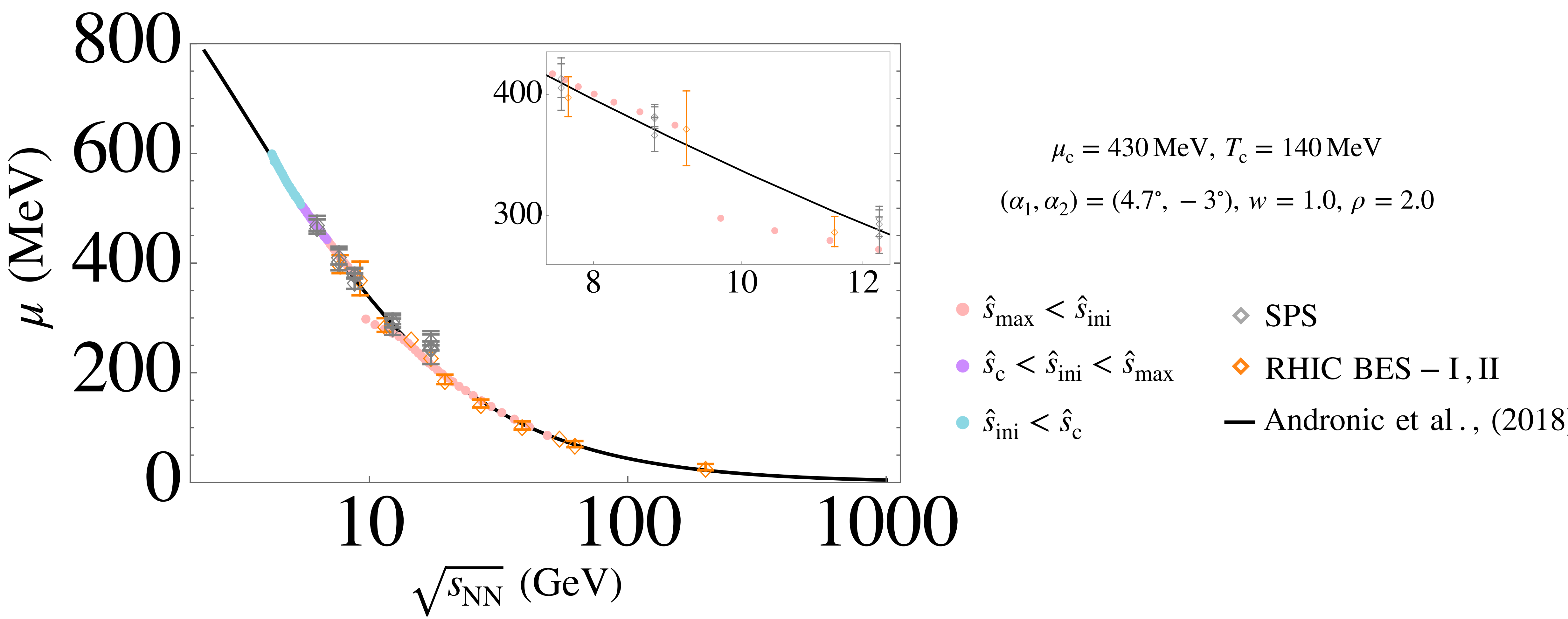
1. The attraction toward the maximum point
2. Multiple intersections that the adiabatic process cannot reach
3. Chemical potential jump along the freeze-out curve

Criticality on hydrodynamic trajectories: A non-fluctuational signature

Non-fluctuational signature

M. Pradeep, NS, M. Stephanov, and H.-U. Yee (QM 2026)

Current experiments rule out $\delta\mu_{\rm jump} > 70\,{\rm MeV}$



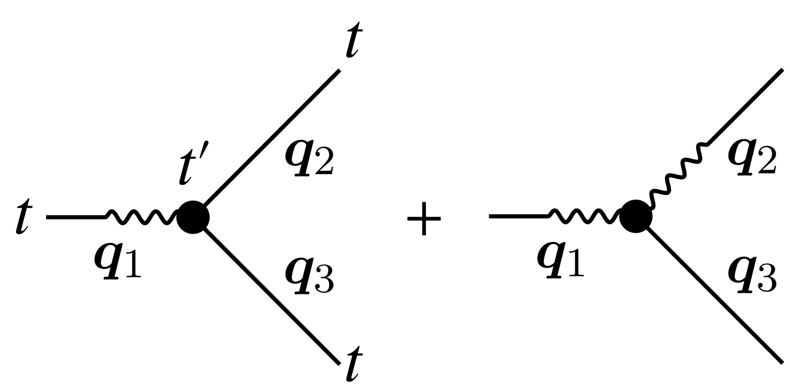
Fluctuation dynamics: our approach

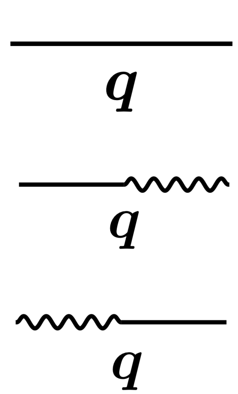
NS and Yin (2022)

- Non-equilibrium effective field theory Hong-Liu, at al (2016)
- Consistent with conservation laws and the fluctuation-dissipation relation

Ex) 3pt

$W_3(t;\boldsymbol{q}_1,\ldots,\boldsymbol{q}_3) =$





$= G_{\mathrm{rr}}(t,t',\boldsymbol{q})$

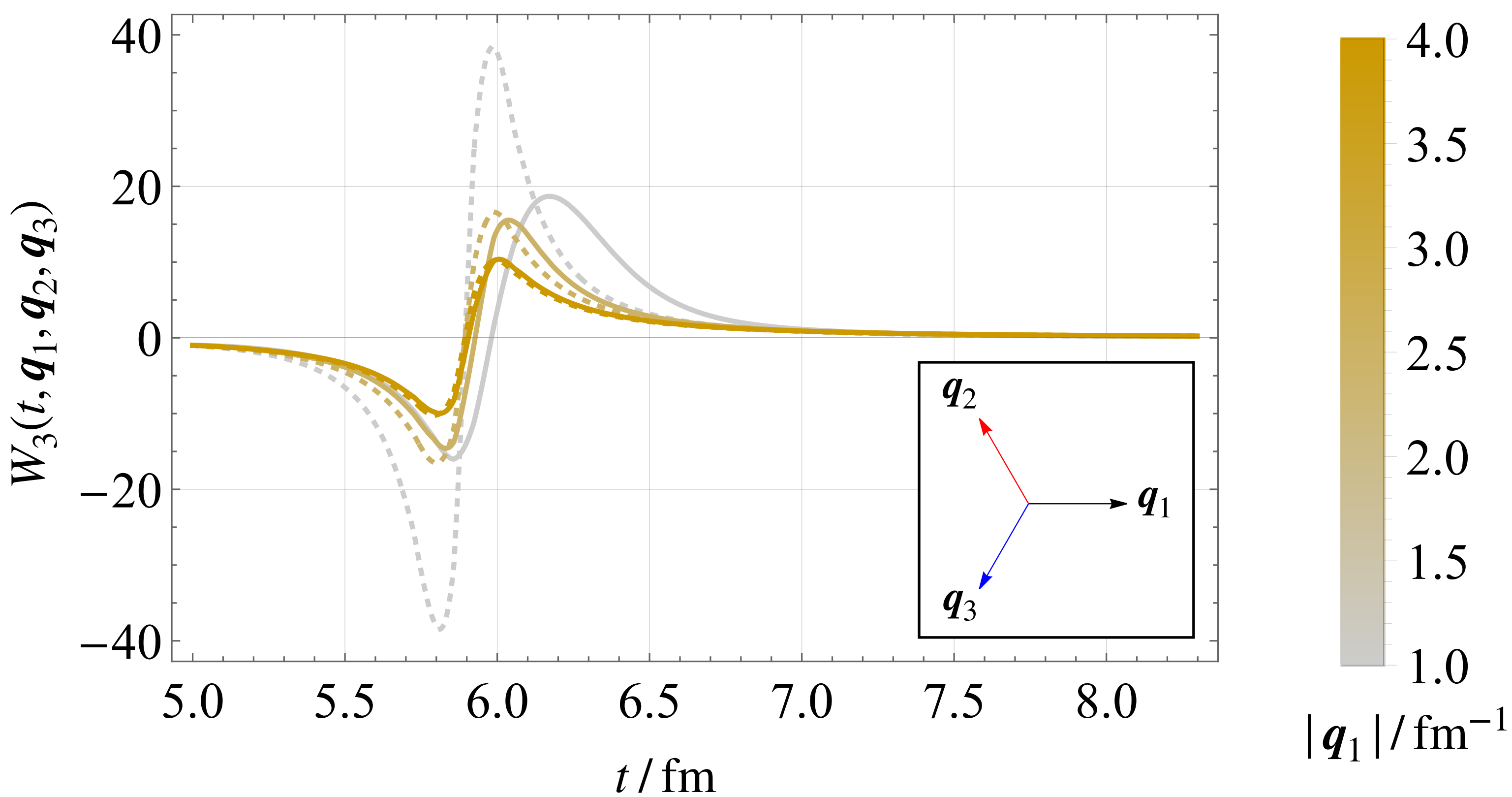
$= G_{\mathrm{ra}}(t,t',\boldsymbol{q}) = G_{\mathrm{ar}}(t',t,\boldsymbol{q})$

$= G_{\mathrm{ar}}(t,t',\boldsymbol{q})$

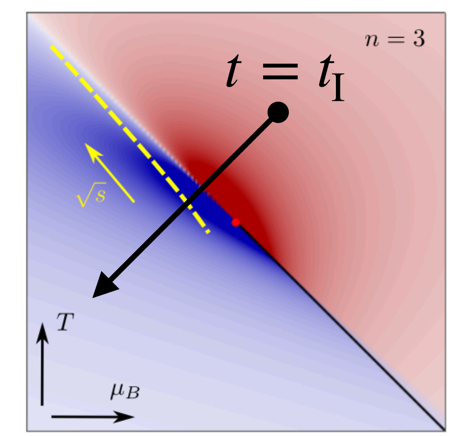
+ vertices

Unequal-time 2pt correlators obeying Schwinger-Dyson equations under arbitrary hydrodynamic evolution

Toy model calculation



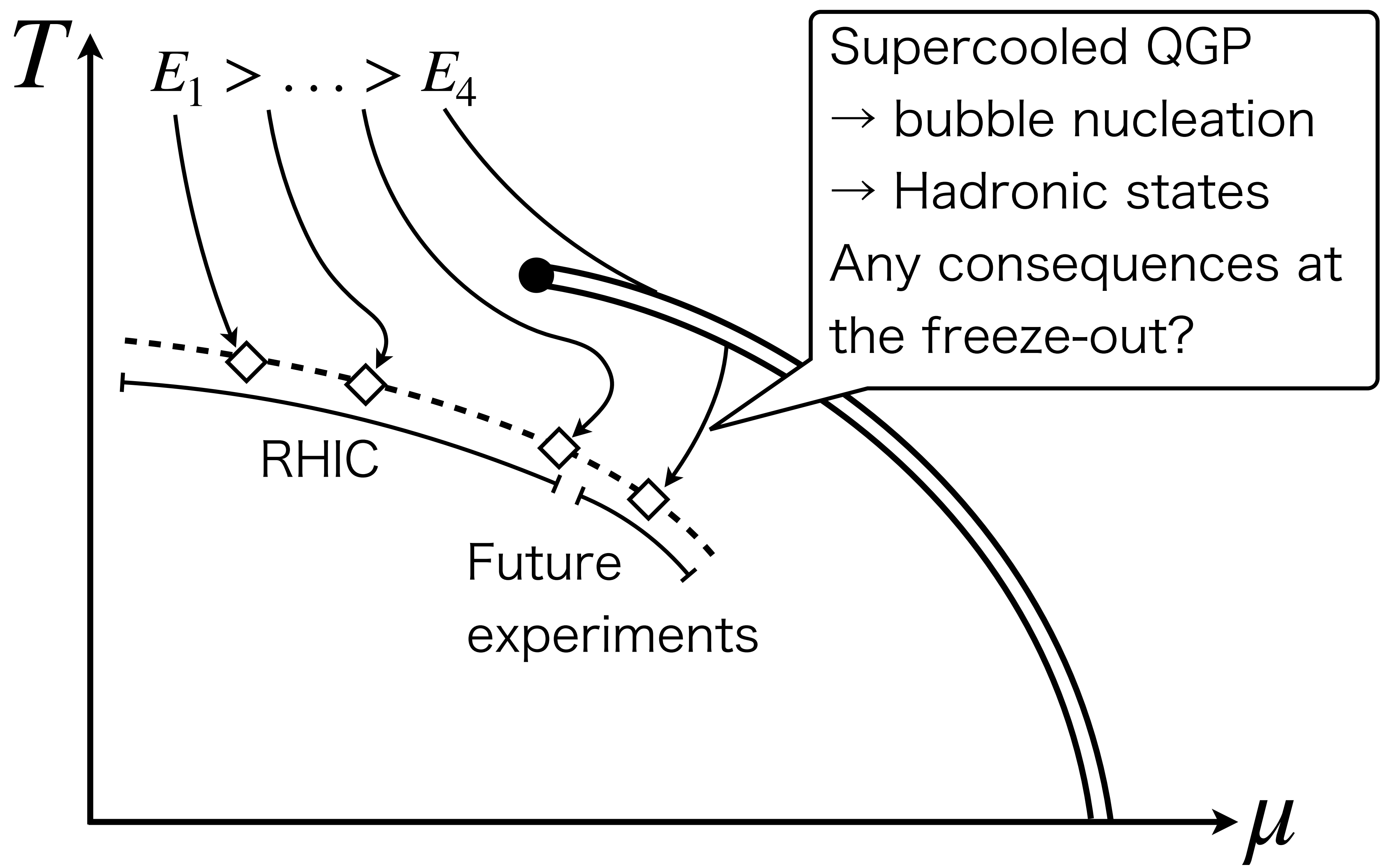
Solid: the solution, Dashed: equilibrium value



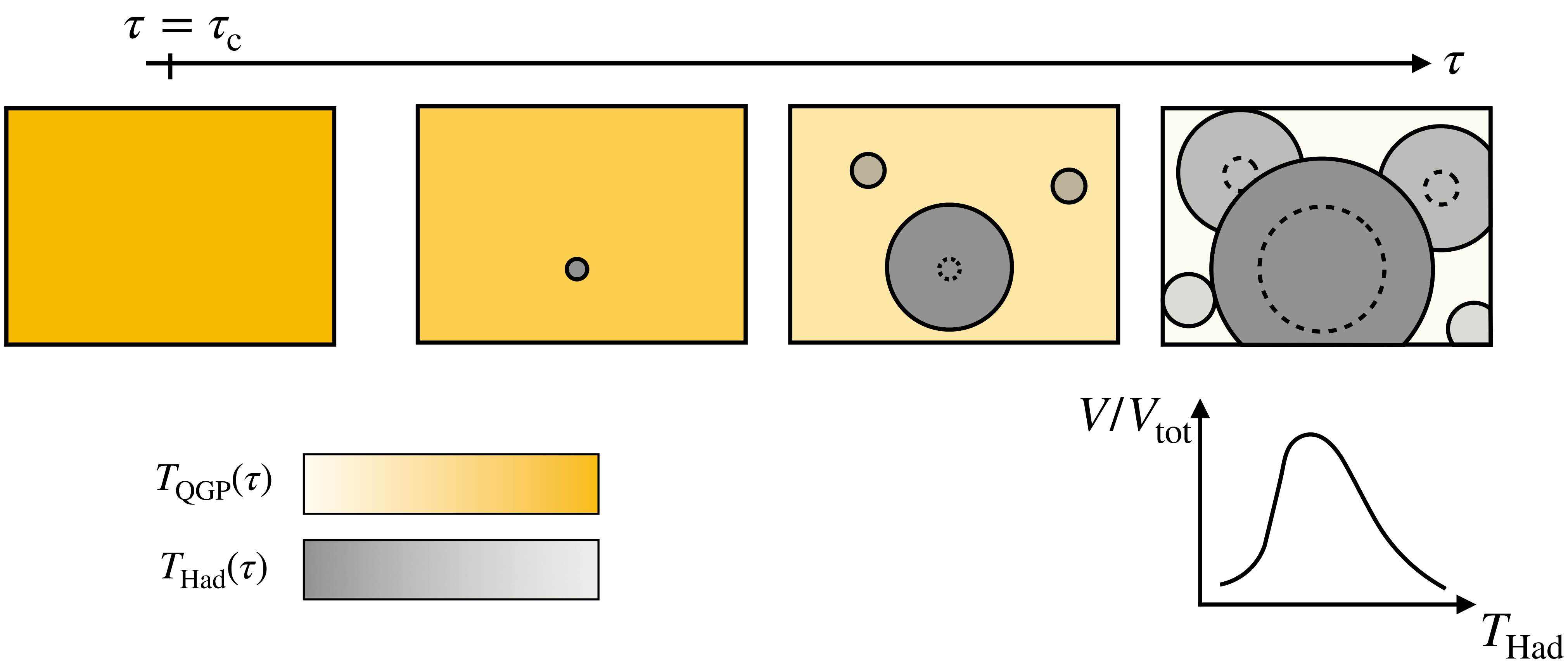
$$T(t) \sim T_\mathrm{I} \left(\frac{t}{t_\mathrm{I}}\right)^{-c_s^2}$$

- Nonequilibrium nature: peak (i) drops (ii) leg behind background evolution (memory effect)
- Long-lived contribution from low-wavelength $(|\boldsymbol{q}| \to 0)$?
- Application to ideal hydrodynamics (ongoing)

From fluctuations to first-order dynamics



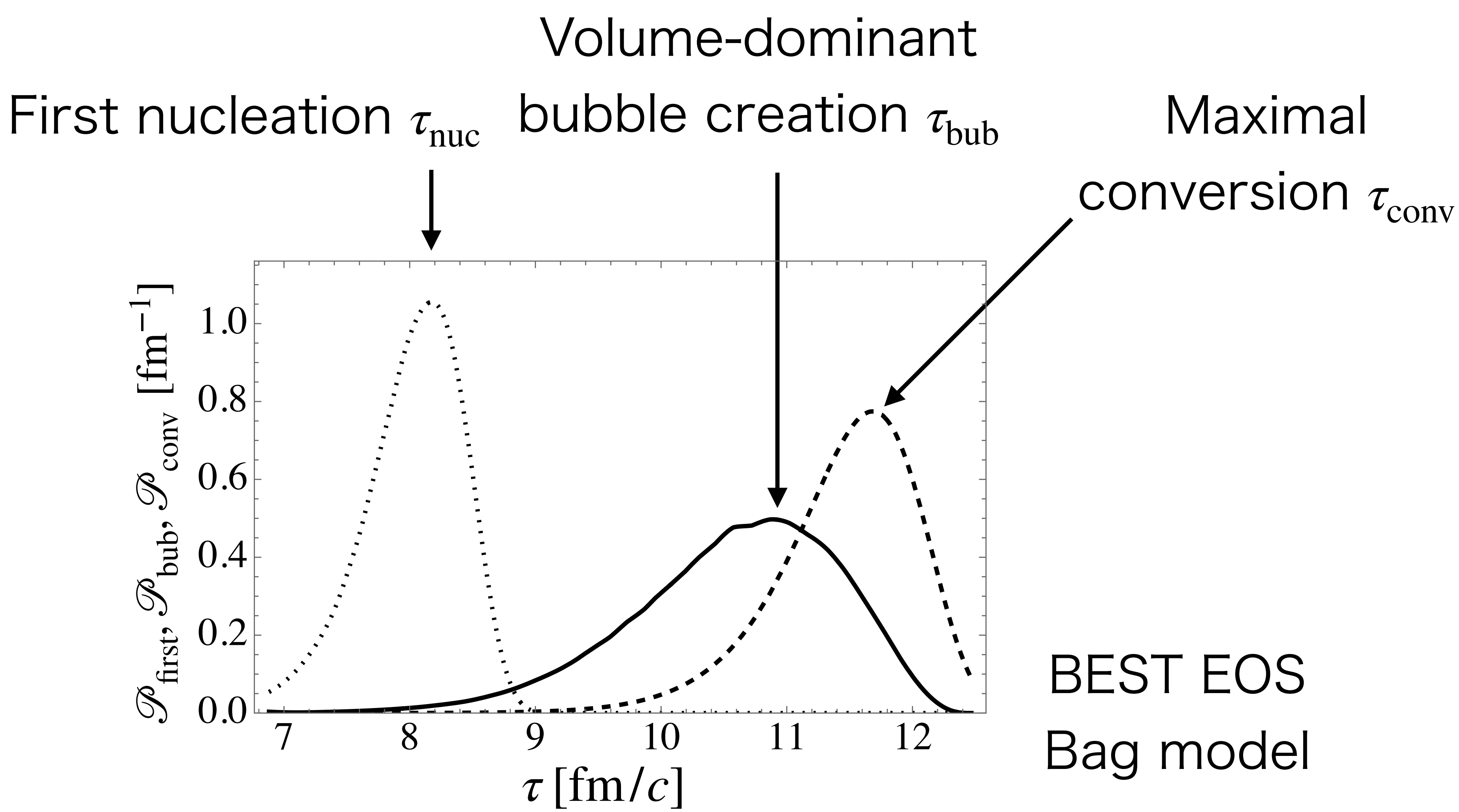
Bubble nucleation and growth in expanding media



- Expanding media $T_{QGP}(\tau) \to \tau$ dependent hadronic states $T_{Had}(\tau)$
- Distribution in (μ_{Had}, T_{Had}) (fractional volume weighted)

Three characteristic stages

NS, Stephanov, Yee (in preparation)

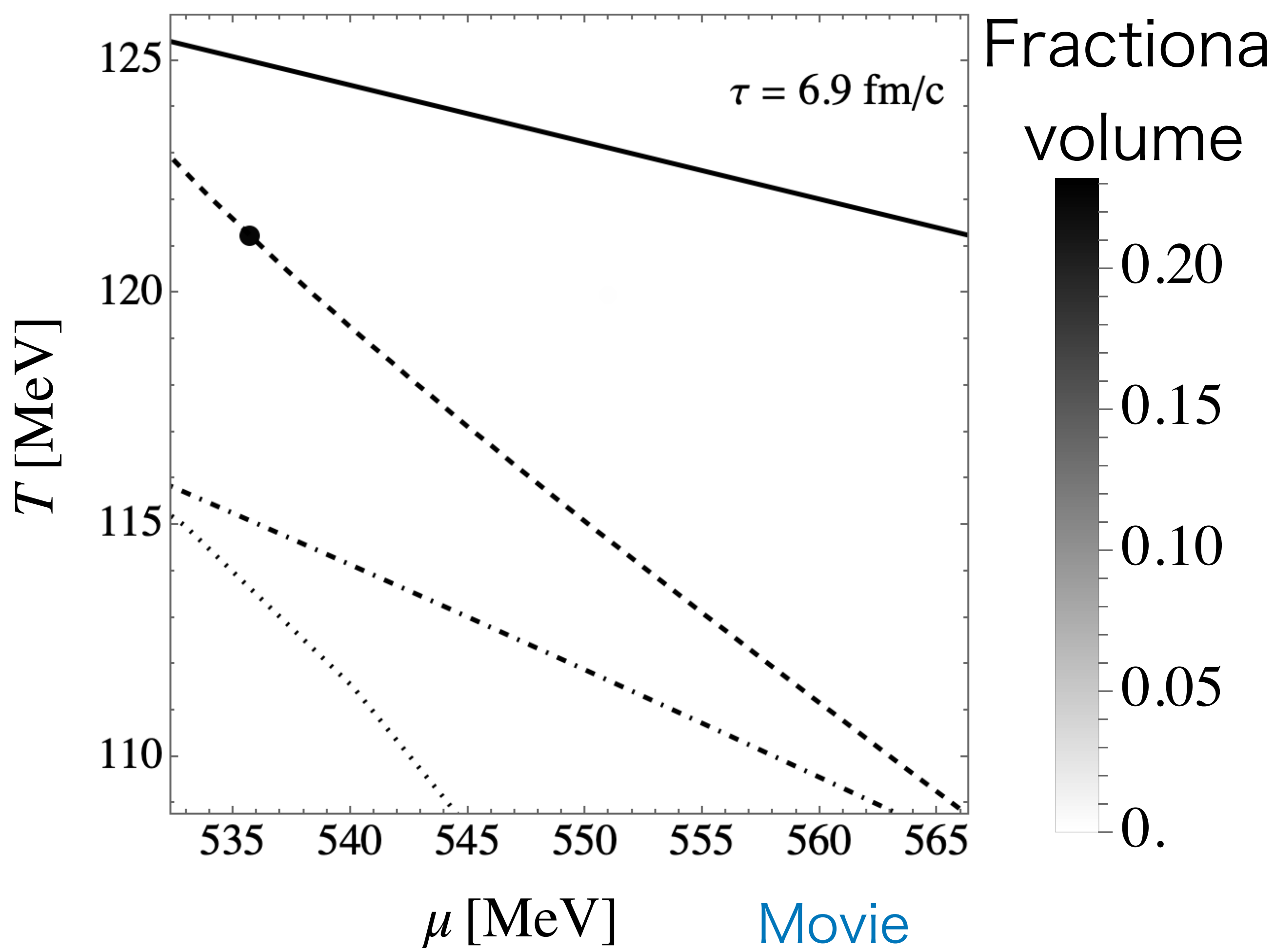


BEST EOS
Bag model

General hierarchy: $\tau_{\rm nuc} < \tau_{\rm bub} < \tau_{\rm conv}$

Hadronic state distribution

NS, Stephanov, Yee (in preparation)



Movie

Phase transition

supercooled QGP

Freeze-out curve

Metastable boundary

BEST EOS

$\mu_c = 450\,\text{MeV}$ $T_c = 136\,\text{MeV}$

$\alpha_1 = 7^\circ$ $\alpha_2 = -5^\circ$

$\rho = 1.0$ $w = 0.5$

Phase conversion generates hadronic fluid configurations beyond simple hydrodynamic evolution

Summary

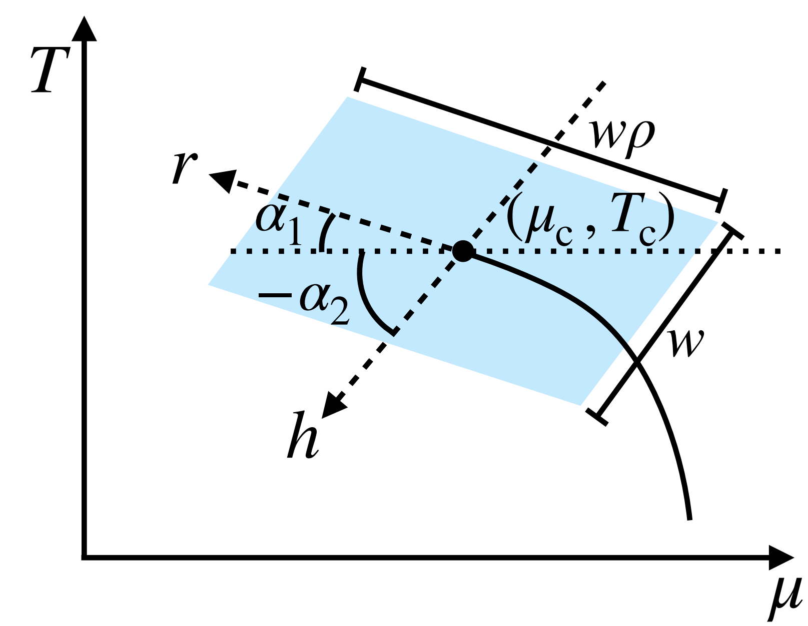
- The QCD critical point modifies hydrodynamic trajectories through the equation of state, potentially leaving observable signatures beyond fluctuations, such as freeze-out trajectory jumps. Quantitative predictions for flow observables require realistic hydrodynamic simulations.
- Quantitative comparisons with experiment require nonequilibrium fluctuation dynamics; simulations directly connecting such dynamics to experimental observables remain limited.
- At sufficiently low collision energies, hydrodynamics must be supplemented by phase-conversion dynamics, resulting in a nontrivial distribution of hadronic states.

Backup

Thermodynamics near QCD critical point

Nonaka and Asakawa (2005), Palotto et al. (2020)

- Gibbs free energy from the 3D Ising model $G(h,r)$
- Mapping parameters of QCD



$$\frac{\mu-\mu_c}{T_c} = -w(r\rho\cos\alpha_1 + h\cos\alpha_2)$$

$$\frac{T-T_c}{T_c} = w(r\rho\sin\alpha_1 + h\sin\alpha_2)$$

$\longrightarrow$ $h(\mu,T),\, r(\mu,T)$

$\longrightarrow$ Pressure: $P(T,\mu) - P_c = T_c^4 G(h(\mu,T), r(\mu,T)) + \cdots$

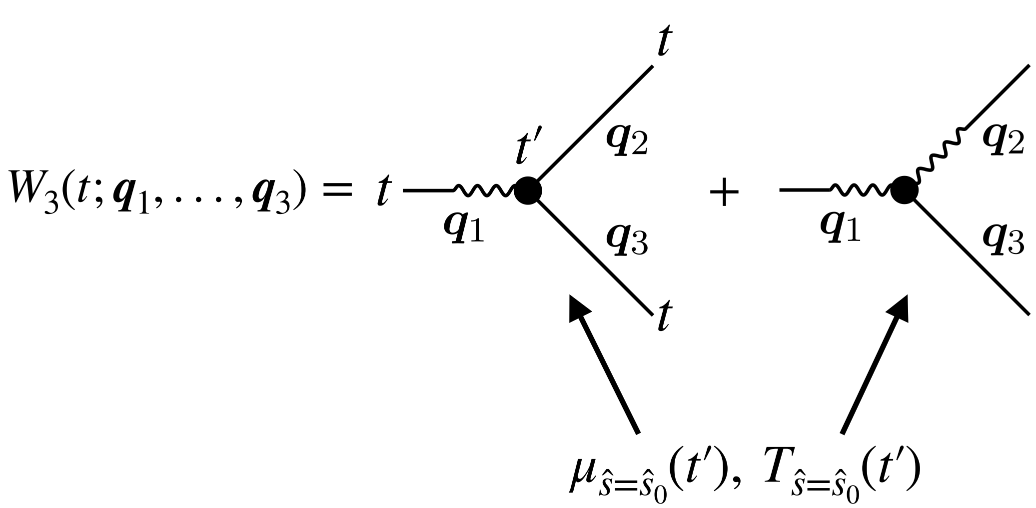
- Crossover from chiral limit $|\alpha_2 - \alpha_1| \sim \mathcal{O}(m_q^{2/5})$ Pradeep and Stephanov (2019)

Compute observables and determine by experimental data

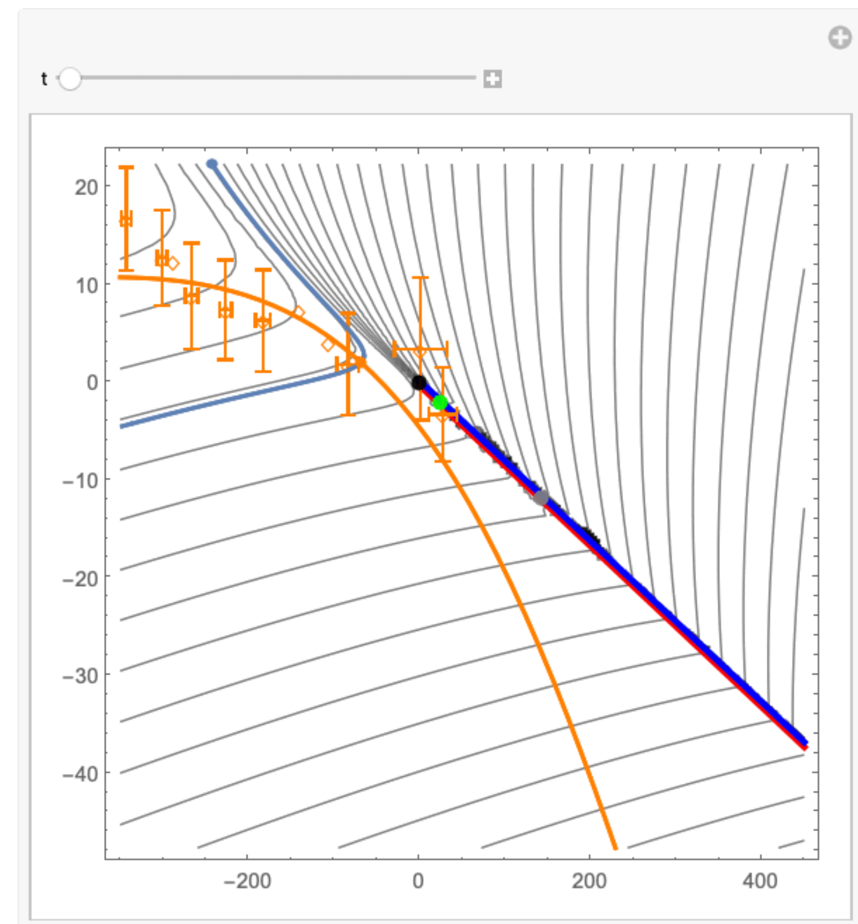
Baryon fluctuation dynamics in heavy-ion collisions

NS (in progress)

- Ideal background evolution near the critical point



- Limit the location of the critical point from experimental data
- Evaluate the memory effect



$\mu_{\hat{s}=\hat{s}_0}(t),\ T_{\hat{s}=\hat{s}_0}(t)$ along contours

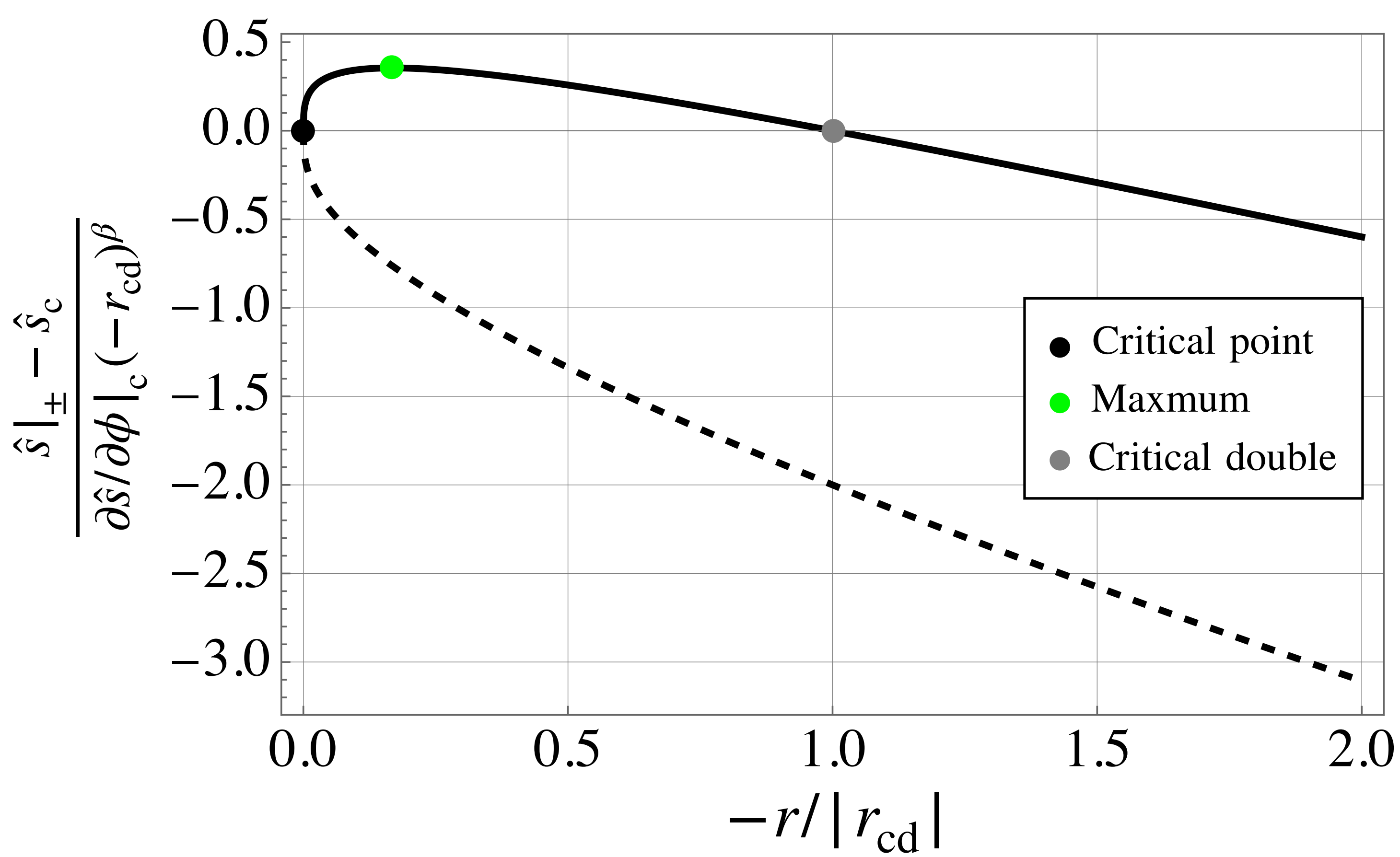
Orange: freeseout points

Non-monotonic specific entropy

Pradeep, Sogabe, Stephanov, and Yee (2024)

$$\hat{s}|_{\pm}(r) - \hat{s}_c = \pm \left.\frac{\partial \hat{s}}{\partial \phi}\right|_{\varepsilon,\,c} (-r)^{\beta} + \left.\frac{\partial \hat{s}}{\partial \varepsilon}\right|_{\phi,\,c} \tilde{\varepsilon}(0)(-r)^{1-\alpha} + \cdots \qquad (\alpha = 0.11, \quad \beta = 0.326)$$

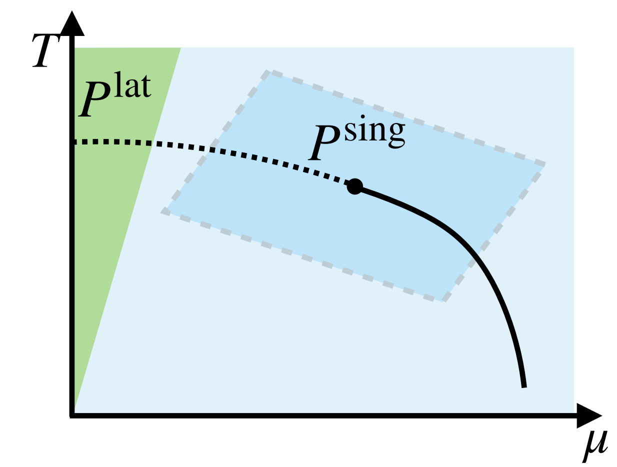
Relevant operators: $\phi \sim \pm(-r)^{\beta}$ $\qquad \varepsilon = \tilde{\varepsilon}(0)(-r)^{1-\alpha} \qquad (h = 0)$



Beam Energy Scan Theory (BEST)

Parotto et al. (2020)

- Matching with lattice QCD data



$$P = P^{\rm norm}(\mu, T) + P^{\rm sing}(\mu, T)$$

$T^4 c_n^{\rm norm} \equiv T^4 c_n^{\rm lat} - T_{\rm c}^4 c_n^{\rm sing}$ from $P^{\rm norm}(\mu, T)$

$$P^m(\mu, T) = T^4 \left[c_0^m(T) + c_2^m(T) \left(\frac{\mu}{T}\right)^2 + c_4^m(T) \left(\frac{\mu}{T}\right)^4 + \cdots \right] \quad (m = {\rm lat}\,,\ {\rm sing}\,,\ {\rm norm})$$

- T′ method: More than a technical refinement

Baryon fluctuation out of equilibrium

- Time evolution of nth cumulant: $W_n(t; \boldsymbol{q}_1, \ldots, \boldsymbol{q}_{n-1}) \equiv \langle \delta n(t, \boldsymbol{q}_1) \ldots \delta n(t, \boldsymbol{q}_n) \rangle_{\rm cum}$

  Under background hydrodynamic evolution
- Phenomenological approach (diffusive Langevin equation)

  An, Basar, Stephanov, and Yee (2021)
- Nonequilibrium effective field theory (EFT) NS and Yin (2022)
  - Systematic expansion (derivative, fields)
  - Field theoretical technique

$$\langle \delta n(t, \boldsymbol{q}_1) \ldots \delta n(t, \boldsymbol{q}_n) \rangle_{\rm connected} \propto \int \mathscr{D}n \mathscr{D}\theta_{\rm a}\, e^{iI_{\rm eff}[n,\theta_{\rm a}]} \delta n(t, \boldsymbol{q}_1) \ldots \delta n(t, \boldsymbol{q}_n)$$

$I_{\rm eff}[n, \theta_{\rm a}]$? $\theta_{\rm a}$? Diagram rules ?

Nonequilibrium EFT

- Microscopic: QCD with density matrix evolution (Schwinger Keldysh path)

$$e^{\frac{i}{\hbar}W[A_1,A_2]} = \int_{\rho_{t=0,\beta}} \mathscr{D}\Phi_1 \mathscr{D}\Phi_2 \, e^{\frac{i}{\hbar}S_{\mathrm{QCD}}[\Phi_1;A_1]-\frac{i}{\hbar}S_{\mathrm{QCD}}[\Phi_2;A_2]+\text{Gauge Fix.}} \qquad (\Phi = \{q, \bar{q}, G\})$$

A^μ : $\mathrm{U}(1)_\mathrm{B}$ gauge field, $\rho_{t=0,\beta}$: equilibrium state, β : Inverse temp.

- EFT:

$$e^{\frac{i}{\hbar}W[A_r,A_a]} = \int \mathscr{D}\theta_r \mathscr{D}\theta_a \, e^{iI_{\mathrm{eff}}[\theta_r,\theta_a;A_r,A_a]}$$

- Low-energy and wavelength ($\hbar$ expansion)

Classical: $A_r = \dfrac{A_1 + A_2}{2}$ Fluctuation: $\hbar A_a = A_1 - A_2$

- Degrees of freedom： θ_r, θ_a (θ_r related to μ)

Constraints from micro.

Lecture notes by Glorioso and Liu (2018)

1. Baryon symmetry: $A_{\mathrm{r/a},\mu} \to A_{\mathrm{r/a},\mu} - \partial_\mu \alpha, \quad \theta_{\mathrm{r/a}} \to \theta_{\mathrm{r/a}} + \alpha$

   $I_{\mathrm{eff}}[\theta_\mathrm{r}, \theta_\mathrm{a}; A_\mathrm{r}, A_\mathrm{a}] = I_{\mathrm{eff}}[\mathscr{A}_\mathrm{r}, \mathscr{A}_\mathrm{a}];$ Gauge invariants: $\mathscr{A}_{\mathrm{r/a}\mu} = A_{\mathrm{r/a}\mu} + \partial_\mu \theta_{\mathrm{r/a}}$

2. Kubo-Martin-Schwinger (KMS) symmetry

   Classical limit of a generating functional identity: $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}W[A_{1\mu}(t), A_{2\mu}(t)]} = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}W[(-1)^\mu A_{1\mu}(-t), (-1)^\mu A_{2\mu}(-t-\mathrm{i}\hbar\beta_0)]}$

$$\mathscr{A}_{\mathrm{r}\mu}(t) \to (-1)^\mu \mathscr{A}_{\mathrm{r}\mu}(-t) + \mathscr{O}(\hbar)\,; \quad \mathscr{A}_{\mathrm{a}\mu} \to (-1)^\mu \Big[\mathscr{A}_{\mathrm{a},\mu} - \mathrm{i}\beta\hbar(\partial_t \mathscr{A}_{\mathrm{r}\mu})\Big](-t) + \mathscr{O}(\hbar^2)$$

   Time reversal: $(-1)^\mu = 1 \ (\mu = t),\ -1 \ (\mu = x, y, z)$

3. Unitarity ⟶ $I_{\mathrm{eff}}[\mathscr{A}_\mathrm{r}, \mathscr{A}_\mathrm{a}]$ Leading order: $\mathscr{O}(\mathscr{A}_\mathrm{a})$

4. Shift "symmetry:" $\theta_\mathrm{r} \to \theta_\mathrm{r} + \Lambda(\boldsymbol{x})$

   Reduce redundant degrees of freedom

   ∵ Chemical potential (time derivative) $\mu \equiv \mathscr{A}_{\mathrm{r}0} = A_{\mathrm{r}0} + \partial_0 \theta_\mathrm{r}$

Effective Lagrangian

NS and Yin (2021)

- Baryon number: $n = J^t_{\mathrm{r}} = \dfrac{\delta I_{\mathrm{eff}}}{\delta A^t_{\mathrm{a}}}$, $\mu \equiv \mathscr{A}_{\mathrm{r}t}$, No external field $(A_{\mathrm{r/a},\mu} = 0)$, $\hbar = 1$

$$\mathscr{L}_{\mathrm{eff}} = n\partial_t\theta_{\mathrm{a}} - \lambda(n)\nabla\theta_{\mathrm{a}} \cdot (\beta\nabla\mu - \mathrm{i}\nabla\theta_{\mathrm{a}}) + \text{higher orders}$$

θ_{a} necessary for keeping the KMS symemtry (~ Langevin noise)

- Equation of motion：

$$\frac{\delta I_{\mathrm{eff}}}{\delta\theta_{\mathrm{a}}} = 0 \;\Leftrightarrow\; \partial_t n + \nabla\cdot\boldsymbol{J} = 0; \quad \boldsymbol{J} = -\lambda(n)\nabla\mu + 2\mathrm{i}\lambda(n)\nabla\theta_{\mathrm{a}} + \cdots$$

Diffusive Langevin + higher order terms

Pertuvative expansion

NS and Yin (2021)

- Scale separation between the background hydrodynamics and the fluctuation
- Fluctuation from the background evolution: $\delta n = n - n_0\,,\ \delta\theta = \theta_{\rm a}$

$$\mu = \frac{1}{\chi}\delta n + \xi^2 \nabla^2 \delta n + \cdots\ , \quad \lambda(n) = \lambda + \lambda' \delta n + \cdots$$

Susceptibility : $\chi = \dfrac{\partial n_0(\mu, T)}{\partial \mu}$, correlation length: $\xi(\mu, T)$, Diffusion rate

$\lambda(\mu, T)$: All fixed by background evolution

$$\mathscr{L}_{\rm eff} = (\mathscr{L})_2 + (\mathscr{L})_3 + (\mathscr{L})_4 + \cdots$$

Diffusion eq. + Noise → $(\mathscr{L})_2$

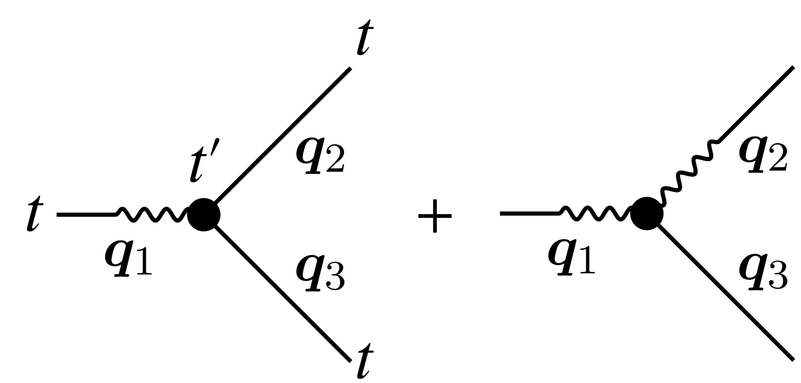
Nonlinear couplings/noise → $(\mathscr{L})_3$, $(\mathscr{L})_4$

Nonequilibrium fluctuation dynamics

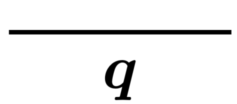
NS and Yin (2021)

- Diagram calculation

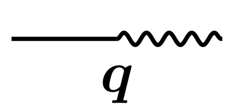
$W_3(t; \boldsymbol{q}_1, \ldots, \boldsymbol{q}_3) =$



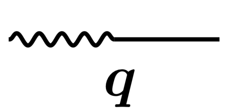
- Feynman rules:



$= G_{\mathrm{rr}}(t, t', \boldsymbol{q})$



$= G_{\mathrm{ra}}(t, t', \boldsymbol{q}) = G_{\mathrm{ar}}(t', t, \boldsymbol{q})$



$= G_{\mathrm{ar}}(t, t', \boldsymbol{q})$

+ vertices

- Schwinger-Dyson equations：

$$-\partial_t G_{\mathrm{ar}}(t, t', \boldsymbol{q}) + \Gamma(t, \boldsymbol{q}) G_{\mathrm{ar}}(t, t', \boldsymbol{q}) = \delta(t - t')$$

$$\partial_t G_{\mathrm{rr}}(t, t', \boldsymbol{q}) + \Gamma(t, \boldsymbol{q}) G_{\mathrm{rr}}(t, t', \boldsymbol{q}) = \lambda(t) \boldsymbol{q}^2 G_{\mathrm{ar}}(t, t', \boldsymbol{q})$$

$$G_{\mathrm{ar}}(t' - 0, t', \boldsymbol{q}) = 1\,, \quad G_{\mathrm{rr}}(t_{\mathrm{I}}, t_{\mathrm{I}}, \boldsymbol{q}) = \bar{W}_2(t_{\mathrm{I}}, \boldsymbol{q})$$

$$\Gamma(t, \boldsymbol{q}) \equiv \lambda(t) \boldsymbol{q}^2 \bar{W}_2^{-1}(t, \boldsymbol{q})\,, \quad \bar{W}_2^{-1}(t_{\mathrm{I}}, \boldsymbol{q}) = \chi^{-1}(t_{\mathrm{I}}) + \xi(t_{\mathrm{I}})^2 \boldsymbol{q}^2$$

Arbitral background evolution: $\chi = \chi(\mu(t), T(t)), \xi = \xi(\mu(t), T(t)),$ etc.

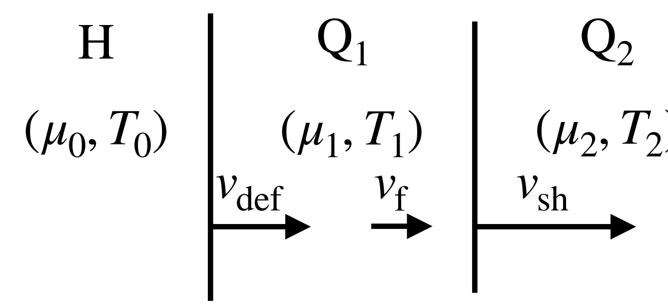
Hydrodynamic matching

- Energy, momentum, and baryon number conservations

  → Hadronic and reheated QGP states

$$(\mu_2, T_2) \xrightarrow[\hat{s}_1 \equiv \frac{s_1}{n_1}]{\text{3 eqs.}} (\mu_1, T_1) \xrightarrow{\text{3 eqs.}} (\mu_0, T_0)$$

$v_{\rm sh}, v_{\rm f}$ at (μ_1, T_1); $v_{\rm def}$ at (μ_0, T_0)



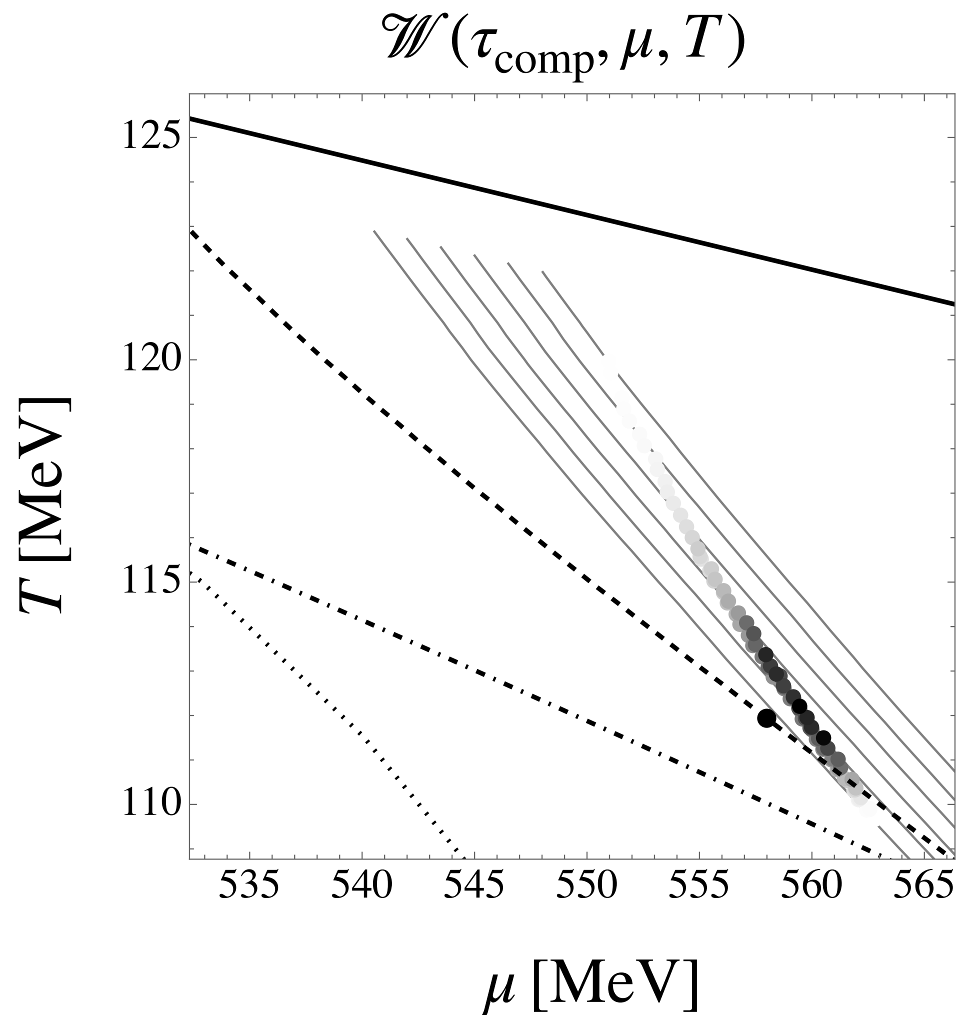
$\mathcal{S}_{\hat{s}_1}(\tau) \equiv (\mu_0, T_0, \mu_1, T_1, v_{\rm def}, v_{\rm f}, v_{\rm sh})$: one-parameter family of solutions

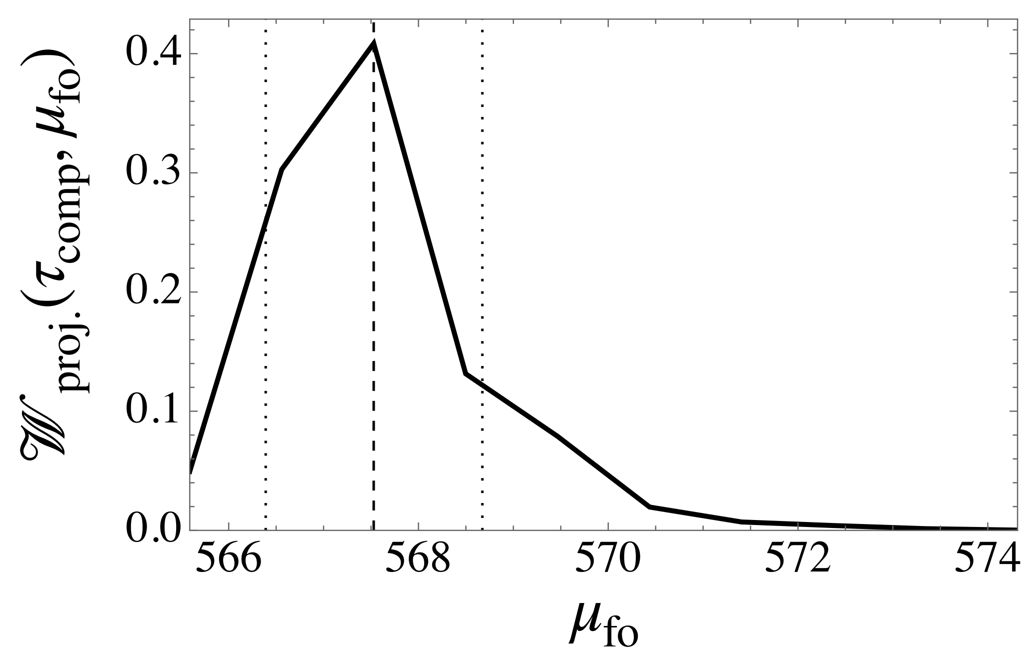
- Assuming all bubbles $\mathcal{S}_{\hat{s}_1}(\tau)$ created stochastically in space

- Compute the size of bubbles growing with $v_{\rm def}$

Hadronic state evolution

NS, Stephanov, Yee (in preparation)





Hadronic state distribution after full conversion $\mathscr{W}(\tau_{comp}, \mu, T)$

→ Ideal adiabatic evolution (gray contours)

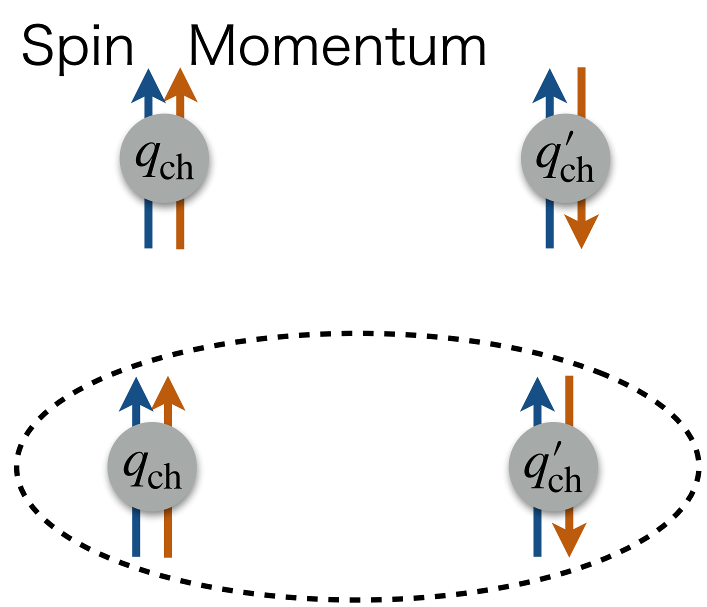
→ Projection to the freeze-out curve

Chiral physics in Color Superconductors (CSC)

Chiral fermions

QGP

CSC



Copper Pairing

Chiral charges: $q_{\rm ch}, q'_{\rm ch} = \pm 1/2$

Phenomena

- Chiral magnetic effect
- Chiral vortical effect
- ⋯

- Topologically protected nodes of pairing gap function
- Topological gapless excitation

Sogabe and Yi, arXiv:2411.08005

Opens new directions for chiral physics in dense QCD matter

A puzzle in spin-one CSC

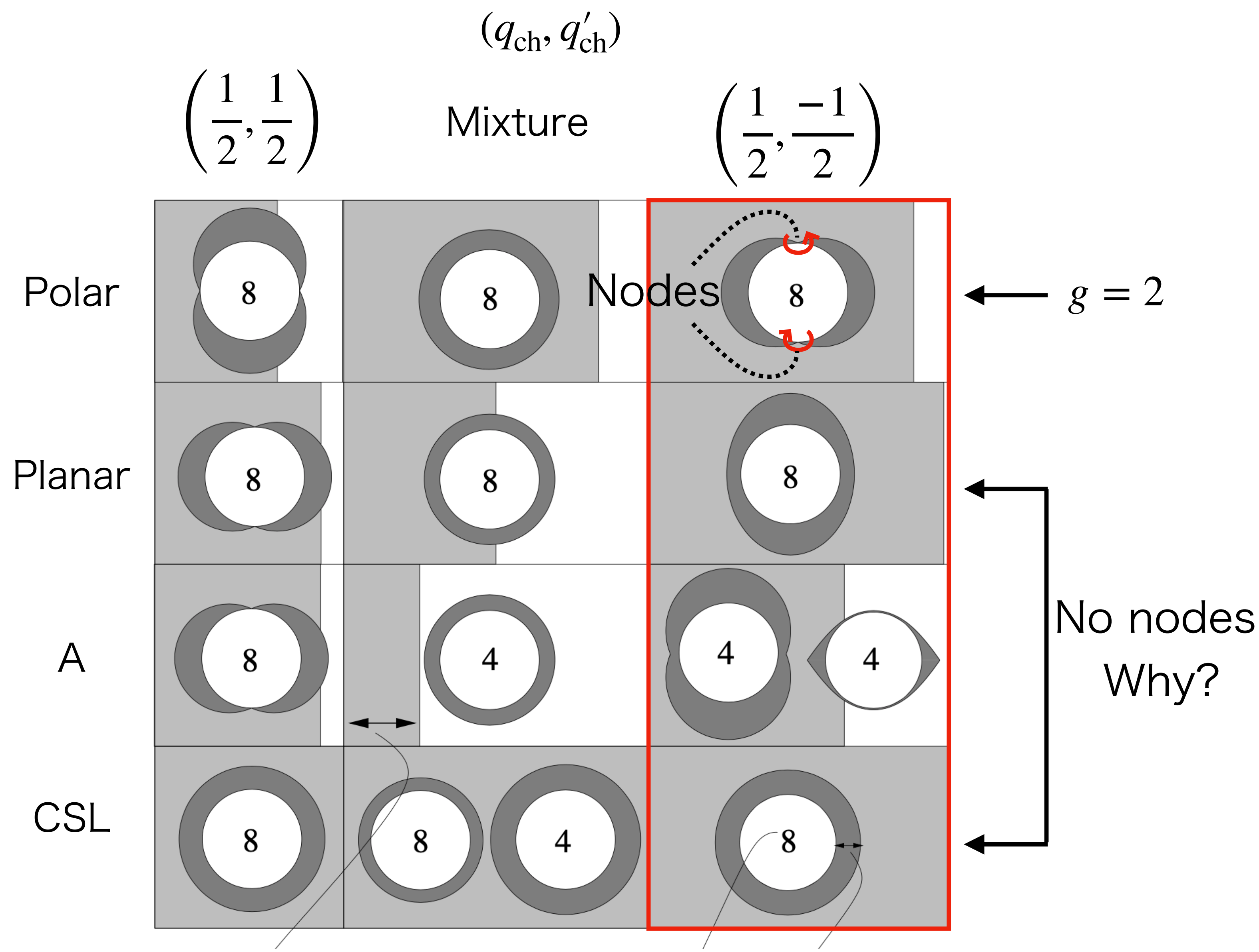
What was known in condensed matter physics:

$$2\Delta q_{\text{ch}} \equiv 2(q_{\text{ch}} - q'_{\text{ch}}) = g$$

g : Circulation around nodes (topological number)

Li and Haldane (2018)
Murakami and Nagaosa (2003)

Topologically protected nodes when $0 \neq 2\Delta q_{\text{ch}} = g$



Schäfer (2000), Table from Schmitt (2005)

Pairing gap functions in momentum space

Our solution: Color d.o.f.

Sogabe and Yi, arXiv:2411.08005

Generalized formula including the color degrees of freedom:

$$2\Delta q_{\mathrm{ch}} = g + \Delta q_{0,\mathrm{co}}$$

$$\Delta q_{0,\mathrm{co}} \equiv q_{0,\mathrm{co}} - q'_{0,\mathrm{co}}$$

Berry monopole charges from the color wave function of the gapless excitations

Scenario A

$$2\Delta q_{\mathrm{ch}} = g$$

- Weyl metals Li and Haldane (2018)
- Polar, A phases in spin-one CSC

Scenario B

$$2\Delta q_{\mathrm{ch}} = \Delta q_{0,\mathrm{co}}\,, \quad g = 0$$

- Color-spin-locking, planner phases is spin-one CSC

A new scenario in both condensed matter and nuclear physics

Theory vs. experiment

Stephanov (2024)