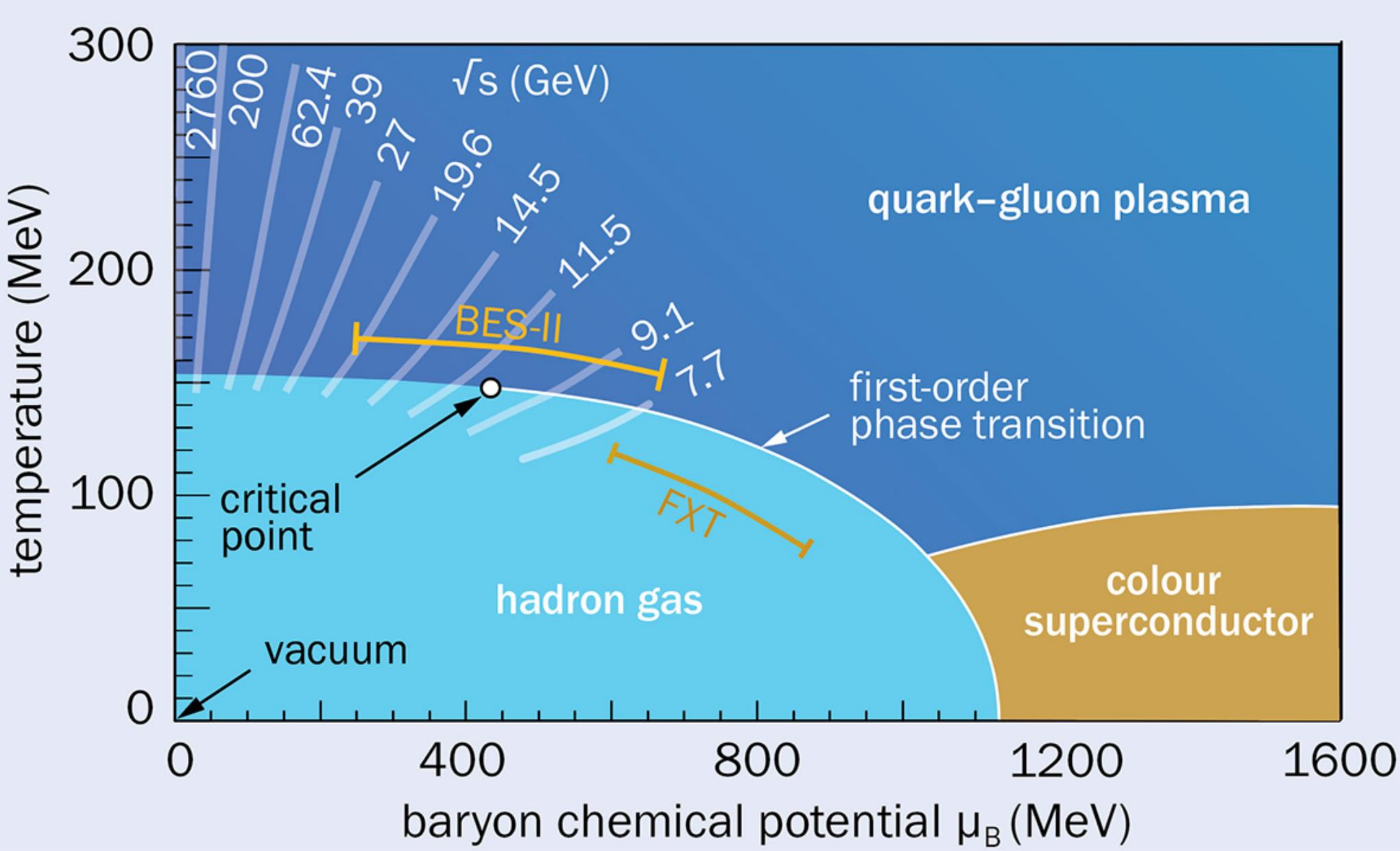
In-in formalism and real-time dynamics in external electric field

Shuhei Minato

(Collaboration with Kenji Fukushima)

Based on: Fukushima, SM, JHEP 05 (2026) 139; arXiv: 2512.19337 [hep-ph]

Introduction and Motivation



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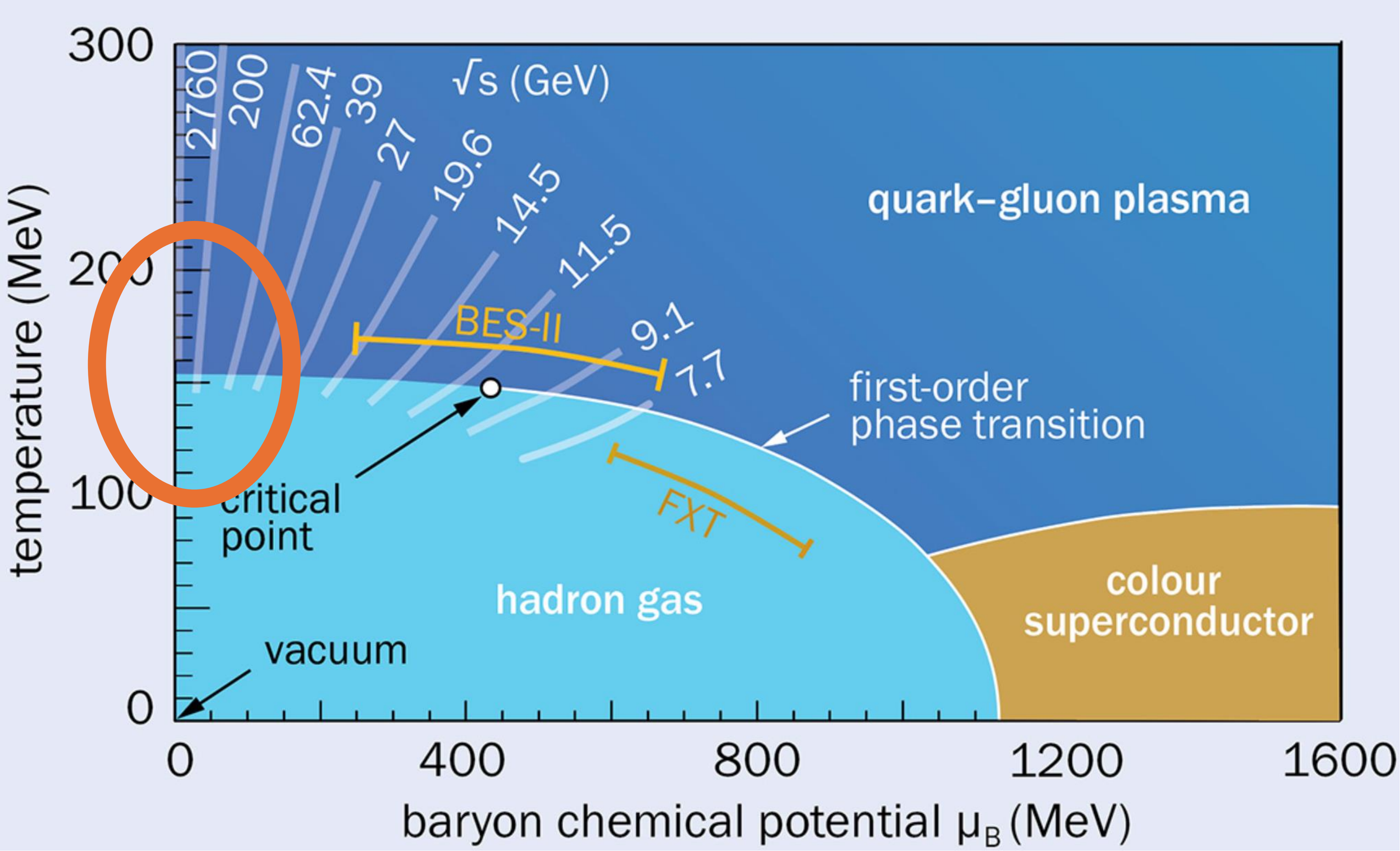

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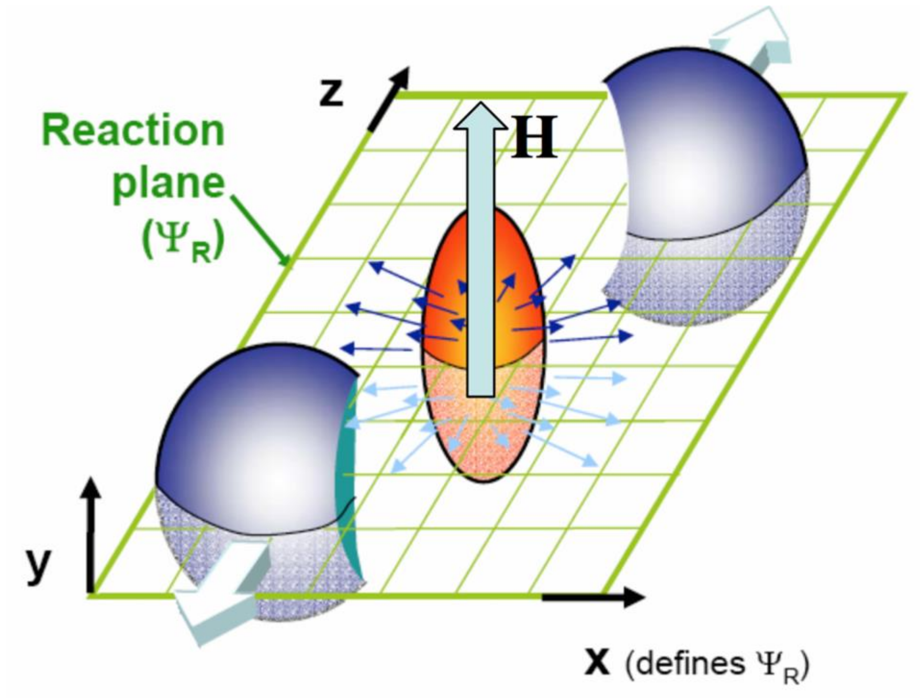
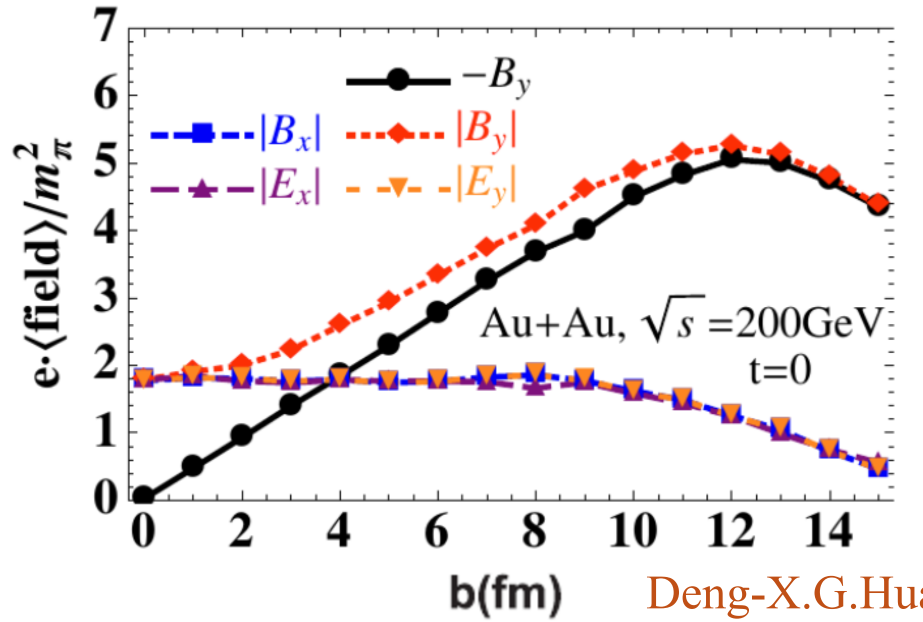
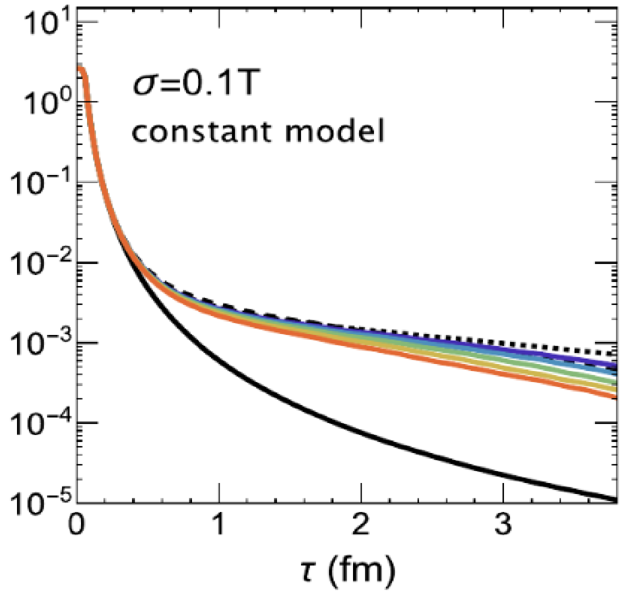




Deng-X.G.Huang (2012)

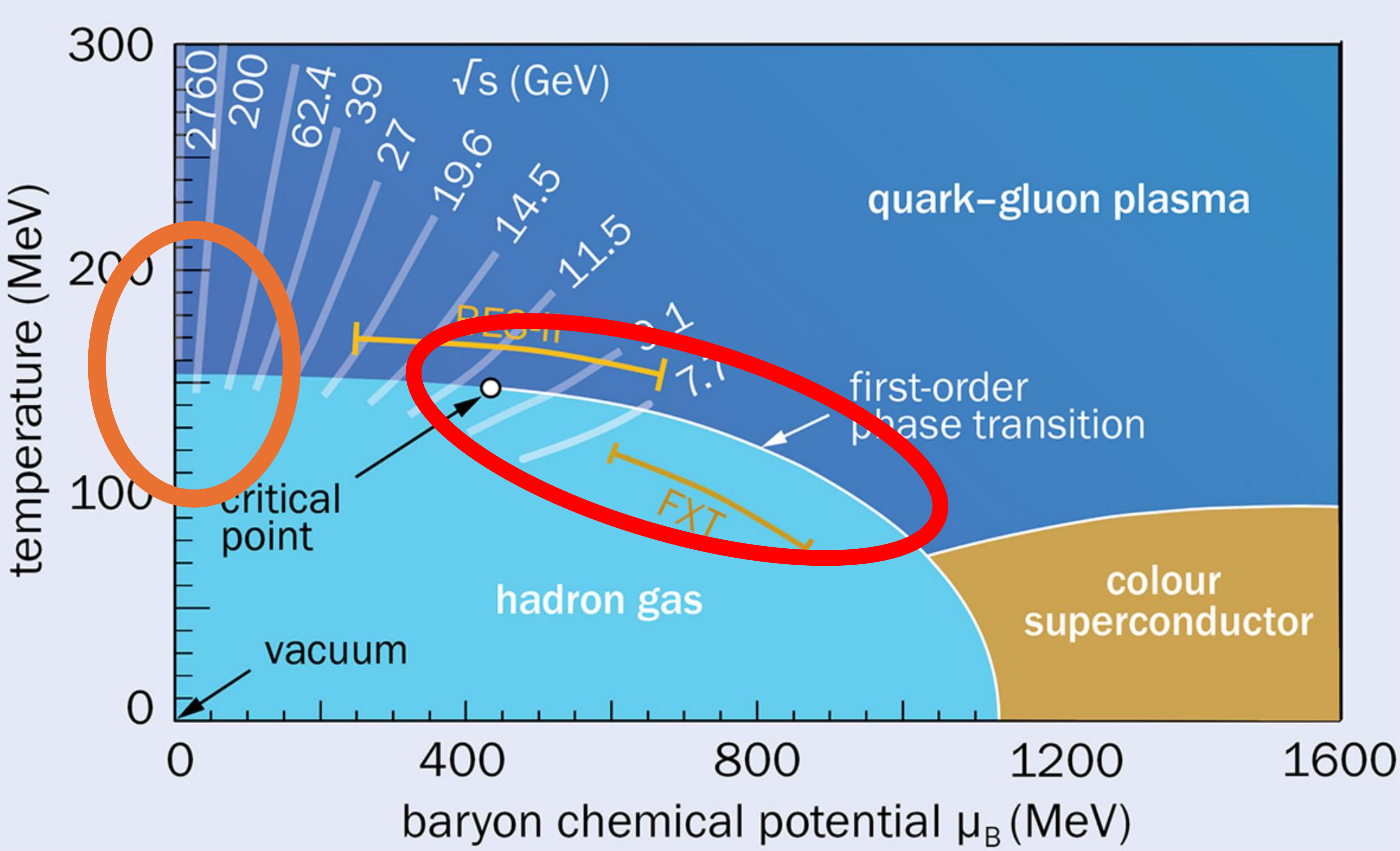


A.Huang+ (2012)

(Lienard-Wiechert + event-by-event fluctuation)

- Strong field $eB \sim 10^{18-20}$Gauss $\sim m_\pi^2$
- Magnetic field survives as small tail
- Electric field acts impulsively

Introduction and Motivation



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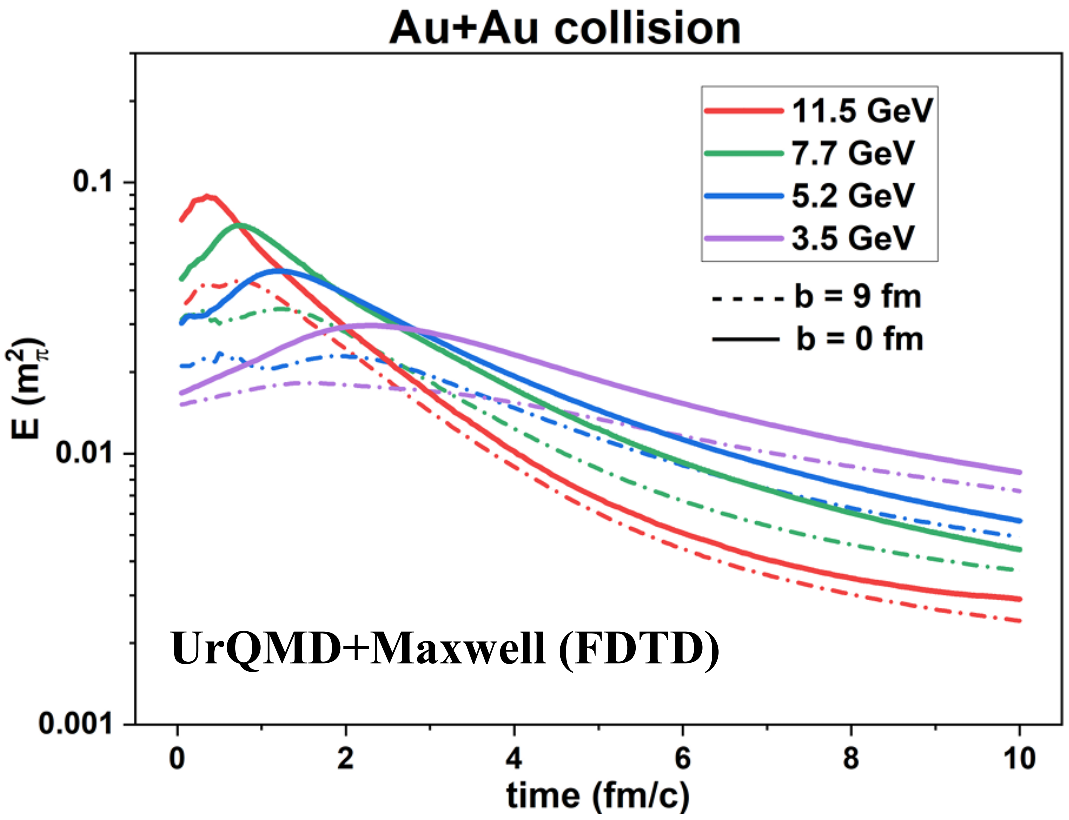
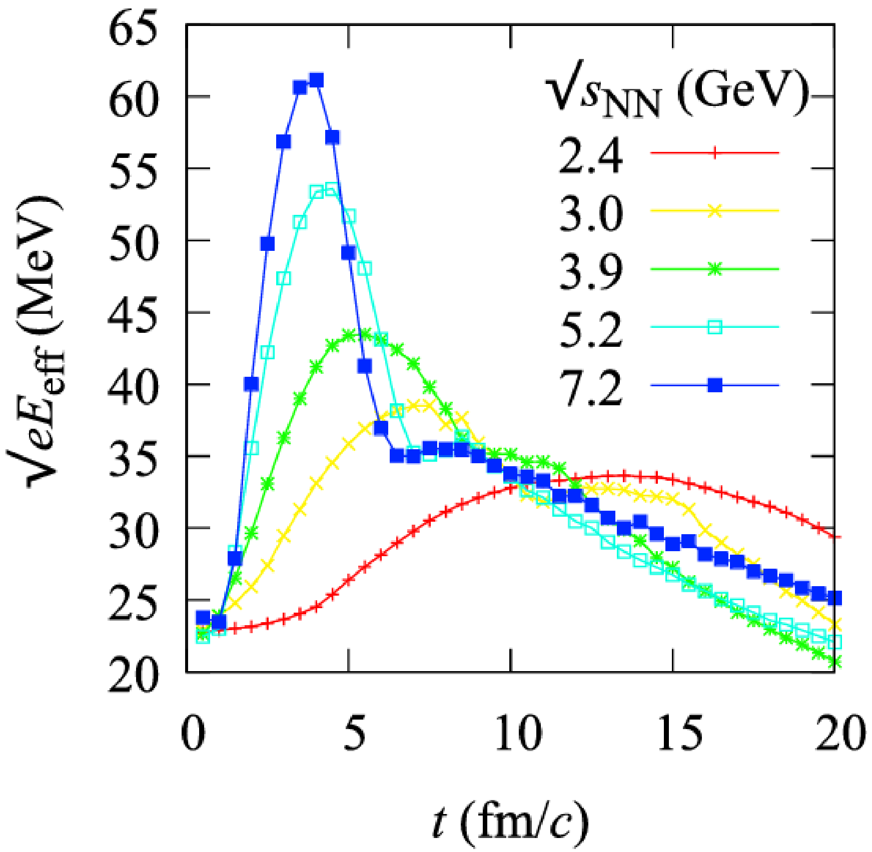


Siddique+ (2025)



JAM Cascade

Taya-Nishimura-Ohnishi (2024)

- Strong electric field $eE \sim (50\mathrm{MeV})^2$
- Long-lived electric field $\tau \sim 10\mathrm{fm/c}$

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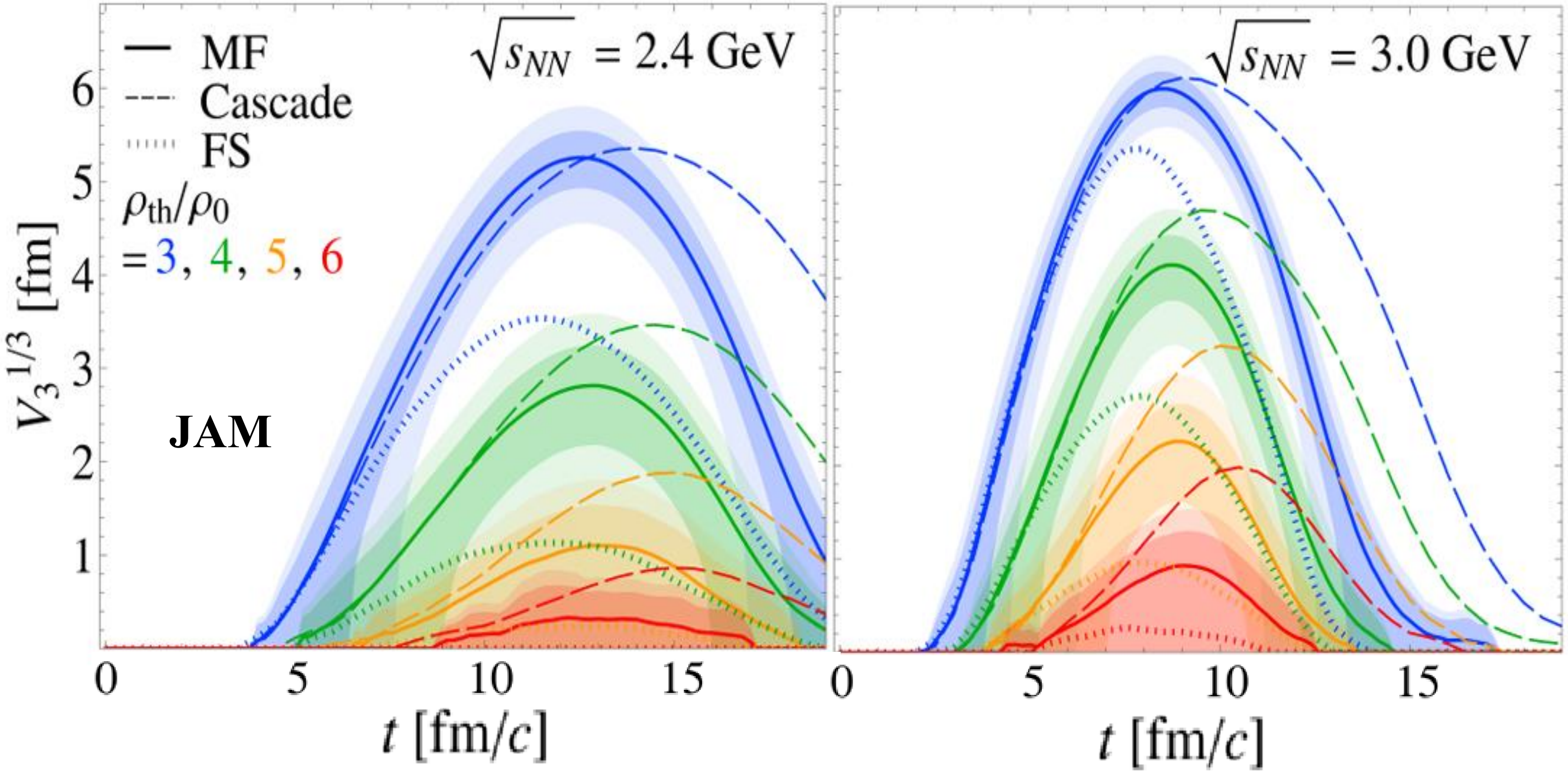
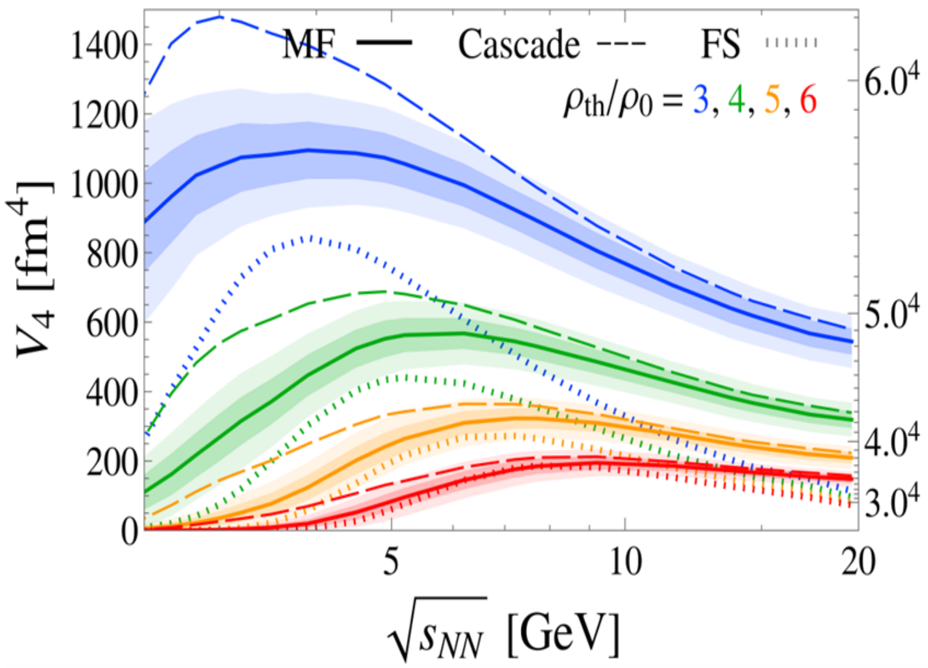




Taya-Jinno-Kitazawa-Nara (2025)

- Strong electric field $eE \sim (50\mathrm{MeV})^2$
- Long-lived electric field $\tau \sim 10\mathrm{fm/c}$
- Long-lived, large volume dense matter $\rho \sim 3\rho_0$

$$V_3(t;\rho_{\rm th}) = \int d^3x\,\gamma(x)\,\Theta(\rho_B(x)-\rho_{\rm th})$$

$$V_4(\rho_{\rm th}) = \int dt\,d^3x\,\Theta(\rho_B(x)-\rho_{\rm th})$$

$$\rho_B(x) = \sqrt{j_B^\mu(x) j_{B\mu}(x)}, \qquad \gamma(x) = \frac{j_B^0(x)}{\rho_B(x)}$$

Introduction and Motivation

- HIC creates QCD matter together with strong electromagnetic field

- Probes of the QCD phase-diagram may be affected by the transient $\boldsymbol{E}$ and $\boldsymbol{B}$

  e.g., Fluctuation / Correlation: Bzdak+ (2020), Ding+ (2024)
  Freeze-out: Fukushima-Hidaka (2016), Vovchenko (2024)
  Dilepton: Rapp+ (2016), Nishimura-Kitazawa-Kunihiro (2023), Taya (2025)

- This motivates a real-time study of QCD matter in external electromagnetic field.

Physics in Magnetic Field

- magnetic catalysis of chiral condensate (low T)

  Shovkovy (2013)

- inverse magnetic catalysis in lattice QCD ($T \sim T_c$)

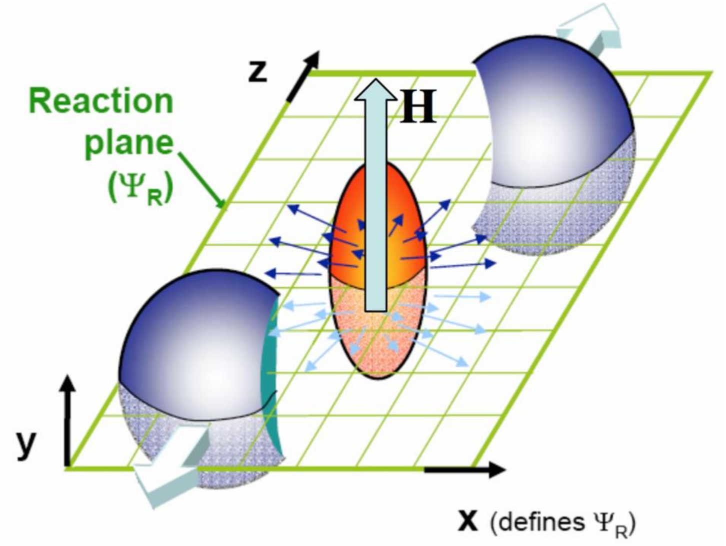
  Bruckmann-Endrodi-Kovacs (2013)

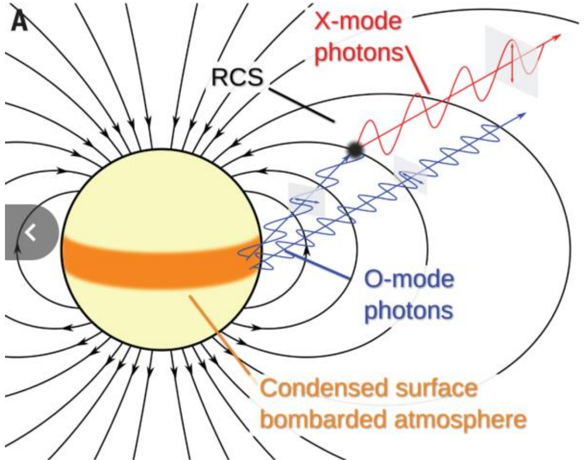
- Chiral Magnetic Effect

  Fukushima-Kharzeev-Warringa (2008)

- Polarized X-ray from Neutron Stars

  Taverna+ (2022)

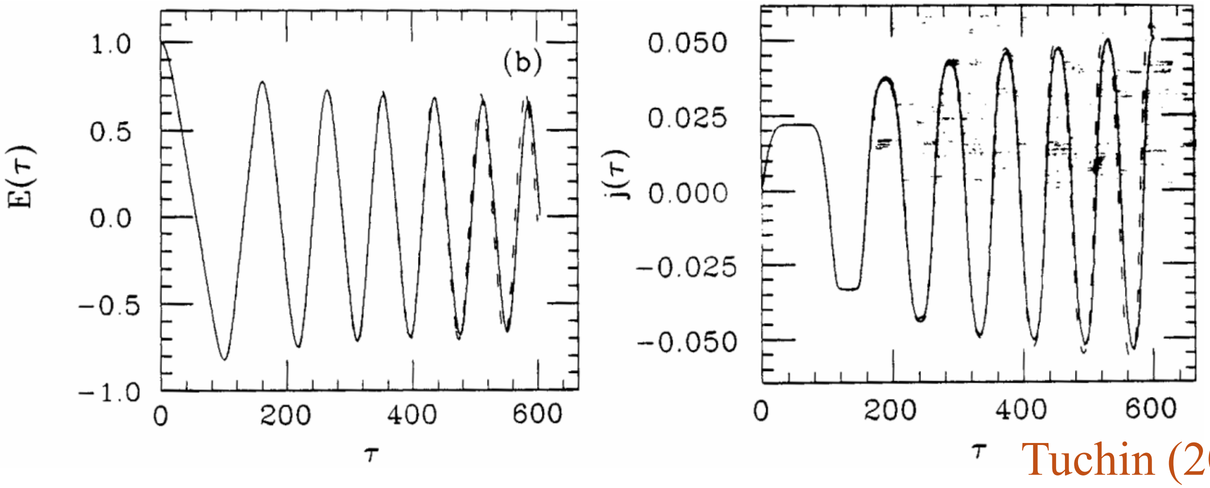




Physics in Electric Field

- Schwinger mechanism

  Schwinger (1951)



Tuchin (2013)

- Kinetic approaches

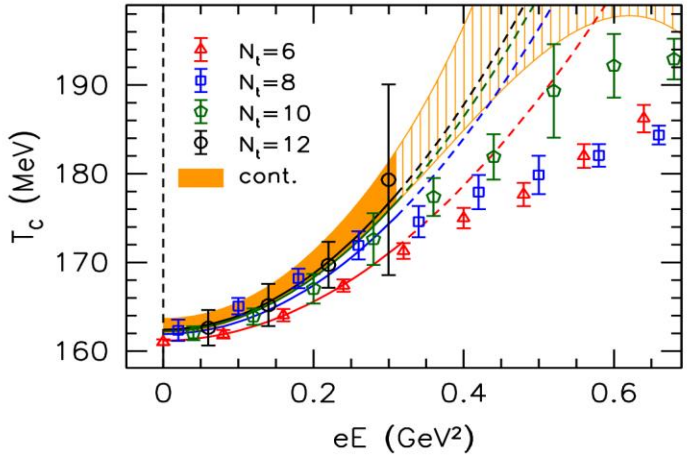
  Stephanov-Yin (2012), Hidaka-Pu-Wang-Yang (2022)

$$\sqrt{G}\partial_t f + \sqrt{G}\dot{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{x}} f + \sqrt{G}\dot{\boldsymbol{p}} \cdot \nabla_{\boldsymbol{p}} f = C[f], \quad \sqrt{G} = 1 + \hbar \boldsymbol{B} \cdot \boldsymbol{\Omega}$$

$$\sqrt{G}\dot{\boldsymbol{x}} = \boldsymbol{v} + \hbar \boldsymbol{E} \times \boldsymbol{\Omega} + \hbar(\boldsymbol{v} \cdot \boldsymbol{\Omega})\boldsymbol{B}$$

- Lattice QCD simulations

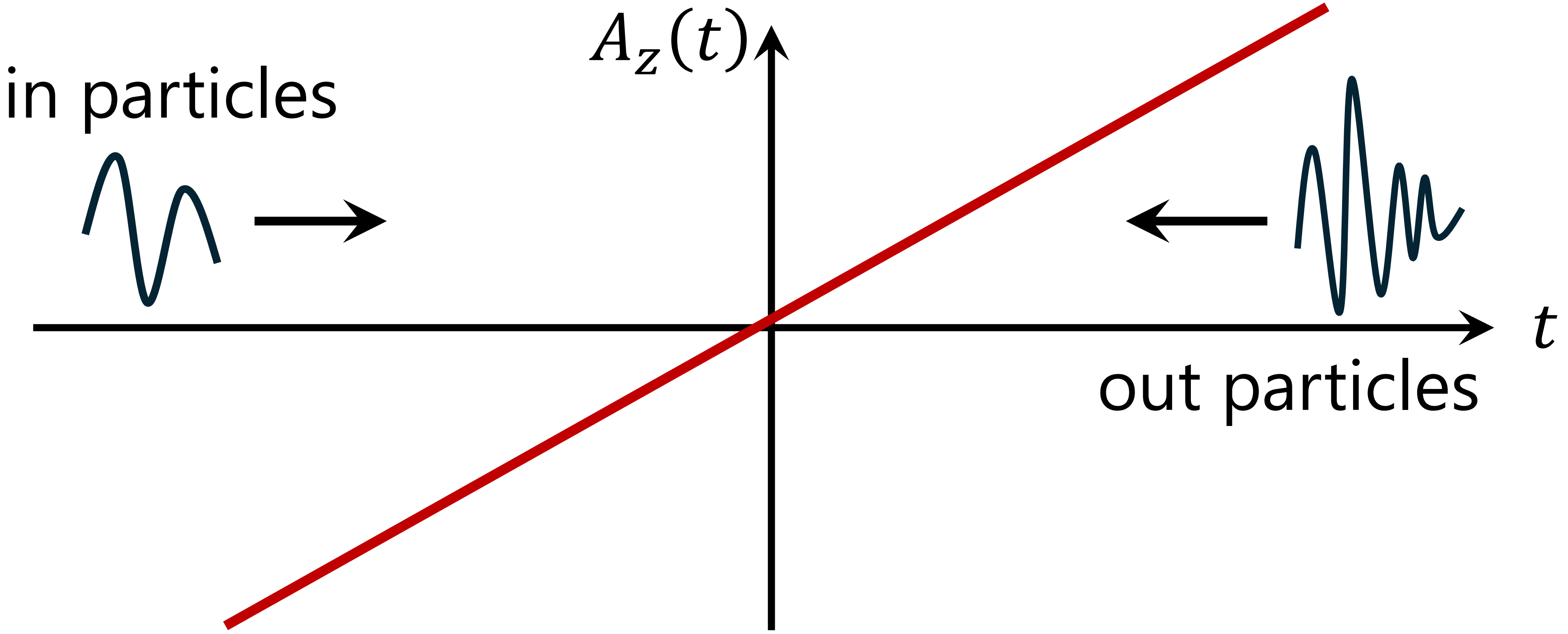
  Yamamoto (2013), Endrodi (2022)



In / Out states

- Time-dependent Hamiltonian (temporal gauge)

$$A_z(t) = -eEt, \qquad H = H(A(t))$$



Unstable vacuum

- The notion of the vacuum is time dependent

$$|0_{\mathrm{in}}\rangle \neq |0_{\mathrm{out}}\rangle$$

$$|\langle 0_{\mathrm{out}}|0_{\mathrm{in}}\rangle|^2 \neq 1$$

- Decay rate

$$P = 1 - |\langle 0_{\mathrm{out}}|0_{\mathrm{in}}\rangle|^2 = \frac{e^2E^2}{4\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp\left(-n\frac{\pi m^2}{eE}\right)$$

(for $\boldsymbol{B} = 0$)

Expectation value: in-in formalism

- Two kinds of observables

$$\langle 0_{\mathrm{out}} | \hat{O}(t) | 0_{\mathrm{in}} \rangle \longleftrightarrow \langle 0_{\mathrm{in}} | \hat{O}(t) | 0_{\mathrm{in}} \rangle$$

Transition amplitude

in-out formalism

Expectation value

in-in formalism

How can we treat the Schwinger effect in these formalism?

See: Fukushima-Gelis-Lappi (2009), Gelis-Tanji (2015)

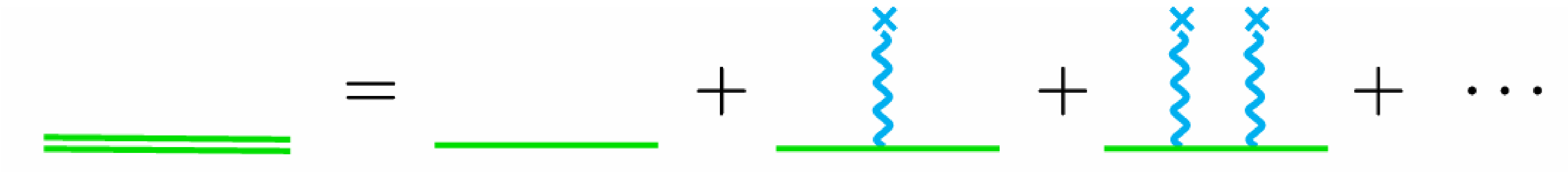
Schwinger's approach: in-out formalism

- Schwinger propagator for homogeneous EM fields

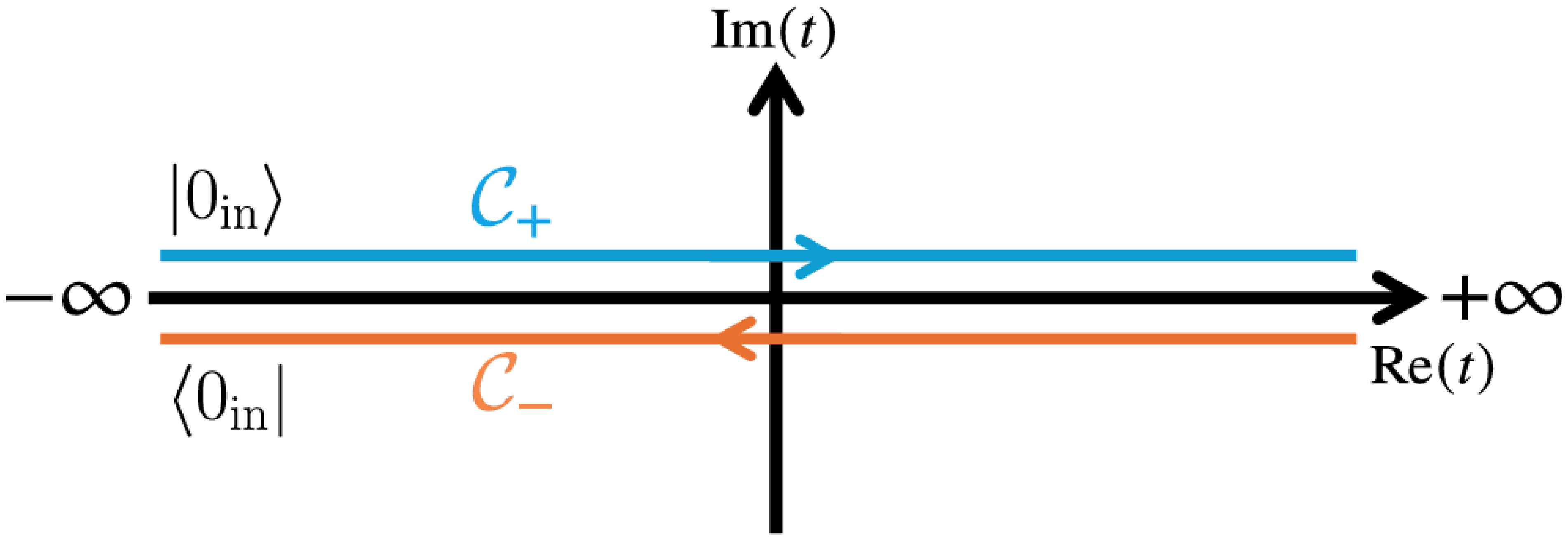
$$
\begin{aligned}
S(p|B,E) \;=\; & \int_0^\infty ds \Big[\frac{1-\gamma^1\gamma^2 \tan(q_f B s)}{\cosh^2(q_f E s)} \not{p}_\parallel + \frac{1+\gamma^0\gamma^3 \tanh(q_f E s)}{\cos^2(q_f B s)} \not{p}_\perp \\
& + m\big(1-\gamma^1\gamma^2 \tan(q_f B s)\big)\big(1+\gamma^0\gamma^3 \tanh(q_f E s)\big) \Big] \\
& \times \exp\Big(-i m^2 s + i \frac{p_\perp^2}{q_f B} \tan(q_f B s) + i \frac{p_\parallel^2}{q_f E} \tanh(q_f E s) \Big)
\end{aligned}
$$

Hattori-Itakura-Ozaki (2023)

= Resummed propagator in external electromagnetic field



Generalization to in-in formalism ?



- 2×2 propagator
- 2×2 vertices

Resummation is technically involved

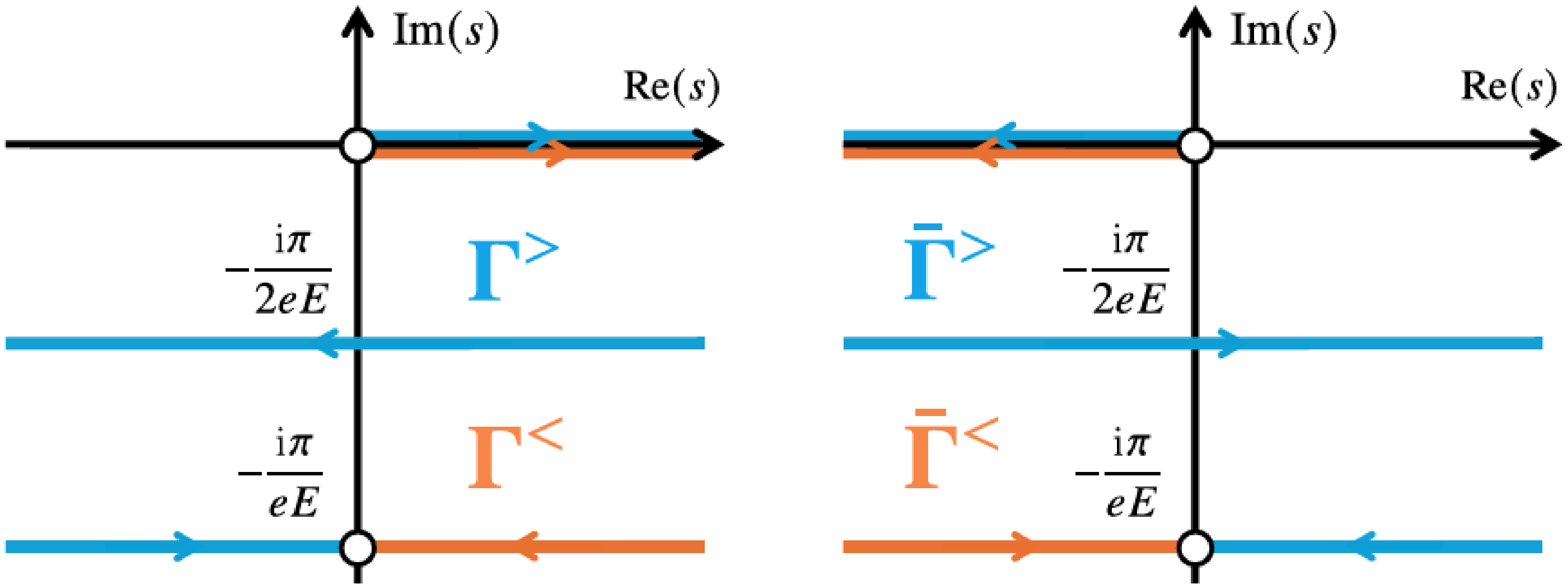
Real-time propagators for in-in formalism

E. S. Fradkin · D. M. Gitman
Sh. M. Shvartsman

Quantum
Electrodynamics

with Unstable Vacuum

Real-time propagators for in-in formalism



$$S^{--}(x,y) = \mathrm{i}(\mathrm{i}\not{D}_x + m)\left[\theta(z_3)\int_{\Gamma^{>}} \mathrm{d}s\, g(s;x,y) + \theta(-z_3)\int_{\Gamma^{<}} \mathrm{d}s\, g(s;x,y)\right]$$

in-in propagator = complexified proper-time contours ?

Fradkin-Gitman-Shvartsman (1991)

In-in formalism with resummation

Fukushima-Minato (2026)

- Key observation:

$$|0_{\mathrm{in}}\rangle = \mathcal{N}\,\hat{\mathscr{U}}_{\mathrm{D}}^{\dagger}\,|0_{\mathrm{out}}\rangle$$

$$\Big\downarrow \quad \hat{\mathscr{U}}_{\mathrm{D}}^{\dagger} := \exp\left[\sum_s \int_{\boldsymbol{p}} \sigma_{\boldsymbol{p}}^{*}\, \hat{a}^{(s)\dagger}_{\mathrm{out},\boldsymbol{p}}\, \hat{b}^{(s)\dagger}_{\mathrm{out},-\boldsymbol{p}}\right]$$

$$\langle 0_{\mathrm{in}}|\hat{O}(x^0)|0_{\mathrm{in}}\rangle = |\mathcal{N}|^2 \langle 0_{\mathrm{out}}|\hat{\mathscr{U}}_{\mathrm{D}}\,\hat{O}(x^0)\,\hat{\mathscr{U}}_{\mathrm{D}}^{\dagger}|0_{\mathrm{out}}\rangle$$

Resummation of diagrams

In-in formalism with resummation

Fukushima-Minato (2026)

Our findings:

In systems with an unstable vacuum, the reference vacuum for the loop expansion is not unique

1. An expansion around $|0_{\mathrm{in}}\rangle$ requires the infinite summation of diagrams
2. An expansion around $|0_{\mathrm{out}}\rangle$ **already resums** the effects of the background electromagnetic field

Intuitive understanding:

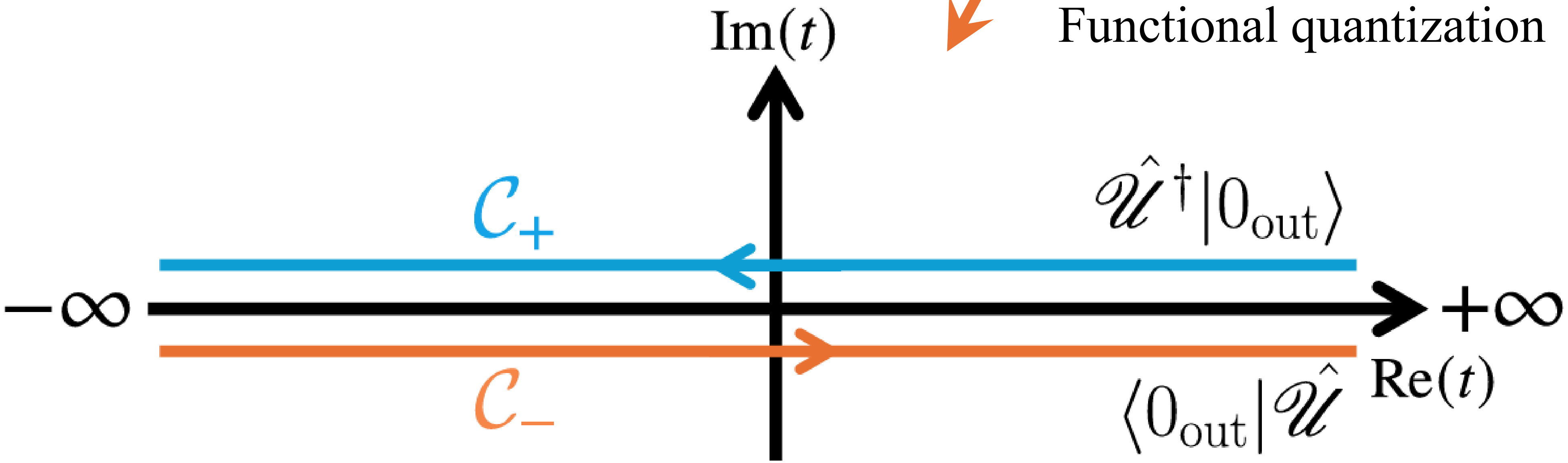
We are usually interested in the physical quantities near $|0_{\mathrm{out}}\rangle$, so the loop expansion around $|0_{\mathrm{out}}\rangle$ makes things simpler

In-in formalism with resummation

Fukushima-Minato (2026)

$$\langle 0_{\text{in}}|\hat{O}(x^0)|0_{\text{in}}\rangle = |\mathcal{N}|^2 \langle 0_{\text{out}}|\hat{\mathscr{U}}_{\text{D}}\,\hat{O}(x^0)\,\hat{\mathscr{U}}_{\text{D}}^{\dagger}|0_{\text{out}}\rangle$$

Functional quantization



Boundary wavefunctional

$$\Psi_{\text{D}}(\psi_+(+\infty)) = \langle \psi_+(+\infty), \bar{\psi}_+(+\infty); +\infty|\hat{\mathscr{U}}_{\text{D}}^{\dagger}|0; \text{out}\rangle$$

$$\Psi_{\text{D}}^*(\psi_-(+\infty)) = \langle 0; \text{out}|\hat{\mathscr{U}}_{\text{D}}|\psi_-(+\infty), \bar{\psi}_-(+\infty); +\infty\rangle$$

In-in formalism with resummation

Fukushima-Minato (2026)

$$\Psi_{\mathrm{D}}(\psi_+(+\infty)) = \langle \psi_+(+\infty), \bar{\psi}_+(+\infty); +\infty | \hat{\mathscr{U}}_{\mathrm{D}}^{\dagger} | 0; \mathrm{out} \rangle$$

$$\Psi_{\mathrm{D}}^{*}(\psi_-(+\infty)) = \langle 0; \mathrm{out} | \hat{\mathscr{U}}_{\mathrm{D}} | \psi_-(+\infty), \bar{\psi}_-(+\infty); +\infty \rangle$$

with $\quad \hat{\mathscr{U}}_{\mathrm{D}}^{\dagger} := \exp\left[\sum_s \int_{\boldsymbol{p}} \sigma_{\boldsymbol{p}}^{*}\, \hat{a}_{\mathrm{out},\boldsymbol{p}}^{(s)\dagger}\, \hat{b}_{\mathrm{out},-\boldsymbol{p}}^{(s)\dagger}\right]$

These wavefunctionals can be written in terms of $\psi(\infty), \bar{\psi}(\infty)$ with coefficients depending on E

yields an E-depending self-energy at $t = \infty$

Performing some straightforward but tedious calculations…

In-in formalism with resummation

Fukushima-Minato (2026)

$$Z[\eta_a,\bar\eta_a] = \int_{C_{-\infty}} [\mathrm{d}\psi_a][\mathrm{d}\bar\psi_a]\, \exp\Big[-\mathrm{i}S_+(\psi_+,\bar\psi_+) + \mathrm{i}S_-(\psi_-,\bar\psi_-) + \mathrm{i}\int \mathrm{d}^4x\, c^{ab}\big(\bar\eta_a\psi_b + \bar\psi_a\eta_b\big)\Big]$$

$$S_+(\psi,\bar\psi) := \boxed{S_{-\mathrm{i}\epsilon}} + \boxed{\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y\;\bar\psi(x)\,\bar\Sigma_{\mathrm{D}}(x,y)\,\psi(y)}$$

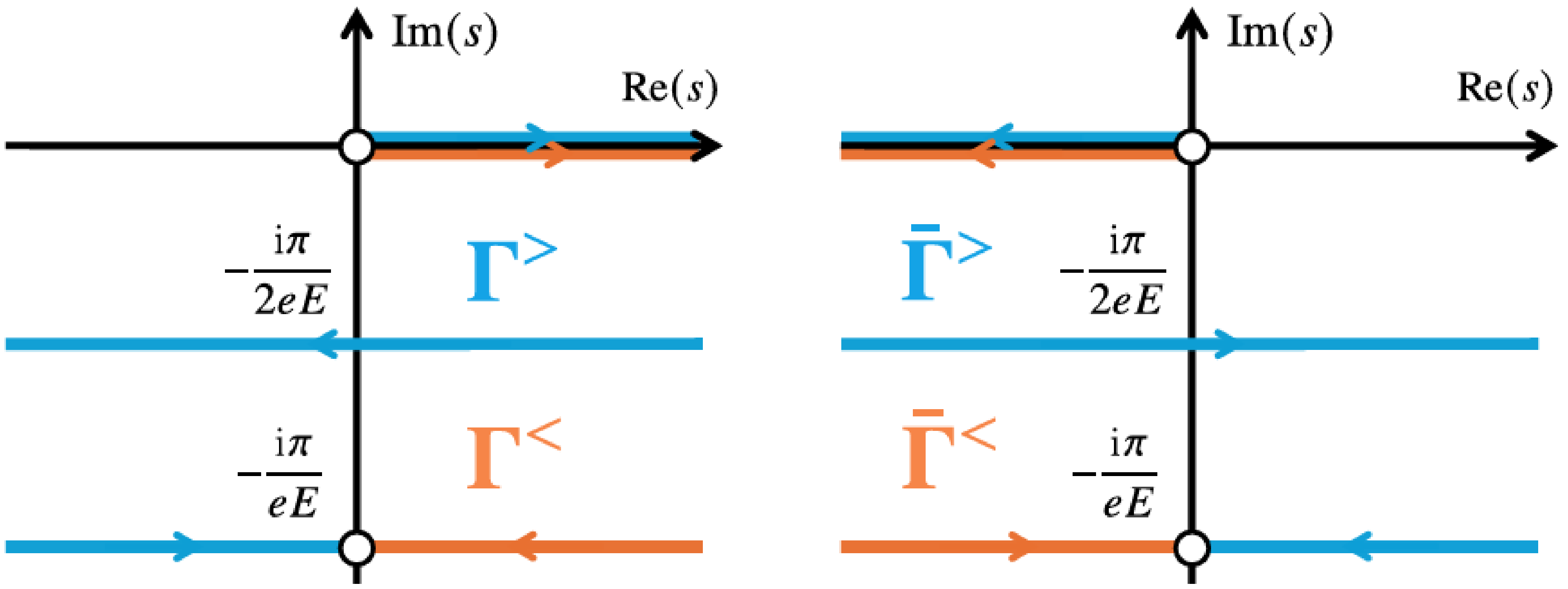
$$S_-(\psi,\bar\psi) := \boxed{S_{+\mathrm{i}\epsilon}} - \boxed{\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y\;\bar\psi(x)\,\Sigma_{\mathrm{D}}(x,y)\,\psi(y)}$$

Dirac action w/ interaction terms **from boundary wavefunctional**

Additional integral contour = nonlocal self-energy from boundary wavefunctional

In-in formalism with resummation

Fukushima-Minato (2026)



Additional integral contour = nonlocal self-energy from boundary wavefunctional

Simple Application

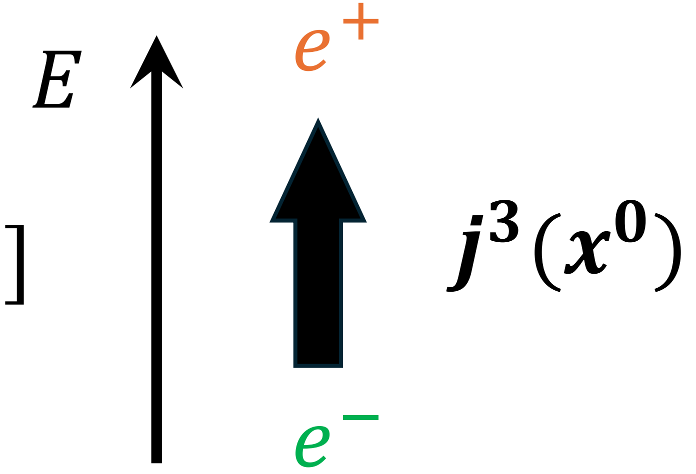
Fukushima-Minato (2026)

This formalism correctly includes the pair-production effect

$$j^3(x^0) = \bar{\psi}\gamma^3\psi$$

$$\langle j^3(x^0)\rangle = \lim_{x \to y} \mathrm{Tr}_S\left[\gamma^3 S_{\mathrm{in-out}}(x,y)\right]$$

$$= 0$$



See also Copinger-Fukushima-Pu (2018)

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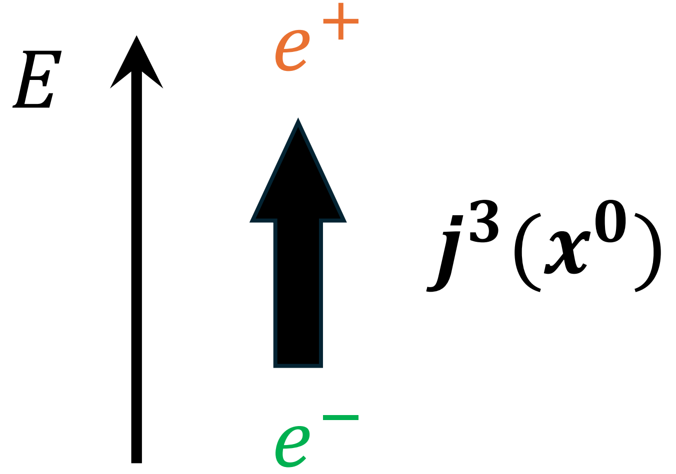
$$\langle j^3(x^0)\rangle = \lim_{x\to y} \mathrm{Tr_s}\left[\gamma^3 S^{--}(x,y)\right]$$

$$= \Lambda \frac{eE\,x^0}{4\pi^3} \mathrm{e}^{-\pi m^2/(eE)}$$

Pair created current



See also Copinger-Fukushima-Pu (2018)

Conclusion

- A loop expansion depends on the choice of reference vacuum
- In an electric field, the vacuum instability makes the choice of reference vacuum important
- We recast the in-in generating functional as a loop expansion around the out vacuum, which builds pair-production effects into propagators.
- Physical applications are left for future work.