Motivations and possible strategies for proving the Yang-Mills gap with Tensor RG

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YITP QCTN 11/20/2026



Motivations and strategies

- Yang-Mills gap: great challenge for mathematical physicists but can something new be learned by lattice gauge theorists ?
- Possible by-product: better understanding of the way the gap can be reduced or closed by adding enough fermionic matter
- In a recent Nature Reviews Physics on the Yang-Mills Millennium problem, M. Douglas suggested that “Conceivably, the Tensor Renormalization Group (TRG) will be adequate, especially if it can accommodate duality arguments."
- Three possible strategies (Lattice 26):
  1. Design a model with a related gap and where the TRG can be applied more easily (Dyson, Tomboulis)
  2. Reproduce the basic steps in numerical calculations of the gap, using dual variables with the TRG and try to find upper and lower bounds on the gap.
  3. Connect the gap in the spectrum with the gap in the imaginary part of the zeros of the partition function (not pinching the real coupling constant axis.)



Terry Tomboulis attempt with Migdal-Kadanoff

LONG DISTANCE DYNAMICS AND CONFINEMENT VIA RG DECIMATIONS

Modern Physics Letters A
Vol. 24, No. 34 (2009) 2717–2730

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We review a recently developed framework employing computable Renormalization Group (RG) decimations for gauge theories in the lattice regularization. They provide upper and lower bounds at every scale for free energies and some order parameters. By interpolating between these bounds representations of the exact quantities are obtained at progressively longer scales (coarser lattices). In the case of the SU(2) gauge theory in four dimensions RG flow to the confining strongly coupled regime is obtained for any initial coupling; whereas for the U(1) theory a fixed point is reached for small initial coupling.

Criticized by K. Ito and E. Seiler, arXiv 0901.4246:

It was proven a little later that in 4 dimensions the MK RG drives lattice $SU(N)$ Yang-Mills theory, but also compact lattice QED to the strong coupling fixed point [11]. This signals confinement for these models, and is therefore misleading for the abelian model, which is known to have a deconfining transition [12, 13].

This fact raised problems for Tomboulis's approach, because it was not clear how his inequalities would distinguish between the groups $SU(N)$ and $U(1)$, especially since in his short letter there were no details given concerning the proof of the crucial inequalities. In fact there even re-



Exact coarse graining ($\sim$ Tao Xiang's HOTRG)

PHYSICAL REVIEW B **87**, 064422 (2013)

Accurate exponents from approximate tensor renormalizations

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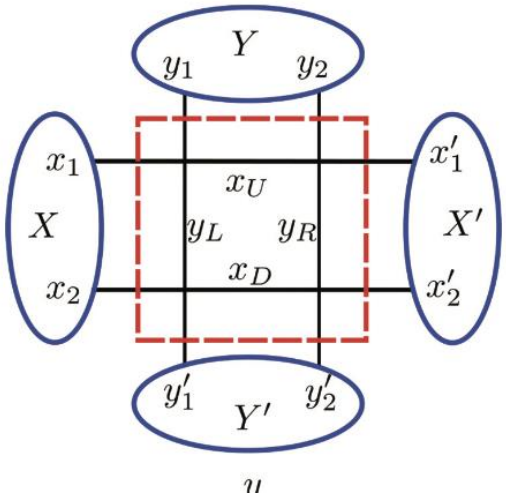
(Received 18 November 2012; published 27 February 2013)

We explain the recent numerical successes obtained by Tao Xiang's group, who developed and applied tensor renormalization group methods for the Ising model on square and cubic lattices, by the fact that their new truncation method sharply singles out a surprisingly small subspace of dimension two. We show that in the two-state approximation, their transformation can be handled analytically, yielding a value of 0.964 for the critical exponent ν much closer to the exact value 1 than the 1.338 value obtained in the Migdal-Kadanoff approximation. We propose two alternative blocking procedures that preserve the isotropy and improve the accuracy to $\nu = 0.987$ and 0.993, respectively. We discuss applications to other classical lattice models, including models with fermions, and suggest that it could become a competitor for Monte Carlo methods suitable for accurate calculations of critical exponents, taking continuum limits, and the study of near-conformal systems in arbitrarily large volumes.



n

We now use this reformulation to blockspin. First, we follow Migdal[12] by using an isotropic procedure. We consider a square block enclosing four sites and sum over the states, inside the block, associated with the nearest-neighbor links joining these four points. This defines a new rank-4 tensor $T'_{XX'YY'}$, where each index now takes four values:

$$T'_{X(x_1,x_2)X'(x'_1,x'_2)Y(y_1,y_2)Y'(y'_1,y'_2)} = \sum_{x_U,x_D,x_R,x_L} T_{x_1x_Uyy_L} T_{x_Ux'_1y_2y_R} T_{x_Dx'_2y_Ry'_2} T_{x_2x_Dy_Ly'_1} , \quad (2)$$

where $X(x_2,x_2)$ is a notation for the product states. For reasons that will become clear later, we use the convention $X(0,0) = 1$, $X(1,1) = 1$, $X(1,0) = 3$, $X(0,1) = 4$. Later, we also use the ket notation $|00\rangle$ for $X = 1$, etc. This is represented graphically in Fig. 1.

The new tensor can be used to define an *exact* expression of the partition function of the same form as (1); however, the number of states proliferates as 2^{2^n} after n steps and approximations are needed in order to get an expression useful for practical purposes.

Exact blocking for spin and gauge models

PHYSICAL REVIEW D **88,** 056005 (2013)

Exact blocking formulas for spin and gauge models

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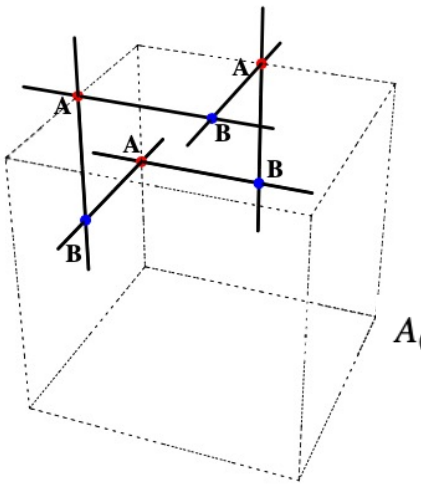
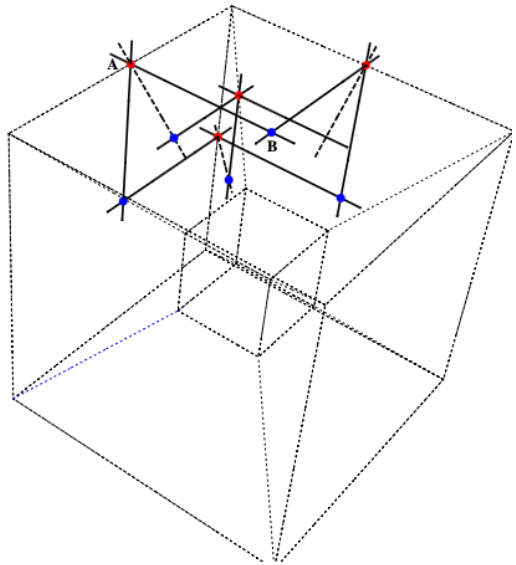
(Received 26 July 2013; published 3 September 2013)

Using the example of the two-dimensional (2D) Ising model, we show that in contrast to what can be done in configuration space, the tensor renormalization group formulation allows one to write exact, compact, and manifestly local blocking formulas and exact coarse-grained expressions for the partition function. We argue that similar results should hold for most models studied by lattice gauge theorists. We provide exact blocking formulas for several 2D spin models [the $O(2)$ and $O(3)$ sigma models and the SU(2) principal chiral model] and for the three-dimensional gauge theories with groups Z_2, U(1) and SU(2). We briefly discuss generalizations to other groups, higher dimensions and practical implementations.

$$Z = \mathrm{Tr}\prod_l A^{(l)}_{n_1 \ldots n_{2(D-1)}} \prod_p B^{(p)}_{m_1 m_2 m_3 m_4}.$$



$$A_{(r_1,m_1,n_1)(r_2,m_2,n_2)(r_3,m_3,n_3)(r_4,m_4,n_4)} = (F_{r_1}(\beta)F_{r_2}(\beta)F_{r_3}(\beta)F_{r_4}(\beta))^{\frac{1}{4}} \times \sum_{r',m',n'} d_{r'}^{-1}(-1)^{m'-n'} C^{r_1 r_2 r'}_{m_1 m_2 m'} C^{r_1 r_2 r'}_{n_1 n_2 n'} C^{r_3 r_4 r'}_{m_3 m_4 -n'}$$



(5.26)

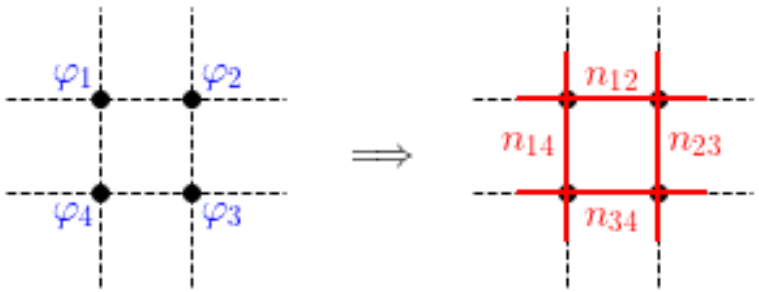
Tensor LFT: From compact to discrete (O(2) case)

$$Z_{O(2)} = \prod_x \int_{-\pi}^{\pi} \frac{d\varphi_x}{2\pi} e^{\beta \sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x)} = \mathrm{Tr} \prod_x T^{(x)}_{n_{x-\hat{1},1}, n_{x,1}, \ldots, n_{x,D}}.$$

$$e^{\beta \cos(\varphi_{x+\hat{\mu}} - \varphi_x)} = \sum_{n_{x,\mu}=-\infty}^{\infty} e^{i n_{x,\mu} \varphi_{x+\hat{\mu}}} I_{n_{x,\mu}}(\beta) e^{-i n_{x,\mu} \varphi_x}.$$

$$\text{Tensor}: \; T^{(x)}_{n_{x-\hat{1},1}, n_{x,1}, \ldots, n_{x-\hat{D},D}, n_{x,D}} = \sqrt{I_{n_{x-\hat{1},1}} I_{n_{x,1}}, \ldots, I_{n_{x-\hat{D},D}} I_{n_{x,D}}} \times \delta_{n_{x,\mathrm{out}}, n_{x,\mathrm{in}}},$$

$$\prod_x \int_{-\pi}^{\pi} d\varphi_x \Longrightarrow \sum_{\{n\}}$$

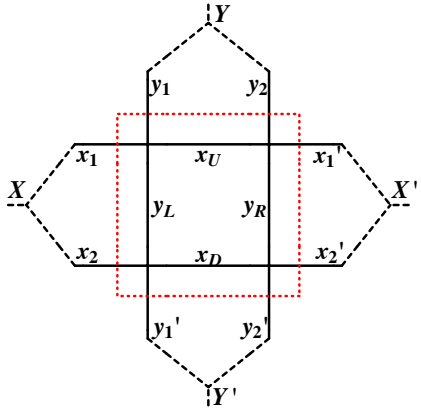


The gauged version is the Abelian Higgs model.

The Tensor Renormalization Group (TRG) method

- Exact blocking (spin and gauge, PRB 87 064422, PRD 88 056005)

Unique feature: the blocking separates the degrees of freedom inside the block (integrated over), from those kept to communicate with the neighboring blocks. The only approximation is the truncation in the number of “states" kept.



- Applies to many lattice models: Ising model, $O(2)$ model, $O(3)$ model, $SU(2)$ principal chiral model, Abelian and $SU(2)$ gauge theories, ϕ^4, fermions ..
- Solution of sign problems: complex temperature (PRD 89, 016008), chemical potential (PRA 90, 063603).
- Reviews: YM, Ryo Sakai, Judah Unmuth-Yockey, Tensor lattice field theory for renormalization and quantum computing, arXiv:2010.06539, Rev. Mod. Phys. 94, 025005 (2022). Shinichiro Akiyama, YM, Ryo Sakai, Tensor Renormalization Group for fermions, arXiv:2401.08542, J.Phys. CM 36 (2024).



TRG inspired talks at Lattice 2026

- Lattice Field theories with Tensor Networks, Adwait Naravane
- Tensor renormalization group approach to EN, Gota Tanaka
- Tensor Renormalization Group Study of Step-Scaling Function in the CP(1) Model with a theta Term, Hayato Aizawa
- Tensor-Network Study of Confining Flux Tubes and Effective Strings in (2+1)D Compact U(1) Gauge Theory, Alessio Negro
- Tensor Network States and Lattice Gauge Theories, Erez Zohar
- Gauge-invariant PEPS for pure Z_2 ..., Etsuko Itou and T. Okubo
- Tensor renormalization group for four-dimensional SU(N) gauge theories, Katsumasa Nakayama
- Tensor renormalization group analysis of correlators in Yang-Mills theory, Takaaki Kuwahara
- Phase diagram of the single-flavor GNW model ... Jian-Gang Kong, Shinichiro Akiyama, Tao Shin, Zhiyuan Xie
- Tensor renormalization group study of cold and dense 2 and 3 color QCD ..., Yuto Sugimoto, Shinichiro Akiyama, Yoshinobu Kuramashi



Introductions to Tensor Field Theory

REVIEWS OF MODERN PHYSICS, VOLUME 94, APRIL–JUNE 2022

Tensor lattice field theory for renormalization and quantum computing

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(published 26 May 2022)

The successes and limitations of statistical sampling for a sequence of models studied in the context of lattice QCD are discussed and the need for new methods to deal with finite-density and real-time evolution is emphasized. It is shown that these lattice models can be reformulated using tensorial methods where the field integrations in the path-integral formalism are replaced by discrete sums. These formulations involve various types of duality and provide exact coarse-graining formulas that can be combined with truncations to obtain practical implementations of the Wilson renormalization group program. Tensor reformulations are naturally discrete and provide manageable transfer matrices. Truncations with the time continuum limit are combined, and Hamiltonians suitable for performing quantum simulation experiments, for instance, using cold atoms, or to be programmed on existing quantum computers, are derived. Recent progress concerning the tensor field theory treatment of noncompact scalar models, supersymmetric models, economical four-dimensional algorithms, noise-robust enforcement of Gauss's law, symmetry preserving truncations, and topological considerations are reviewed. Connections with other tensor network approaches are also discussed.

DOI: 10.1103/RevModPhys.94.025005

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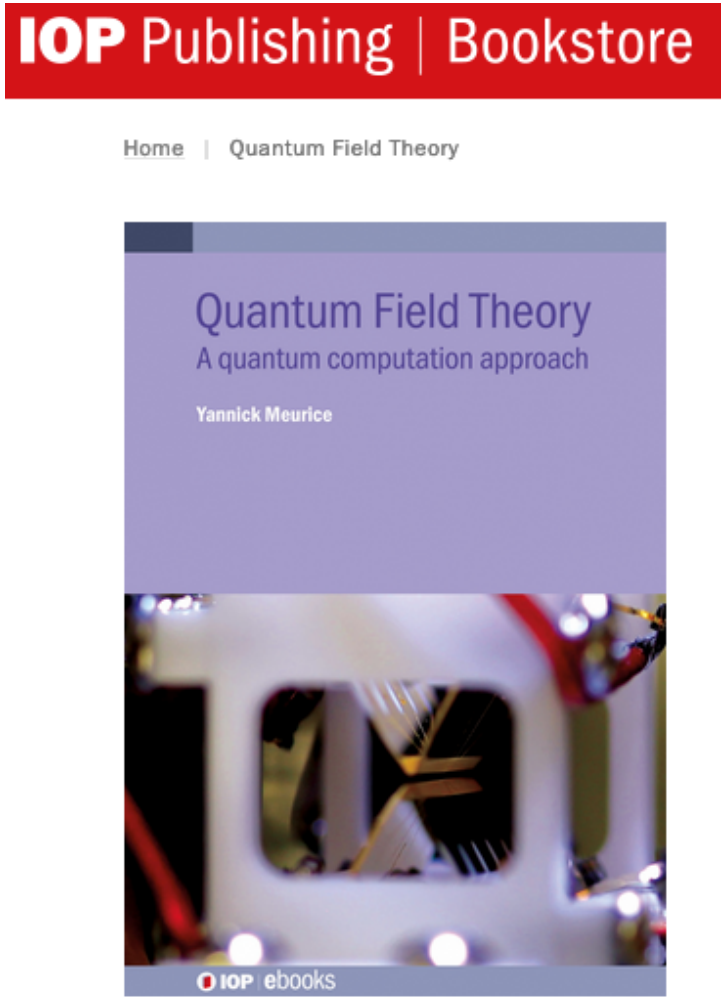
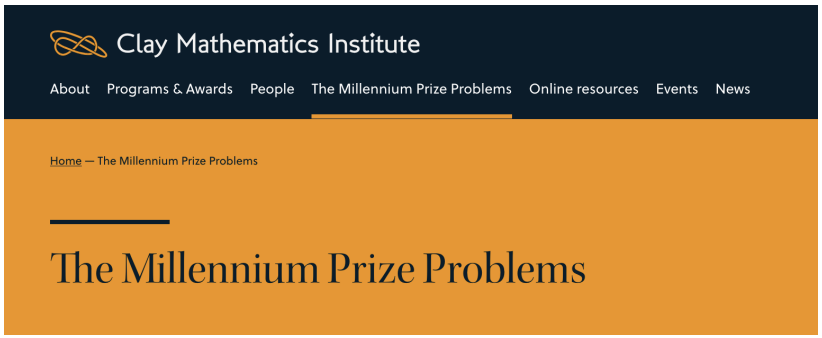
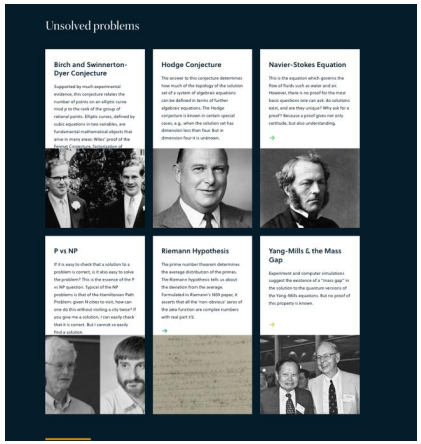


Millennium problem: prove the Yang-Mills gap



In order to celebrate mathematics in the new millennium, The Clay Mathematics Institute of Cambridge, Massachusetts (CMI) established seven *Prize Problems*.



The Birch and Swinnerton-Dyer Conjecture
ANDREW WILES

The Hodge Conjecture
PIERRE DELIGNE

Existence and Smoothness of the Navier–Stokes Equation
CHARLES L. FEFFERMAN

The Poincaré Conjecture
JOHN MILNOR

The P versus NP Problem
STEPHEN COOK

The Riemann Hypothesis
ENRICO BOMBIERI

Quantum Yang–Mills Theory
ARTHUR JAFFE AND EDWARD WITTEN

M. Douglas review: Tensor RG is a promising strategy

nature reviews physics

https://doi.org/10.1038/s42254-025-00909-2

Review article

Check for updates

The Yang–Mills Millennium problem

Michael R. Douglas

Abstract

The Yang–Mills Millennium Prize problem is one of the great challenges of mathematical physics. In the quarter century since it was set, what progress has been made? This Review outlines the problem from a physics point of view, gives its physical background, explains its nature and significance as a problem in mathematics and surveys promising approaches from recent years.

Sections

Introduction

Physical background

Mathematical approaches to Yang–Mills

Other ideas and the role of computation

Outlook

Tensor RG

The block spin RG, although conceptually straightforward to explain, has many technical difficulties in implementation.

Of the many subsequent reformulations of the RG, one which many find particularly promising is the tensor RG. This was originally developed in condensed matter physics but is general; ref. 67 provides a review oriented towards QFT and gauge theory. In the tensor RG, the space of effective actions, and many of the steps involving approximations, are linear.

The starting point is to reformulate the partition function as a tensor network. This has been done for LGT 67,

67. Meurice, Y., Sakai, R. & Unmuth-Yockey, J. Tensor lattice field theory with applications to the renormalization group and quantum computing. *Rev. Mod. Phys*. **94**, 025005 (2022).

Key points of Michael Douglas Review

- Yang–Mills theory is the basis of the standard model of particle physics and describes the strong and weak forces.
- The Yang–Mills Millennium problem is to show that Yang–Mills theory is mathematically well defined and that it has the mass gap property.
- The issue of definition is to prove that the theory has a continuum limit, which is well defined at arbitrarily high energies. This requires renormalization, which has never been made rigorous in the needed generality.
- The mass gap property (no massless particles) is expected because it is true of real-world quantum chromodynamics and it is seen in numerical simulations. It is widely felt that no clear path is known towards proving it.
- Recent mathematical approaches include rigorous stochastic quantization and the rigorous strong coupling expansion. They are part of probability theory, and mathematicians are making significant advances.
- Numerical and computational methods are important in the physical study of Yang–Mills and likely to be used in any rigorous proof. Physicists could contribute significantly by developing more powerful computational renormalization group methods.

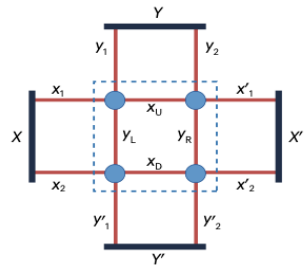


Fig. 3 | Graphical representation of a tensor renormalization group block spin step for the 2D nonlinear sigma model. The lower case x, y represent vectors in the vector space V, and upper case X, Y represent vectors obtained as a tensor product $V \otimes V$ followed by projection down to V. Figure reprinted with permission from ref. 67, APS.

A simpler model: O(3) in 2D Euclidean (AI solved?)

Exponential decay in two-dimensional classical $O(n)$ models

OpenAI

September 23, 2026

Abstract

We prove exponential decay of two-point correlations for the classical nearest-neighbor $O(n)$ model on the square lattice, for every $n \geq 3$ and every finite positive inverse temperature. The estimate is uniform over finite free-boundary subgraphs and bounded nonnegative edge strengths. This resolves positively the all-temperature exponential spin-decay conjecture for these models.

Theorem 1.1. *For every integer $n \geq 3$ and every finite $\beta > 0$, there are constants $A = A(n, \beta) < \infty$ and $m = m(n, \beta) > 0$ such that, for every finite nearest-neighbor square-lattice subgraph G, every strength array $b \in [0, \beta]^E$, and all $x, y \in V$,*

$$0 \leq \langle \sigma_x \cdot \sigma_y \rangle_{G,b}^{(n)} \leq A \exp(-m\|x - y\|_2). \tag{1.2}$$



The hard beginning (Glimm-Jaffe axioms)

OS0. Analyticity
OS1. Regularity
OS2. Euclidean invariance
OS3. Reflection (OS) positivity
OS4. Ergodicity

which we now explain in detail.

OS0 (Analyticity). The functional $S\{f\}$ is entire analytic. Specifically, for every finite set of test functions $f \in \mathscr{D}(R^d)$, $j = 1, 2, \ldots, N$, and complex numbers $z = \{z_1, \ldots, z_N\} \in C^N$, the function

$$z \to S\left\{\sum_{j=1}^{N} z_j \ f_j\right\}$$

is entire on C^N. In other words, $d\mu$ decays faster than any exponential.

OS1[1] (Regularity). For some p, $1 \leq p \leq 2$, for some constant c, and for all $f \in \mathscr{D}(R^d)$,

$$|S\{f\}| \leq \exp c(\|f\|_{L_1} + \|f\|_{L_p}^p). \tag{6.1.5}$$

If $p = 2$, we also assume that $S_2(x, y)$ defined in (6.1.15) is locally integrable.

OS2 (Invariance). $S\{f\}$ is invariant under Euclidean symmetries E of R^d (translations, rotation, and reflections), i.e., $S\{f\} = S\{Ef\}$. Equivalently $d\mu$ is Euclidean invariant, so that $d\mu = E\, d\mu$.



The hard beginning II (from Douglas et al.)

$$\mathcal{A} = \{A(\phi) = \sum_{j=1}^{N} c_j \exp(\phi(f_j)) : \; c_j \in \mathbb{C}, \; f_j \in \mathcal{D}'(\mathbb{R}^D))\}. \tag{3.9}$$

- OS3: Reflection Positivity- The Euclidean symmetries act on $\mathcal{D}'(R^D)$, so that $E\phi(f) = \phi(Ef)$, which defines a unitary condition action on the $L^2(\mu)$. Finally, let $\mathcal{A}^+ \subset \mathcal{A}$ be the set of functionals where $f_j \in C_0(\mathbb{R}^D_+)$ and $C_0(\mathbb{R}^D_+)$ is the space of continuous functions that vanish at infinity in the positive half space. $\mathbb{R}^D_+ := \{(t, \vec{x}) : t > 0\}$ so now if we let $\theta : (t, \vec{x}) \to (-t, \vec{x})$ be the time reflection. Then the content of this axiom states that the inner-product inequality is satisfied:

$$\int (\theta A)^- A \, d\mu = \langle \theta A, A \rangle_{L^2(\mu)} \geq 0. \tag{3.10}$$

- OS4: Ergodicity- The time translation subgroup $T(t)$ acts ergodically, meaning it upholds the uniqueness of the vacuum, on the measure $d\mu$. The time translation subgroup in Euclidean space is the one parameter group of pure time shifts

$$T(t) : x = (t, \vec{x}) \to (t + t', \vec{x}) \qquad t' \in \mathbb{R} \tag{3.11}$$

  so that $T(t + s) = T(t)T(s)$.

  This is equivalent to saying that for all $L^1(\mu)$ functions $A(\phi)$,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t T(s) A(\phi) T(s)^{-1} ds = \int A(\phi) d\mu(\phi), \tag{3.12}$$

  where the limit above should be taken in the L^1 sense.



Help from Lean 4

Formalization of QFT

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arXiv:2603.15770

ABSTRACT

A foundational result in constructive quantum field theory is the construction of the free bosonic quantum field theory in four-dimensional Euclidean spacetime and the proof that it satisfies the Glimm–Jaffe axioms, a variant of the Osterwalder–Schrader axioms. We present a formalization of this result in the Lean 4 interactive theorem prover. The project is intended as a proof of concept that extended arguments in mathematical physics can be translated into machine-checked proofs using existing AI tools. We begin by introducing interactive theorem proving and constructive quantum field theory, then describe our formalization and the design decisions that shaped it. We also explain the methods we used, including coding assistants, and conclude by considering how AI assisted formalization may influence the future of theoretical physics.

Our original release assumed three results, Minlos' theorem, the nuclear property of Schwartz space, and Goursat's theorem. In subsequent releases from our group and from contributors from the Lean community, these assumptions have been proven (or avoided), so that the OS/GJ axioms are now proven using only Lean and its library Mathlib.



Help from Lean 4

github.com/mrdouglasny/OSforGFF?tab=readme-ov-file 90%

README Apache-2.0 license

Master Theorem

```
theorem gaussianFreeField_satisfies_all_OS_axioms_generic
    {d : ℕ} [Fact (2 ≤ d)] (m : ℝ) [Fact (0 < m)] [GFFPropagator d m] :
    SatisfiesAllOS (gaussianFreeField_free (d := d) m)

theorem gaussianFreeField_satisfies_all_OS_axioms_dim4 (m : ℝ) [Fact (0 < m)] :
    SatisfiesAllOS (μ_GFF 4 m)

theorem gaussianFreeField_satisfies_all_OS_axioms_dim3 (m : ℝ) [Fact (0 < m)] :
    SatisfiesAllOS (μ_GFF 3 m)

theorem gaussianFreeField_satisfies_all_OS_axioms_dim2 (m : ℝ) [Fact (0 < m)] :
    SatisfiesAllOS (μ_GFF 2 m)

theorem gaussianFreeField_satisfies_all_OS_axioms_dim5 (m : ℝ) [Fact (0 < m)] :
    SatisfiesAllOS (μ_GFF 5 m)
```

where `SatisfiesAllOS` bundles OS0 (analyticity), OS1 (regularity), OS2 (Euclidean invariance), OS3 (reflection positivity), OS4 (clustering) and OS4 (ergodicity). The four `_dim4` / `_dim3` / `_dim2` / `_dim5` theorems are the closed-form instances (Bessel K_1, Yukawa, K_0, K_{3/2}) of the first; the corollary `gaussianFreeField_satisfies_all_OS_axioms_of_dim` covers every `d ≥ 2` via the canonical proper-time propagator: the OS3 proper-time Fubini domination runs at boundary-vanishing order `d`, so no upper bound on the dimension is needed.

Functional Programming in Lean

David Thrane Christiansen

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This is a free book on using Lean as a programming language. All code samples are tested with Lean release `4.33.0`.

Contents

Introduction

Acknowledgments

1. Getting to Know Lean

2. Hello, World!

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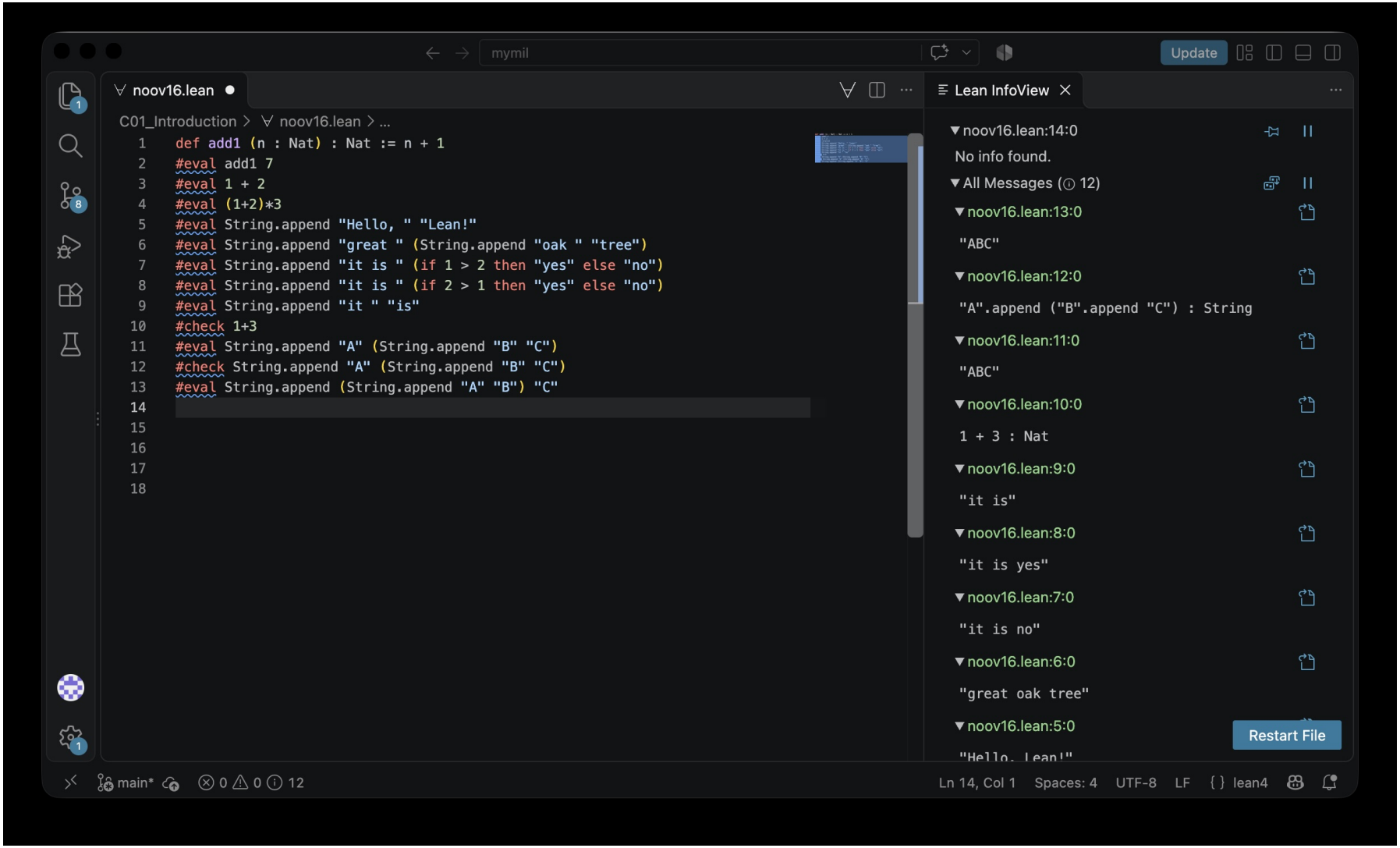
When viewed as a programming language, Lean is a strict pure functional language with dependent types. A large part of learning to program with Lean consists of learning how each of these attributes affects the way programs are written, and how to think like a functional programmer. *Strictness* means that function calls in Lean work similarly to the way they do in most languages: the arguments are fully computed before the function's body begins running. *Purity* means that Lean programs cannot have side effects such as modifying locations in memory, sending emails .....



Learning Lean



Recent developments

- Agentic AI and functional programming (Lean) open the possibility of fast progress on the Yang–Mills gap problem.
- OpenAI submitted Lean code on 28 October 2026. It was assessed by the Yang–Mills group (Douglas, Jaffe, Nissim, Rajasekaran, ...) on 10 November 2026:

  "In the group's judgment, the submission proves that a relativistic QFT satisfying the Wightman axioms can be constructed as a scaling limit of four-dimensional Wilson lattice gauge theory for any simple compact gauge group. It does not prove that the infinite-volume limit is unique, nor that there is a mass gap."

- Starting point: Wilson action (lattice), with Balaban's work on the ultraviolet problem for the continuum limit.
- TRG could provide an alternative method for the renormalization.



Validation of the OpenAI Yang–Mills construction

Based on Lean code submitted October 28, 2026

Summary of results and independent validation

Yang–Mills group Nov 10, 2026 release

2. What the construction provides

Go to page 29

1. The construction starts with the Wilson definition of lattice gauge theory, associates renormalized lattice observables to gauge-invariant polynomials in the curvature F and its covariant derivatives, and shows that their correlations converge to limiting correlations defining a Euclidean QFT. OS Reconstruction is then used to build a relativistic QFT that satisfies their own version of the Wightman axioms `YangMills.Hierarchy.Reconstructed.axioms`. They show that the theory is non-trivial in `YangMills.Hierarchy.Properties.nonfree`.

2. Renormalized local operators corresponding to gauge-invariant polynomials in F and its covariant derivatives are defined, including a conserved stress-tensor. Some short-distance checks are made; see, for example, Theorem 7.1.

3. The construction provides an operator product expansion `YangMills.Hierarchy.Properties.ope` with bounds on the error but does not show that it matches the predictions of a perturbative expansion. The construction also provides the scaling for β in `YangMills.WilsonRegularity.running` required in asymptotic freedom. The construction defines the trace of the stress tensor `YangMills.Hierarchy.centeredTrace` and proves a number of properties about it in `YangMills.Hierarchy.Properties.nonfree`.

4. The construction provides no evidence for or against a mass gap in the theory.



Roadmap

- Motivation
- From partition functions to tensor networks
- Transfer matrix
- The gap at strong coupling
- The $N_\tau \to \infty$
- Preliminary results for 2D $O(3)$
- Outlook



Why the Tensor Renormalization Group?

M. Douglas, review of the Yang–Mills Millennium problem:

> *"Conceivably, the Tensor Renormalization Group (TRG) will be adequate [to prove the existence of a gap], especially if it can accommodate duality arguments."*

Our proposal

- The TRG gives an exact discrete reformulation of nonlinear sigma models and pure gauge theories.
- It defines a discrete transfer matrix with suitable mathematical properties and an obvious mass gap at strong coupling.
- Two types of duality are used (Pontryagin and geometrical).
- Main challenge: define the continuum limit at infinite volume while controlling the gap.
- Numerical TRG, with truncations and coarse graining, can guide rigorous work.



Roadmap: the Osterwalder–Schrader (Glimm–Jaffe) axioms

Axiom	Content	Where it enters in the TRG approach
OS0	Analyticity	Entire character-expansion coefficients
OS1	Regularity	Growth of entire functions, local sources
OS2	Euclidean invariance	Continuum and infinite-volume limits
OS3	Reflection positivity	Positive transfer matrix $\mathbb{T}$
OS4	Ergodicity (clustering)	Unique leading eigenvalue of $\mathbb{T}$, gap

- **Not too complicated:** convergence of sums over representations, regularity and analyticity, existence of $\mathbb{T}$, strong-coupling gap.
- **Hard:** the gap at arbitrarily small coupling and infinite volume, and the continuum limit. This requires renormalization, which the TRG was designed to handle.



Character expansions

- Compact field integrations; link or plaquette interactions expanded in characters with discrete labels:
  - $O(2)$ NLσM, $U(1)$ gauge theory: Fourier series;
  - $O(3)$ NLσM: spherical harmonics;
  - $SU(2)$ gauge theory: representations $j = 0, \frac{1}{2}, 1, \ldots$
- Integrate the fields using orthogonality and the decomposition of products of representations:

$$Z = \int [\mathcal{D}\,\text{Fields}]\; e^{-S[\text{Fields},\,\beta]} = \text{Tr}\left(\prod \text{Tensors}\right)$$

- Infinite sums: convergence must be proven (easy at small β, special care as $\beta \to \infty$, $\beta \propto 1/g^2$).
- Gauge fields integrated out completely: manifestly gauge invariant.
- Coefficients are modified (spherical) Bessel functions, entire in β.



External sources: towards OS0 and OS1

$O(3)$ **NLσM:** insert

$$\prod_x e^{J_x \vec{n}_x \cdot \vec{n}_{J_x}}, \qquad \text{same form as} \qquad \prod_{x,\mu} e^{\beta \vec{n}_x \cdot \vec{n}_{x+\hat{\mu}}}.$$

- Expand in spherical harmonics; the angles of $\vec{n}_{J_x}$ are not integrated over.
- Derivatives with respect to $J_x \vec{n}_{J_x}$ insert $\vec{n}_x$.

$SU(2)$ **lattice gauge theory:** local plaquette couplings

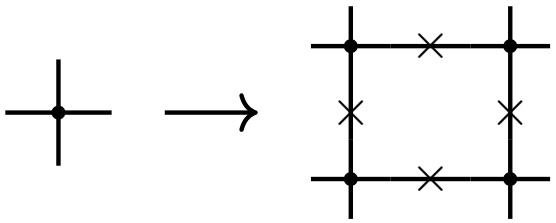
$$\beta_{pl} \;\to\; \beta_{pl} + \delta\beta_{pl},$$

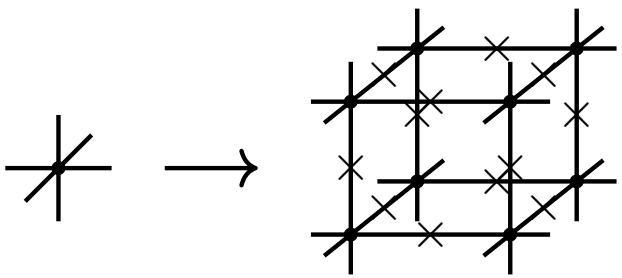
and derivatives with respect to $\delta\beta_{pl}$ insert plaquettes.

Summary

Partition functions with sources are products of contracted tensors whose entries are entire functions with at most exponential growth. This is the starting point for OS0 and OS1, at finite volume and finite β. The lattice regularization breaks the Euclidean invariance (OS2) which needs to be restored in the continuum limit at infinite volume.

Tensor network and transfer matrix

$D = 2$ 

$D = 3$ 

$\times$: contraction (sum over representations)

- Spin models: (hyper)cubic network built from the $2D$ legs of each site tensor.
- Gauge theory: alternating electric and magnetic layers of A and B tensors ("lasagna" for $D = 3$); can be recombined as in a spin model but asymmetrically .
- Single out a Euclidean time direction:

$$Z = \mathrm{Tr}\left(\mathbb{T}^{N_\tau}\right),$$

with N_τ time slices (periodic in time; periodic or open in space).

Transfer matrix and Gauss's law for Abelian gauge th.

PHYSICAL REVIEW D **102,** 014506 (2020)

Discrete aspects of continuous symmetries in the tensorial formulation of Abelian gauge theories

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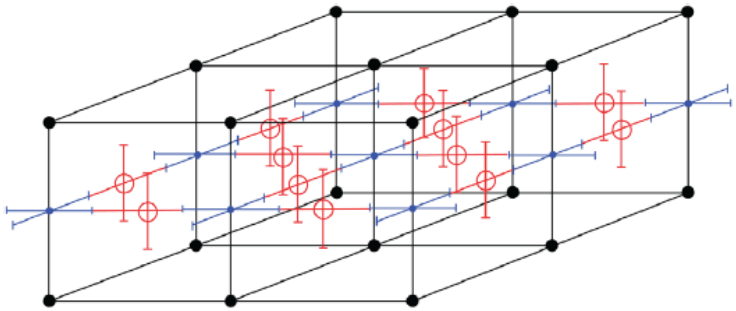
(Received 7 April 2020; accepted 30 June 2020; published 14 July 2020)

We show that standard identities and theorems for lattice models with $U(1)$ symmetry get reexpressed discretely in the tensorial formulation of these models. We also explain the geometrical analogy between the continuous lattice equations of motion and the discrete selection rules of the tensors. We further construct a gauge-invariant transfer matrix in arbitrary dimensions, show the equivalence with its gauge-fixed version in a maximal temporal gauge, and explain how a discrete Gauss's law is always enforced. Moreover, we propose a noise-robust way to implement Gauss's law in arbitrary dimensions, and we reformulate Noether's theorem for global, local, continuous, or discrete Abelian symmetries: for each given symmetry, there is one corresponding tensor redundancy. We discuss semiclassical approximations for classical solutions with periodic boundary conditions in two solvable cases, and we show the correspondence of their weak coupling limit with the tensor formulation after Poisson summation. Finally, we briefly discuss connections with other approaches and implications for quantum computing.



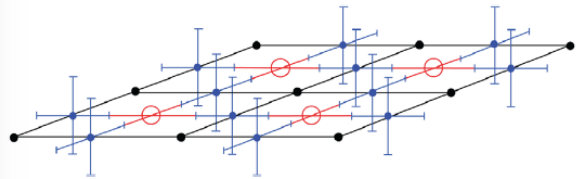
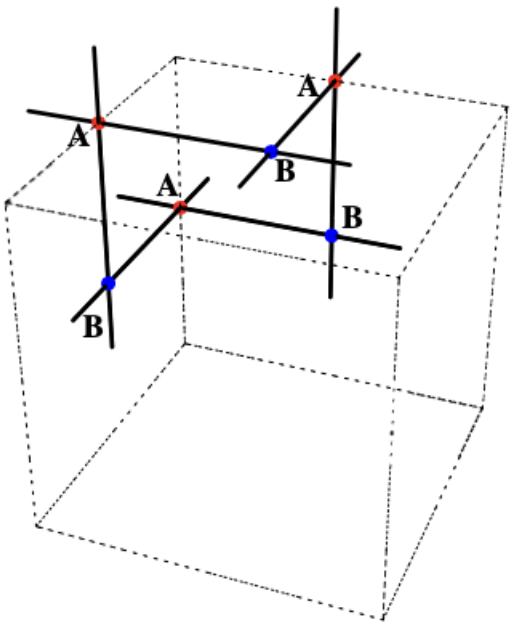


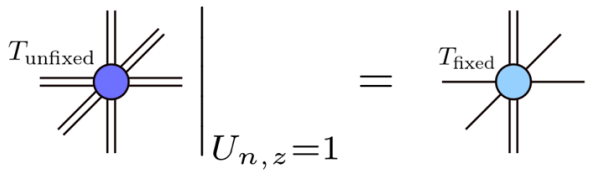
FIG. 3. Magnetic layer of the transfer matrix for $D = 3$ on a time slice. Small circles (blue) are used for the A tensors and large circles (red) for the B tensors.

Deconfinement from Thermal Tensor Networks: Universal CFT signature in (2+1)-dimensional $\mathbb{Z}_N$ lattice gauge theory

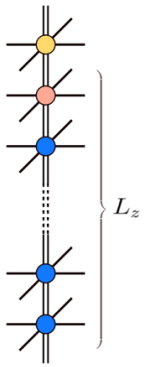
Adwait Naravane,[1] **Yuto Sugimoto**,[2] **Shinichiro Akiyama**,[3,4] **Jutho Haegeman**,[1] **Atsushi Ueda**[1]



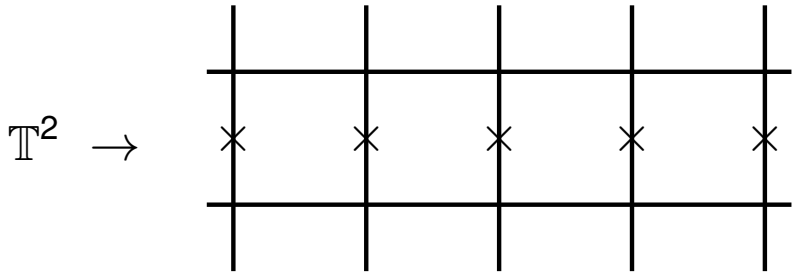
From Y. Liu et al. PRD88



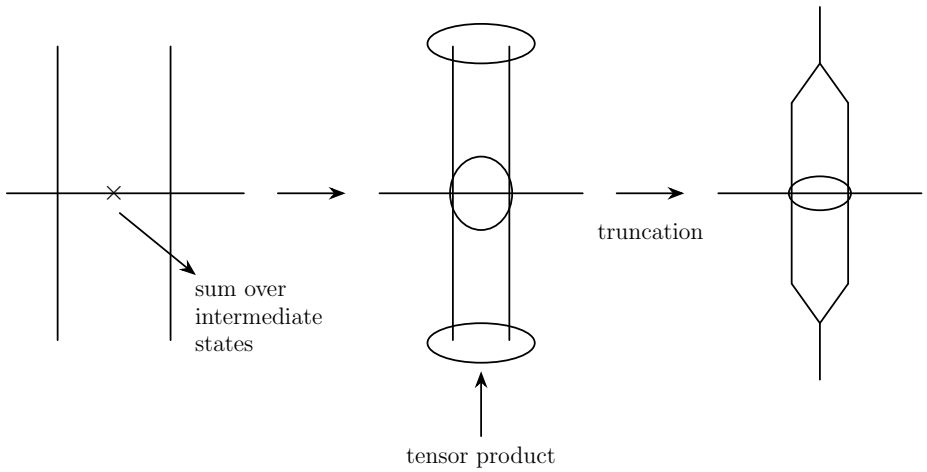
Electric PEPO is removed by imposing the temporal gauge condition.



Hilbert space and coarse graining



- An “electric” Hilbert space between time slices.
- Each vertical leg: infinitely many discrete indices (representations of $O(3)$ or $SU(2)$).
- Hilbert space = spatial tensor product, e.g. $\bigotimes_{\vec{x}} |j, m\rangle_{\vec{x}}$ for $SU(2)$.



- Z reduced to the eigenvalues of $\mathbb{T}$: a hard problem.
- Truncations at finite spatial volume give finite matrices.
- Partial coarse graining followed by SVDs, iterated in a spatially translation-invariant way (see above).

Gap in Hamiltonians with small spatial couplings

Treat the couplings asymmetrically:

- spin: β_τ large (time links), β_s small (space links);
- gauge: $\beta_{sp\,t}$ large (space-time plaquettes), $\beta_{sp\,sp}$ small (space-space plaquettes).

Then, with a model-dependent time lattice spacing a_τ,

$$\mathbb{T} \propto \left(\mathbf{1} - a_\tau H + \cdots\right),$$

and the eigenvectors of $\mathbb{T}$ with the largest eigenvalues are the low-energy states of H.

- NLσM at $\beta_s = 0$: decoupled rotors. Ground state: every rotor in its single-site ground state; gapped, degenerate excitations (single-site triplets), split by small β_s.
- Gauge models without space-space plaquettes: electric Hamiltonian with Gauss's law; ground state with zero electric fields, gapped excitations created by plaquette operators.



A warm-up exercise: isotropic $O(3)$ at small β

- For the isotropic $O(3)$ NLσM, as $\beta \to 0$, $\mathbb{T}$ becomes proportional to the projector onto the state with $I = 0$ on every site.
- Goal: use mathematical tools to show that for sufficiently small β, $\mathbb{T}$ is
  - a positive matrix,
  - with a nondegenerate leading eigenvalue,
  - and a gap with respect to all other eigenvalues.
- This is standard strong-coupling Euclidean analysis; there should be existing literature.



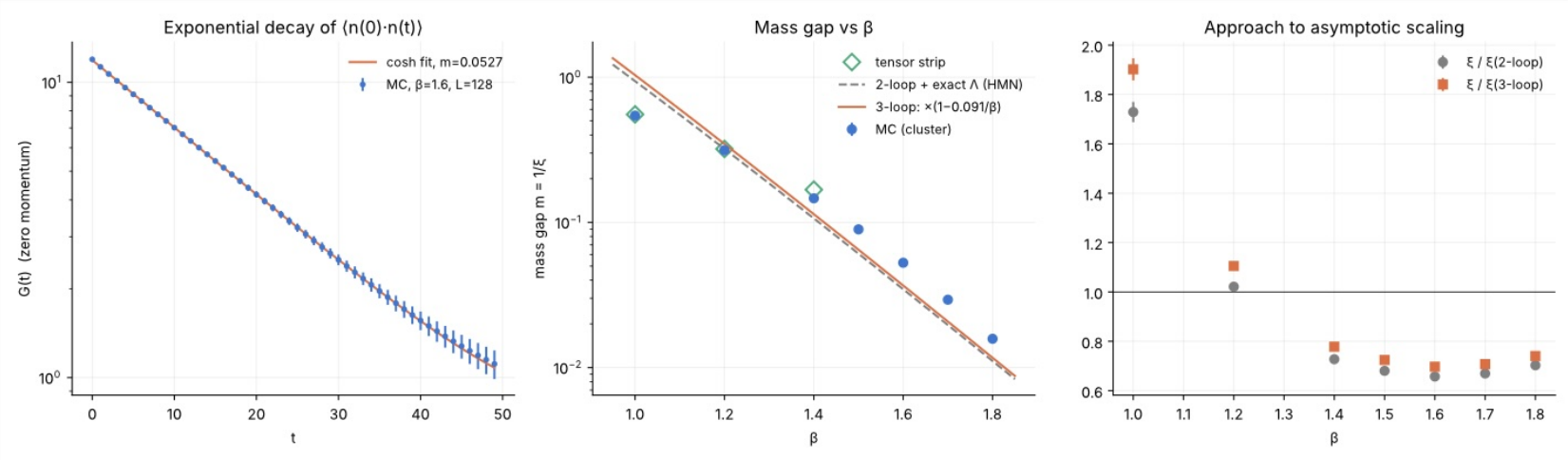
The $N_\tau \to \infty$ limit and asymptotic freedom

- $N_\tau \to \infty$ singles out the largest eigenvalues of $\mathbb{T}$; the key question is the degeneracy of the largest one.
- If the eigenvalues of $\mathbb{T}$ are real and positive with a single largest one, $\mathbb{T}^{N_\tau}$ projects on the vacuum, and results equivalent to OS3 and OS4 may follow (Luscher).
- Main question: do these properties persist as the spatial couplings increase?
- For the $O(3)$ NLσM in 2D and $SU(2)$ in 4D, asymptotic freedom makes the lattice spacing exponentially small as β grows: an exponentially large number of sites is needed at fixed physical volume.
- TRG coarse graining can in principle handle this growth.
- For 2D $O(3)$, a rigorous proof of the gap may have been provided (OpenAI 215)



Mass gap from the spin–spin correlator (preliminary)

- Zero-momentum correlator at $\beta = 1.6$, $L = 128$, $D_{\text{bond}} = 40$, spherical harmonics truncated at $l \leq 3$; fit to $G(t) = A\cosh\big(m\,(t - L/2)\big)$.
- Compared with cluster Monte Carlo and the three-loop formula [22]:
  $\xi = \frac{1}{m} = C_\xi \frac{e^{2\pi\beta}}{2\pi\beta}\left[1 - \frac{0.091}{\beta} + O(\beta^{-2})\right]$, $C_\xi = 2^{-5/2}\,\frac{e^{1-\pi/2}}{8} \approx 0.01249$.



(a) Correlator at $\beta = 1.6$ with cosh fit. (b) Mass gap vs β: tensor strip, cluster MC, two- and three-loop formulas. (c) Ratio of measured ξ to the two- and three-loop predictions.

Model: $SU(3)$ with 12 equal mass Dirac fermions

The fermionic part of the lattice action is

$$S_F = \frac{1}{2}\sum_x \bar{\psi}_x \left(\gamma_\mu \Delta^\mu + m_q\right)\psi_x + h.c., \tag{1}$$

The gauge part of the action is given by the Wilson action

$$S_G = \beta \sum_p \left[1 - \frac{1}{3}\mathrm{Re}\left(\mathrm{Tr} U_p\right)\right] \tag{2}$$

where $\beta = 6/g^2$. We use unimproved Kogut-Susskind's staggered formulation (based on D. Sinclair's code). The plaquette histograms are then used to build the partition function, written in terms of the density of states $\Omega(E)$ as

$$Z(\beta) = \int dE\, \Omega(E) e^{-\beta N_p E}. \tag{3}$$

This is done using the Ferrenberg-Swendsen algorithm.



Expected phase diagram in the (m_q, β) plane

For details and references see arXiv: 2606.14642

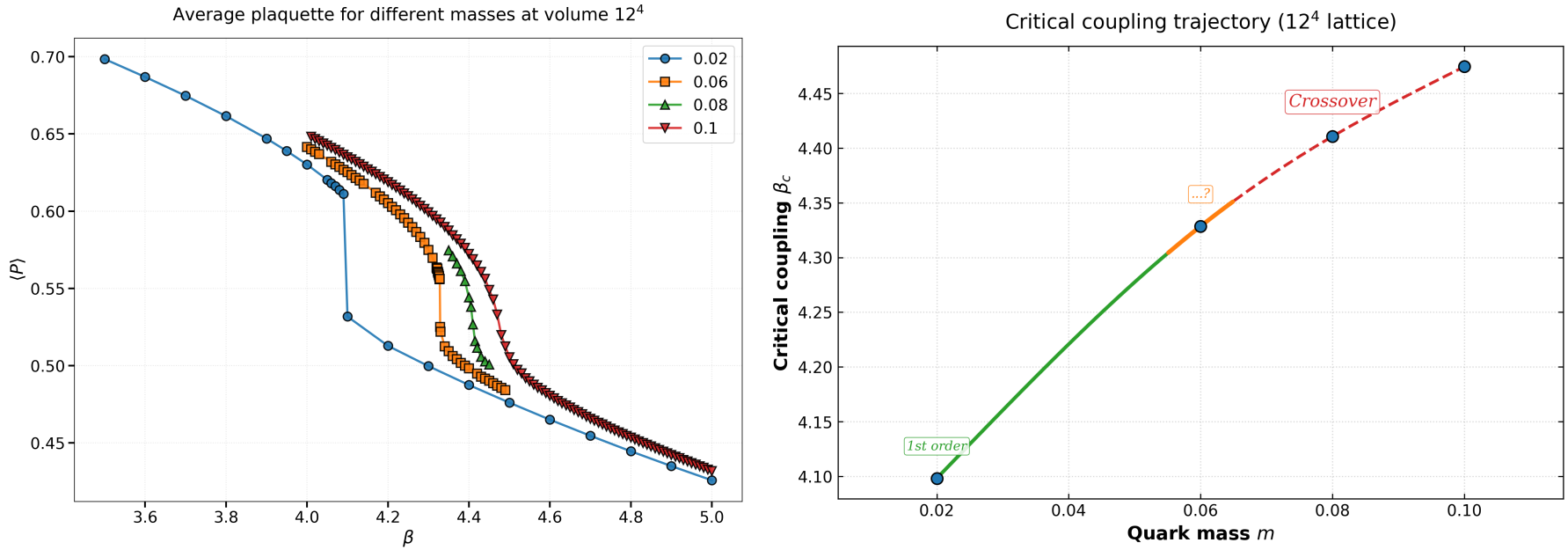
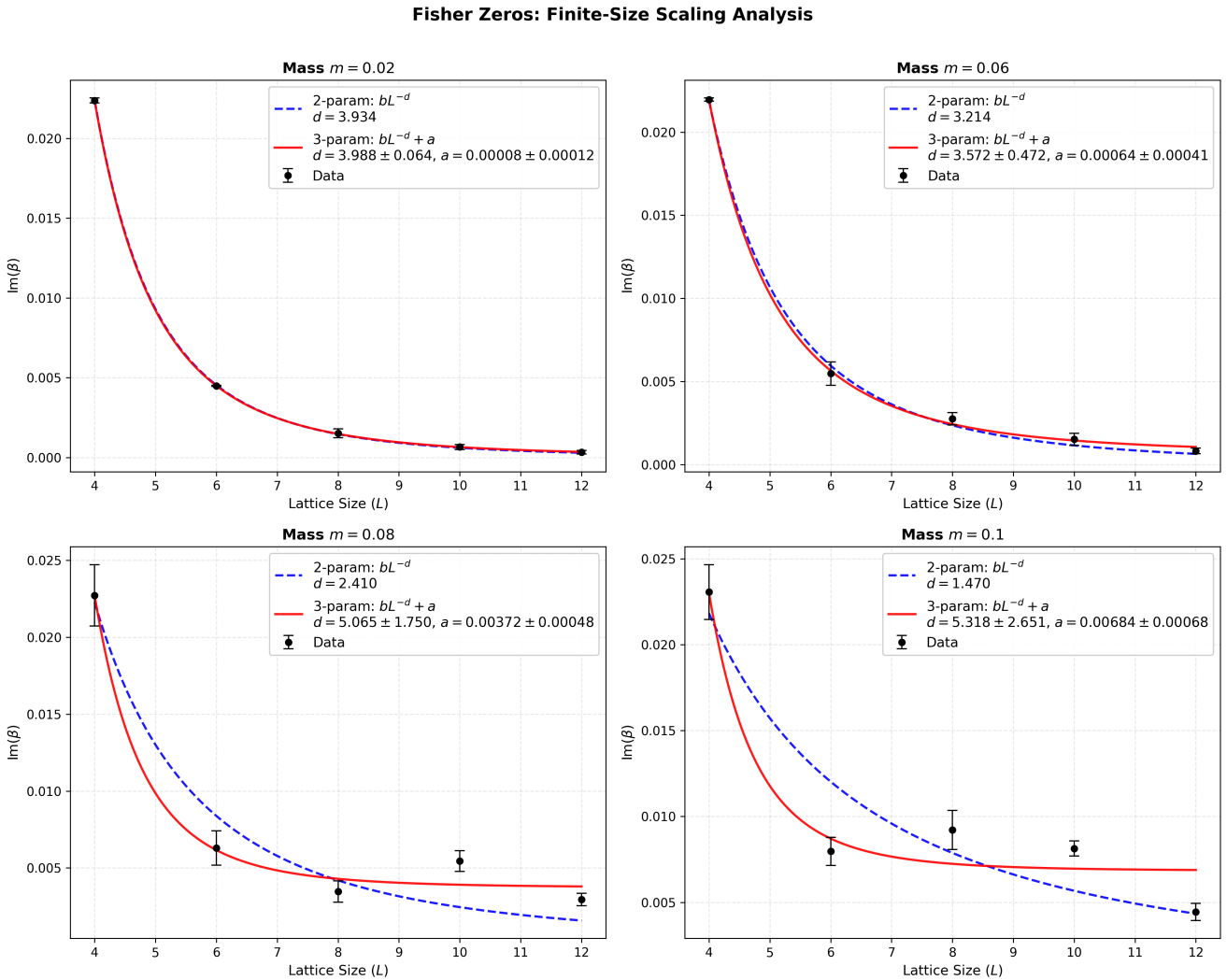


Figure: Left: Distribution of average plaquette values as a function of β for $L = 12$ and masses $m_q = 0.02,\ 0.06,\ 0.08$, and 0.1. Right: Expected phase diagram for 12 flavor $SU(3)$ in the (m_q, β) plane.

Lowest zero as a function of L with 4 masses

$Im\beta$: 2 parameter fit (blue, bL^{-d}) vs. 3 parameters (red, $bL^{-d} + a$)
For details and references see arXiv: 2606.14642



Fisher zeros in O(2)

PHYSICAL REVIEW D **89,** 016008 (2014)

Controlling sign problems in spin models using tensor renormalization

Alan Denbleyker,[1] Yuzhi Liu,[1,2,4] Y. Meurice,[1] M. P. Qin,[3] T. Xiang,[3] Z. Y. Xie,[3] J. F. Yu,[3] and Haiyuan Zou[1,*]

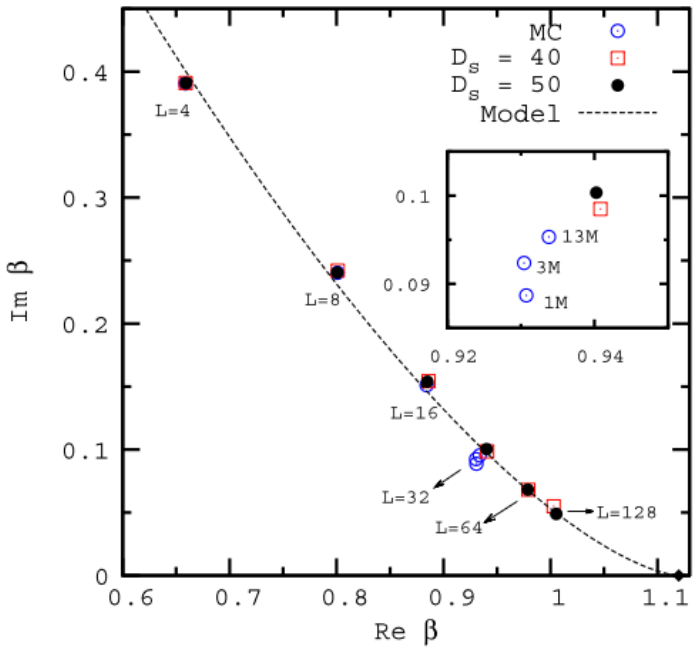


FIG. 9 (color online). Zeros of XY model with linear size $L = 4$, 8, 16, 32, 64, and 128 (from up-left to down-right) calculated from HOTRG with $D_s = 40$, and 50 and zeros with $L = 4$, 8, 16, and 32 from MC. The curve is a model for trajectory of the lowest zeros.

$$Z(\beta_z) = f(z_0) = 0, \tag{11}$$

the values of zeros for different volumes map to the same z_0 at large L.

From the scaling behavior of the correlation length for a Kosterlitz-Thouless (K-T) transition [24,25],

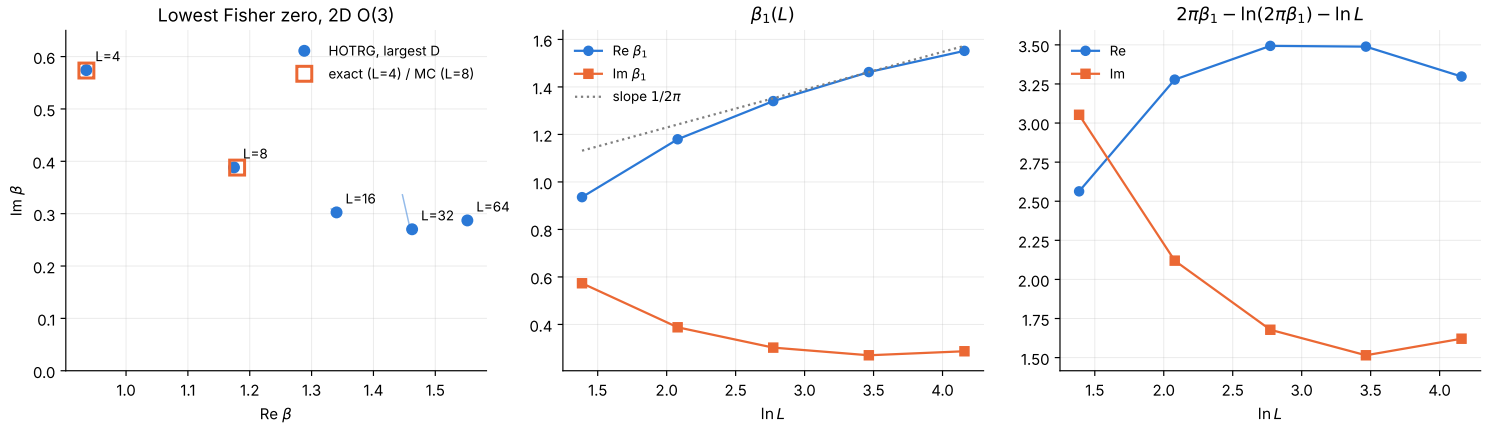
$$\xi = A \exp(b/\sqrt{t}), \tag{12}$$

where $t = \beta_c - \mathrm{Re}\beta - i\mathrm{Im}\beta$ for $\mathrm{Re}\beta < \beta_c$ and β_c is the K-T transition coupling. By taking the imaginary part as a perturbation from the real part and from the leading order equation for real and imaginary t, one can obtain the relation between $\mathrm{Re}\beta_z$ and $\mathrm{Im}\beta_z$ as

$$\mathrm{Im}\beta_z = \frac{2a}{b}(\beta_c - \mathrm{Re}\beta_z)^{3/2}, \tag{13}$$

where $a = \mathrm{Im}(\ln(z_0/A))$.

Fisher zeros for O(3) (preliminary)



- Lowest zero for $L = 4$–64 (HOTRG); agrees with exact ($L = 4$) and Monte Carlo ($L = 8$) values, as done for $O(2)$ by Denbleyker et al..
- Expected scaling $\xi/L \simeq z_0$: $\mathrm{Re}\,\beta_1$ grows by $(\ln 2)/(2\pi) \approx 0.110$ per doubling of L; $\mathrm{Im}\,\beta_1$ tends to a nonzero constant (no pinching of the real axis).
- Observed: increments 0.24, 0.16, 0.12 in $\mathrm{Re}\,\beta_1$; $\mathrm{Im}\,\beta_1 = 0.573,\ 0.389,\ 0.302,\ 0.270$ for $L = 4, 8, 16, 32$, suggesting a limit near 0.25. ($L = 64$ provisional.)
- Contrast: for $O(2)$ the lowest zero approaches the real axis (Kosterlitz–Thouless).

Summary

- The TRG gives an exact discrete reformulation of spin and gauge models, with a natural transfer matrix and Hilbert space.
- Natural first steps for mathematicians, mathematical physicists and Lean users: convergence, analyticity and regularity (OS0, OS1), existence and positivity of $\mathbb{T}$, and the strong-coupling gap.
- The hard part: keeping the gap (positivity (OS3) and nondegeneracy of ground state (OS4)) at small coupling (large β) and infinite volume, and reaching the continuum limit with Euclidean invariance (OS2). This needs renormalization, which is what the TRG is designed for.
- Numerical experiments with approximate transfer matrices can guide rigorous work; start with lower-dimensional models such as the 2D $O(3)$ model.
- Preliminary 2D $O(3)$ results (mass gap, Fisher zeros) are encouraging regarding β scaling but need to be checked; an independent check is planned with Shinichiro Akiyama.



Existing methods that could be used for the gap

- Positivity tests for transfer matrices [24].
- Tensor implementations of q-deformed $SU(2)$ lattice gauge theory [25, 26, 27, 28].
- All-mode approach for general TRG [29].
- Gauge-invariant PEPS ansatz [30].
- Axial-gauge calculations for $3+1$ $SU(3)$ [31].
- Investigation of multiparticle states [32].
- S. Itatani *et al.*, SVD for spectral decomposition (Lattice 2026).
- Recent TRG results: see [33].

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Postdocs at U. Iowa and BNL (2 separate jobs)

The Theoretical Particle Physics group at the University of Iowa is seeking candidates for a postdoctoral position funded through the HEP-QIS Quantum Outposts. The position is for one year starting in August 2027 and could be extended for a second and third year based on performance and funding availability. The salary will be based on the University of Iowa guidelines: https://www.grad.uiowa.edu/postdoc/postdoctoral-scholars-salary. The successful candidate is expected to work closely with Yannick Meurice's group on topics related to quantum computations/simulations in the context of lattice gauge theory.

Application review will begin December 1, 2026 and will continue until the position is filled.

Applicants should submit a curriculum vitae and research statement using the link https://jobs.uiowa.edu/postdoc/view/4699.

They should arrange to have at least two recommendation letters sent directly to Yannick Meurice (yannick-meurice@uiowa.edu).

Yannick Meurice

for the C2QA-NT post-doc position.
The job ad is now out:
https://bnl.wd1.myworkdayjobs.com/Externa/job/Upton-NY/Postdoc-Research-Associate_JR102799
and will be posted to INSPIRE shortly.

Raju Venugopalan



Thanks to the organizers!



Acknowledgments

We thank the organizers and participants of the long-term workshop “New Frontiers in QCD 2026” and of the workshop “Frontiers of Quantum Computing and Classical Tensor Networks for Gauge Theories” at the Yukawa Institute for Theoretical Physics for many stimulating conversations.

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