Critical behavior of the bosonized Schwinger model

with uniform matrix product states

arXiv:2611.16092 and work in progress

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Frontiers of Quantum Computing and Classical
Tensor Networks for Gauge Theories

Plan of the talk

1. **Schwinger model and bosonization**
   Vacuum structure and the phase transition
2. **Bosonic lattice formulation**
   Infinite-chain ground states and criticality
3. **Symanzik effective theory and continuum limit**
   Lattice corrections, the continuum extrapolation, and the result
4. **New DMRG result (work in progress)**
   New determination from Staggered fermions with open boundary conditions
5. **Summary**

The Schwinger model: QED in 1+1 dimensions

One Dirac fermion coupled to a U(1) gauge field

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left[i\gamma^{\mu}(\partial_{\mu} + igA_{\mu}) - m\right]\psi - \frac{g\theta}{2\pi}F_{01}$$

$$E \equiv F_{01} = \partial_t A_1 - \partial_x A_0$$

Parameter	Meaning
m	Fermion mass
g	Gauge coupling (mass dimension 1)
θ	Angle specifying the background electric field

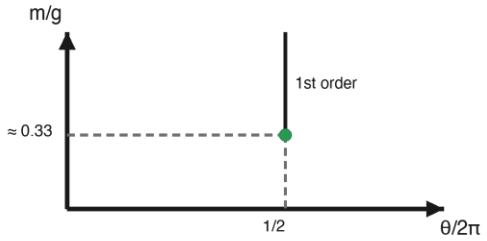
J. Schwinger, Phys. Rev. 128, 2425 (1962)

A simple gauge theory with mass generation,
θ-dependent vacua, and spontaneous symmetry breaking.

Vacuum symmetry breaking at θ = π

At θ = π, charge conjugation is an exact symmetry.

$$\theta = \pi : \qquad C : \; E \mapsto -E$$



Small m/g

A unique C-symmetric vacuum

$$\langle E \rangle = 0$$

Large m/g

Two vacua related by C

$$\langle E \rangle = \pm E_0 \neq 0$$

The transition belongs to the 2d Ising universality class.
Universality fixes the exponents; the critical mass requires a calculation.

Bosonization makes the competing terms explicit

$$H = \int dx \left[\frac{\Pi_\phi^2 + (\partial_x \phi)^2}{2} + \frac{g^2 \phi^2}{2\pi} - mg \frac{e^\gamma}{2\pi^{3/2}} :\cos(2\sqrt{\pi}\,\phi - \theta): \right]$$

S. Coleman, Ann. Phys. 101, 239 (1976)

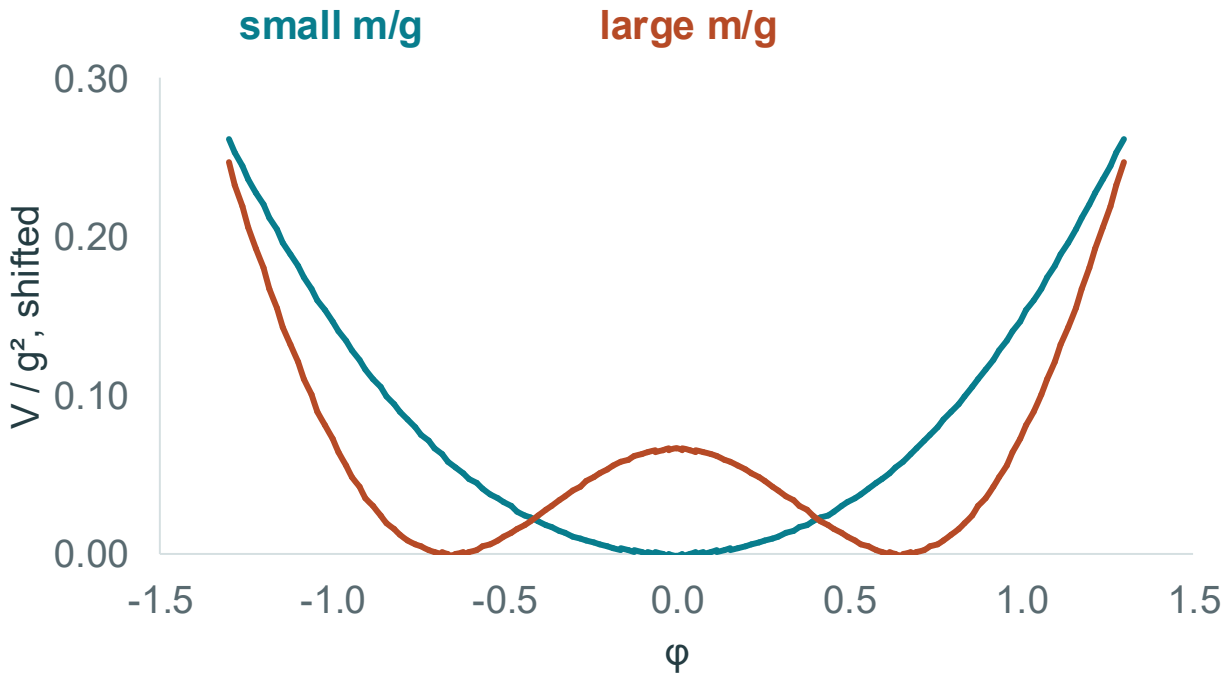
Electric energy

A quadratic well.

$$m = 0: \qquad \mu = \frac{g}{\sqrt{\pi}}$$

Fermion mass

A cosine interaction.
At θ = π: C acts as φ ↦ −φ.



Classical illustration. The quantum transition is in the 2d Ising universality class.

A test of continuum equivalence

Do fermionic and bosonic lattices give the same continuum critical mass?

Staggered formulation

Fermions and gauge links
Gauss-law constraint

Bosonized formulation

A non-compact scalar
Local scalar interactions

Different cutoff effects at finite lattice spacing.
The continuum predictions should agree.

Continuum and lattice Hamiltonians

Continuum

$$H_{\rm cont} = \int dx \left[\frac{1}{2}\Pi_\phi^2 + \frac{1}{2}(\partial_x \phi)^2 + V_{\rm cont}(\phi) \right]$$

Spatial lattice, continuous time

$$H_{\rm lat} = a \sum_j \left[\frac{1}{2} \left(\frac{p_j}{a} \right)^2 + \frac{1}{2} \left(\frac{\phi_{j+1} - \phi_j}{a} \right)^2 + V_{\rm lat}(\phi_j) \right]$$

$$V_r(\phi) = \frac{g^2 \phi^2}{2\pi} - \frac{e^\gamma m g}{2\pi^{3/2}} :\cos(2\sqrt{\pi}\,\phi - \theta):_{0,r}, \qquad r = \text{cont}, \text{lat}$$

$$\phi_j = \phi(ja), \qquad p_j = a\Pi_\phi(ja), \qquad [\phi_j, p_k] = i\delta_{jk}$$

Normal ordering uses the m = 0 vacuum of each regulator.

Related lattice formulations

Seifnashri, arXiv:2601.14359

Belyansky et al., arXiv:2307.02522

Ground state on an infinite bosonic chain

A local basis adapted to the free vacuum

$$\rho_j^{(0)} = \mathrm{Tr}_{k\neq j}\,|\Omega_0\rangle\langle\Omega_0| = \sum_{n=0}^{\infty} w_n |n\rangle\langle n|$$

$$w_n = (1-q)q^n, \qquad 0 \le q < 1, \qquad w_0 \ge w_1 \ge w_2 \ge \cdots$$

$$P_{n_{\max}} = \sum_{n=0}^{n_{\max}} |n\rangle\langle n|, \qquad \mathrm{Tr}\left[(1-P_{n_{\max}})\rho_j^{(0)}\right] = q^{n_{\max}+1}$$

Uniform matrix product states (MPS)

A single MPS tensor repeated at all sites. Minimize the energy density via VUMPS.
The bond dimension D controls the entanglement retained.
Check the validity of chosen n_{max} and D separately at nonzero fermion mass.

Criticality from the correlation length

$$E = \frac{g}{\sqrt{\pi}}\phi, \qquad C: \ \phi \mapsto -\phi \quad (\theta = \pi)$$

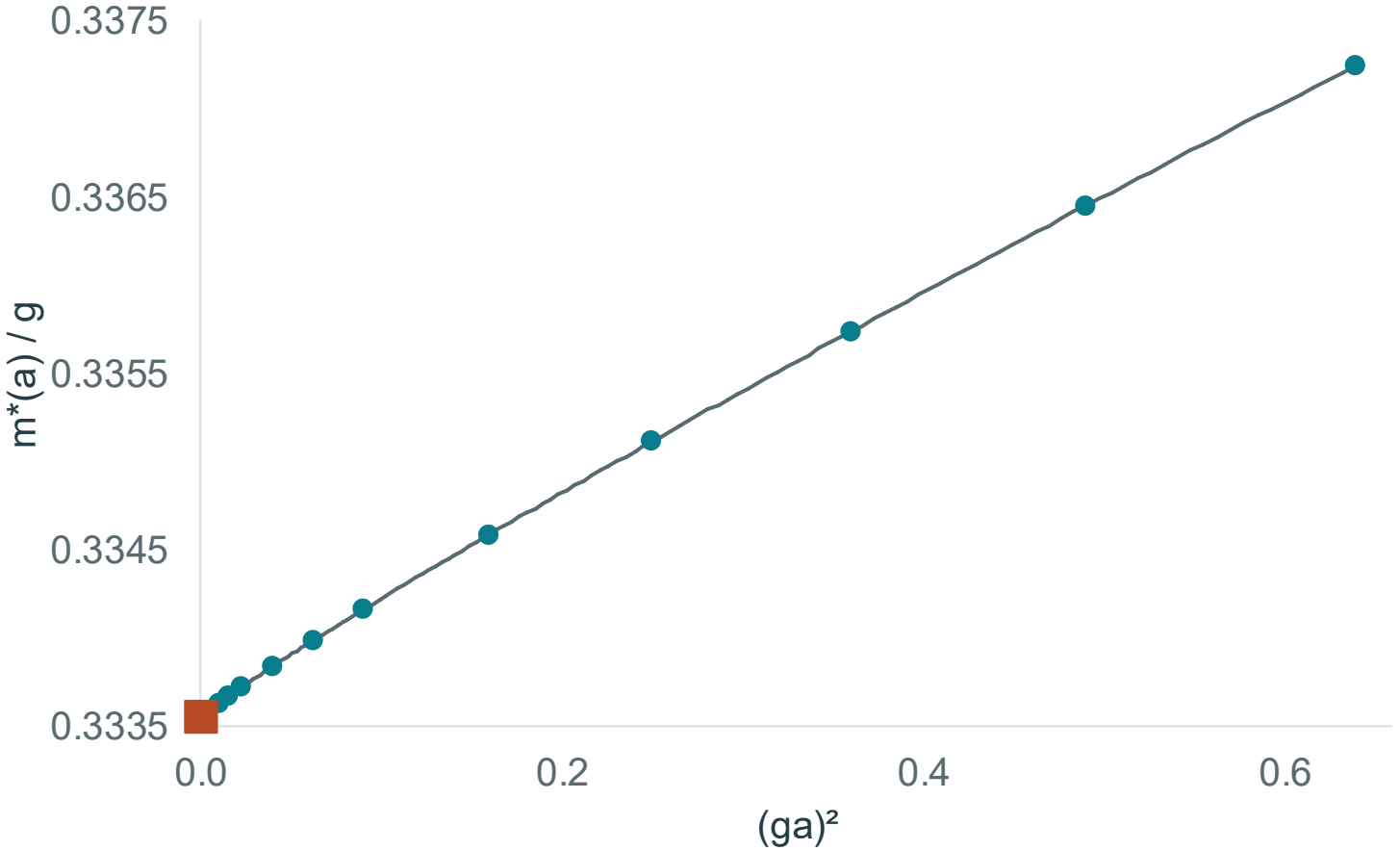
Electric-field 2-point function allows to read off the correlention lengths.

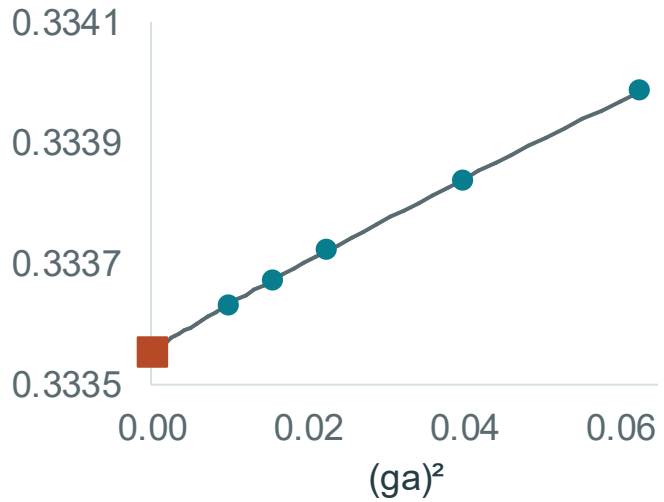
$$\xi_{\text{phys}} \propto |m - m_*(a)|^{-\nu}, \qquad \nu = 1$$

A finite bond dimension D limits the correlation length.
Use several D values to locate the divergence at each lattice spacing.

The continuum limit comes after this critical-mass determination.

The lattice critical mass approaches a continuum limit





Continuum limit

a = 0

Star: extrapolated critical mass

Symanzik effective theory

At distances much larger than a, lattice effects are local corrections to the continuum theory.

$$S_{\text{eff}} = S_{\text{cont}} + \sum_i h_i(a) \int d\tau\, dx\, \mathcal{O}_i(x,\tau)$$

The operators respect lattice symmetries. Their coefficients are fixed by matching long-distance correlation functions.

Example: the free lattice dispersion

$$\frac{4}{a^2}\sin^2\frac{ak}{2} = k^2 - \frac{a^2k^4}{12} + O(a^4k^6) \qquad \delta S = -\frac{a^2}{24}\int d\tau\, dx\, (\partial_x^2\phi)^2$$

A local higher-derivative term reproduces the leading lattice correction.

K. Symanzik, Nucl. Phys. B 226, 187 and 205 (1983)

Lattice corrections and the continuum limit

Charge counting and reflection symmetry allow even powers of a, with possible logarithms.

Assume that the critical mass responds smoothly to the local corrections.

$$\frac{m_*(a)}{g} \sim c_0 + \sum_{n\geq 1} z^{2n} P_{2n}(\ln z), \qquad z = ga, \quad c_0 = \left(\frac{m}{g}\right)_c$$

Primary fit family

$$\frac{m_*(a)}{g} = c_0 + c_2 z^2 + c_{2L} z^2 \ln z + c_4 z^4 + \cdots$$

Vary the fit range and higher powers. Combine fits with weights from the Akaike information criterion (AIC).

Higher logarithms can enlarge the uncertainty.

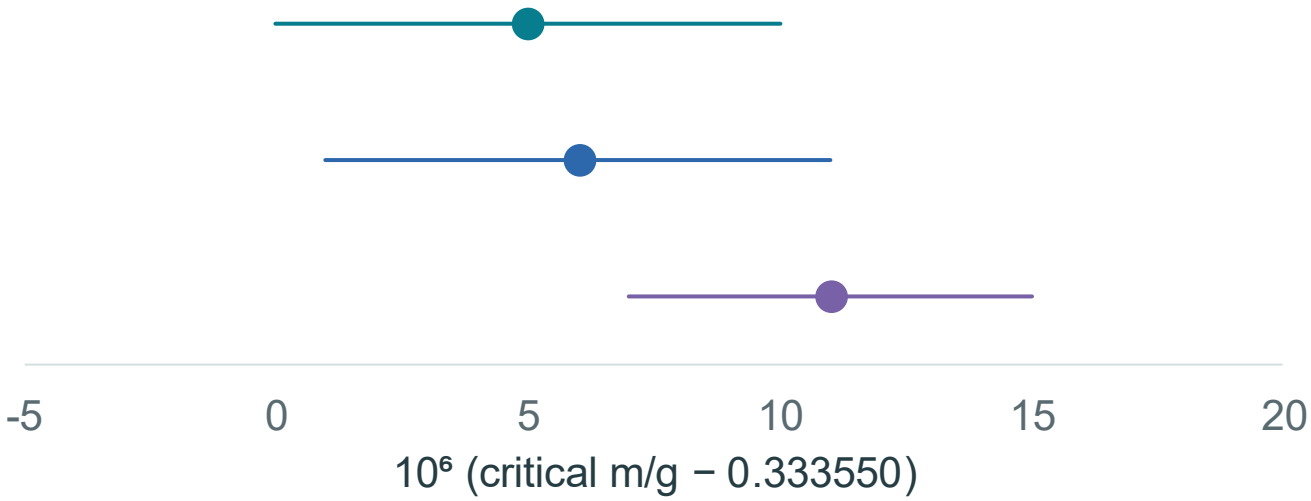
The continuum result agrees with staggered fermions

$$(m/g)_c = 0.333\,555(5)$$ TO, arXiv:2611.16092

This work (TO)
Bosonized VUMPS arXiv:2611.16092

Fujii, Fujikura, Kikukawa,
TO, Pedersen
Staggered VUMPS arXiv:2412.03569

Argüello Cruz, Tarnopolsky, Xin
Staggered DMRG arXiv:2412.01902



Agreement within the uncertainties

Staggered DMRG with OBC: work in progress

Determination	Formulation	Critical m/g
Byrnes, Sriganesh, Bursill, Hamer (2002) arXiv:hep-lat/0202014	Staggered DMRG	0.333 5(2)
Argüello Cruz, Tarnopolsky, Xin arXiv:2412.01902	Staggered DMRG	0.333 561(4)
Fujii, Fujikura, Kikukawa, TO, Pedersen arXiv:2412.03569	Staggered VUMPS	0.333 556(5)
TO arXiv:2611.16092	Bosonized VUMPS	0.333 555(5)

Four orders of magnitude smaller uncertainty than in the three recent determinations.

Summary

The bosonized lattice reproduces the Schwinger critical point.
Its continuum critical mass agrees with staggered fermions.

Symanzik effective theory explains the form of the lattice corrections.

Ongoing DMRG work (tentative):
four orders of magnitude smaller uncertainty
than in the three recent determinations.