mini-workshop on "rheology of dense materials"

March 18, 2026

Scaling theory for critical phenomena in steady shear flow

arXiv:2412.02111 (2024)

Harukuni Ikeda (YITP, Kyoto Univ.)
Hiroyoshi Nakano (ISSP, Tokyo Univ.)

YUKAWA INSTITUTE FOR THEORETICAL PHYSICS

東京大学 物性研究所
THE INSTITUTE FOR SOLID STATE PHYSICS
THE UNIVERSITY OF TOKYO

Introduction

Motivation

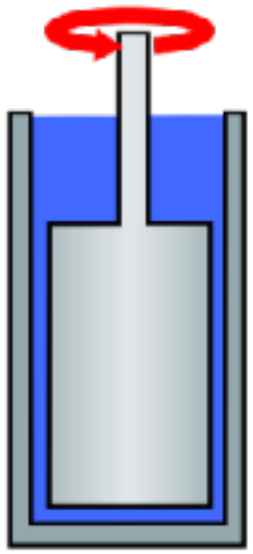
How do external driving forces affect critical phenomena?

This work

Mixing by blender

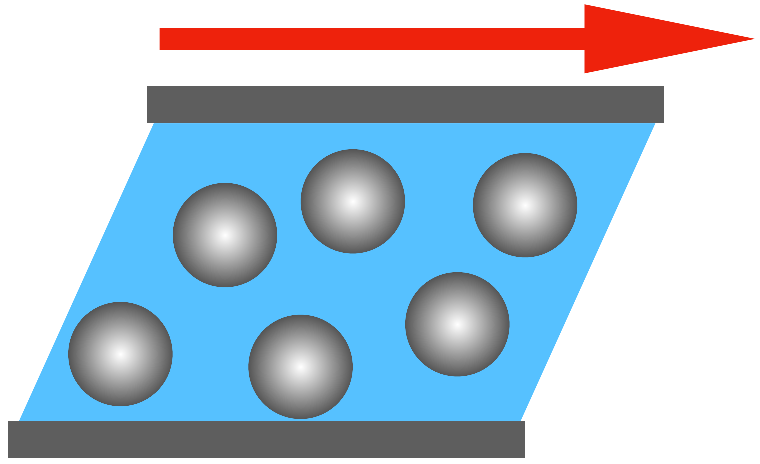
(Irasuto-ya)

Rotational Rheometer



(Wikipedia)

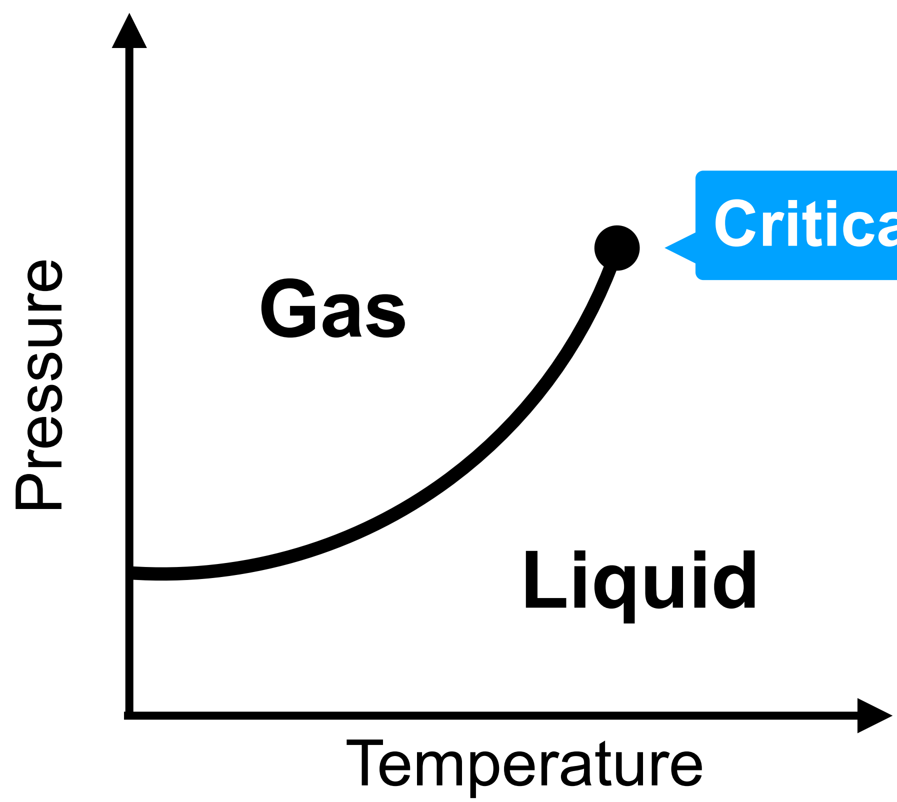
Simple shear



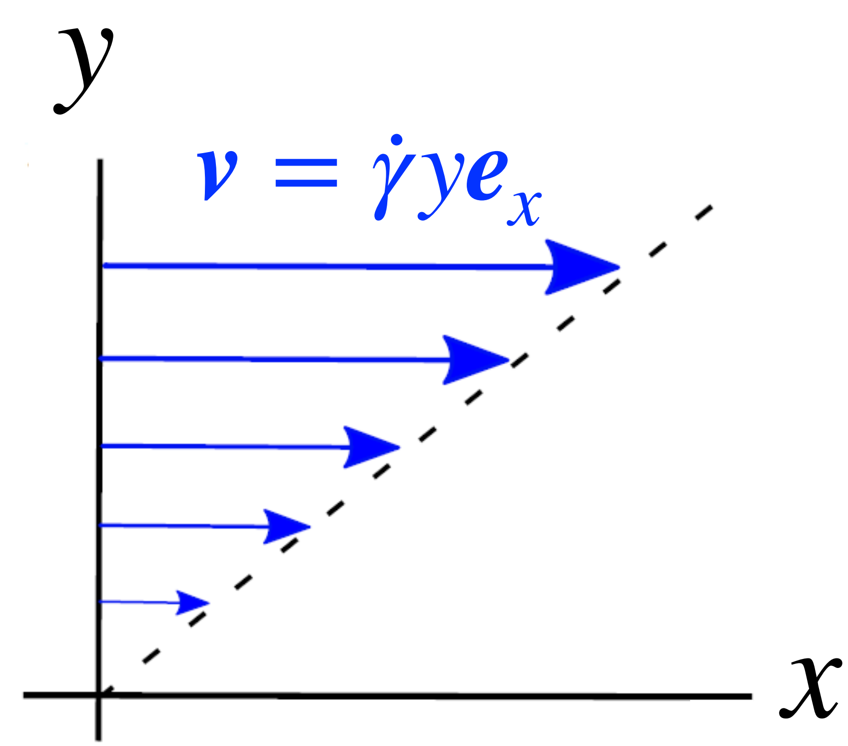
For the simplest example, we here investigate the effects of simple shear.

Critical phenomena in shear

Gas-liquid transition under shear Onuki and Kasaki (1979)



Simple shear



In equilibrium

Upper critical dimension

$d_{\mathrm{up}} = 4$

In shear

Upper critical dimension

$d_{\mathrm{up}} = 2.4$

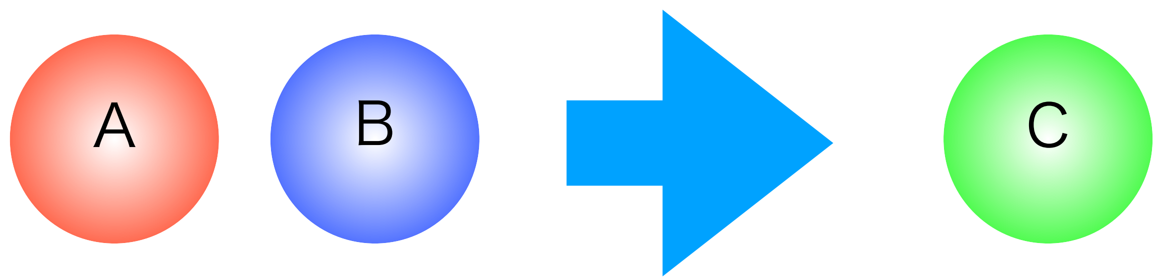
Simple shear reduces the upper critical dimension.

Introduction

Critical phenomena in shear

Chemical reaction in shear R. Reigada *et al* (1996)

A+B → C



In equilibrium

$$\rho_A(t) \propto \begin{cases} t^{-1} & d > 4 \\ t^{-d/4} & d < 4 \end{cases}$$

$$\rightarrow d_{\mathrm{up}} = 4$$

In shear

$$\rho_A(t) \propto t^{-1} \quad d \geq 2$$

$$\rightarrow d_{\mathrm{up}} = 2$$

Simple shear accelerates the chemical reaction and reduces the upper critical dimension.

Introduction

Critical phenomena in shear

O(n) model in shear Nakano, Minami, and Sasa (2021)

$$\frac{\partial \boldsymbol{\phi}(\boldsymbol{x},t)}{\partial t} + \dot{\gamma} y \frac{\partial \boldsymbol{\phi}(\boldsymbol{x},t)}{\partial x} = -\frac{\delta F[\boldsymbol{\phi}(\boldsymbol{x},t)]}{\delta \phi(\boldsymbol{x},t)} + \sqrt{2T}\boldsymbol{\xi}(\boldsymbol{x},t)$$

Advection by simple shear

In equilibrium

Upper critical dimension

$$d_{\mathrm{up}} = 4$$

In shear

Nakano, Minami, and Sasa reported mean-field critical exponents even in d=2.

Simple shear makes the system more mean-field-like.

Introduction

Motivation

Question

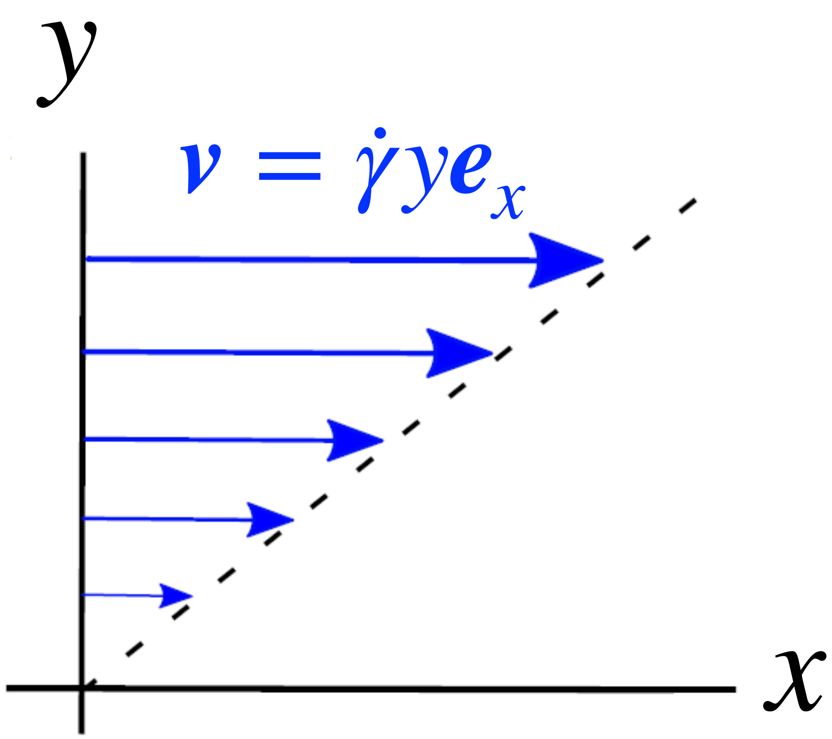
Why does the shear flow reduce the upper critical dimension and make the system more mean-field-like?

Purpose

To answer the above question, we want to construct a scaling theory under steady shear flow.

How to incorporate shear effects?

Settings



Shear flow with constant velocity gradient

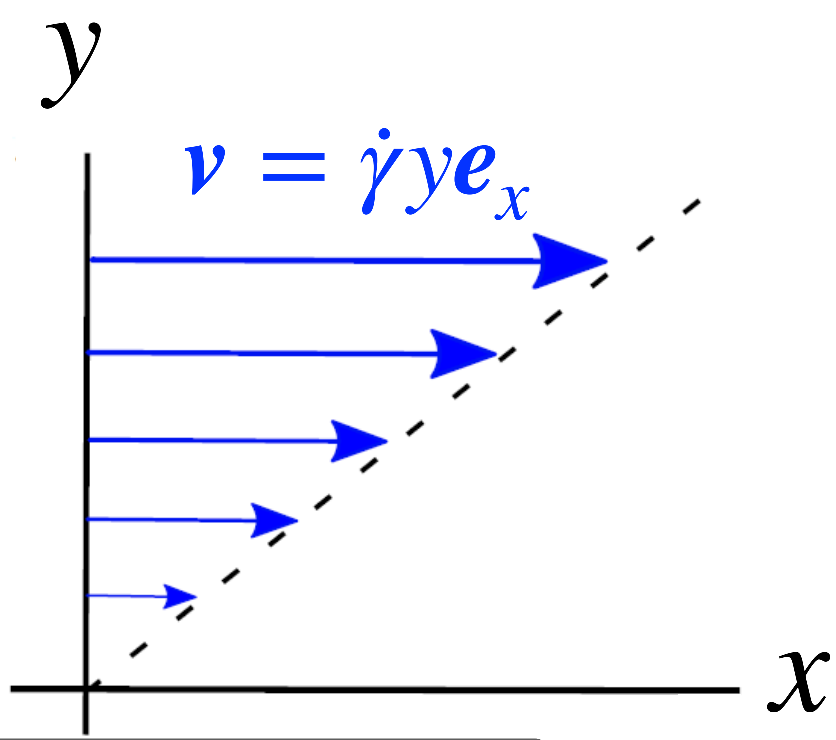
Under shear, the time derivative of a field is replaced as

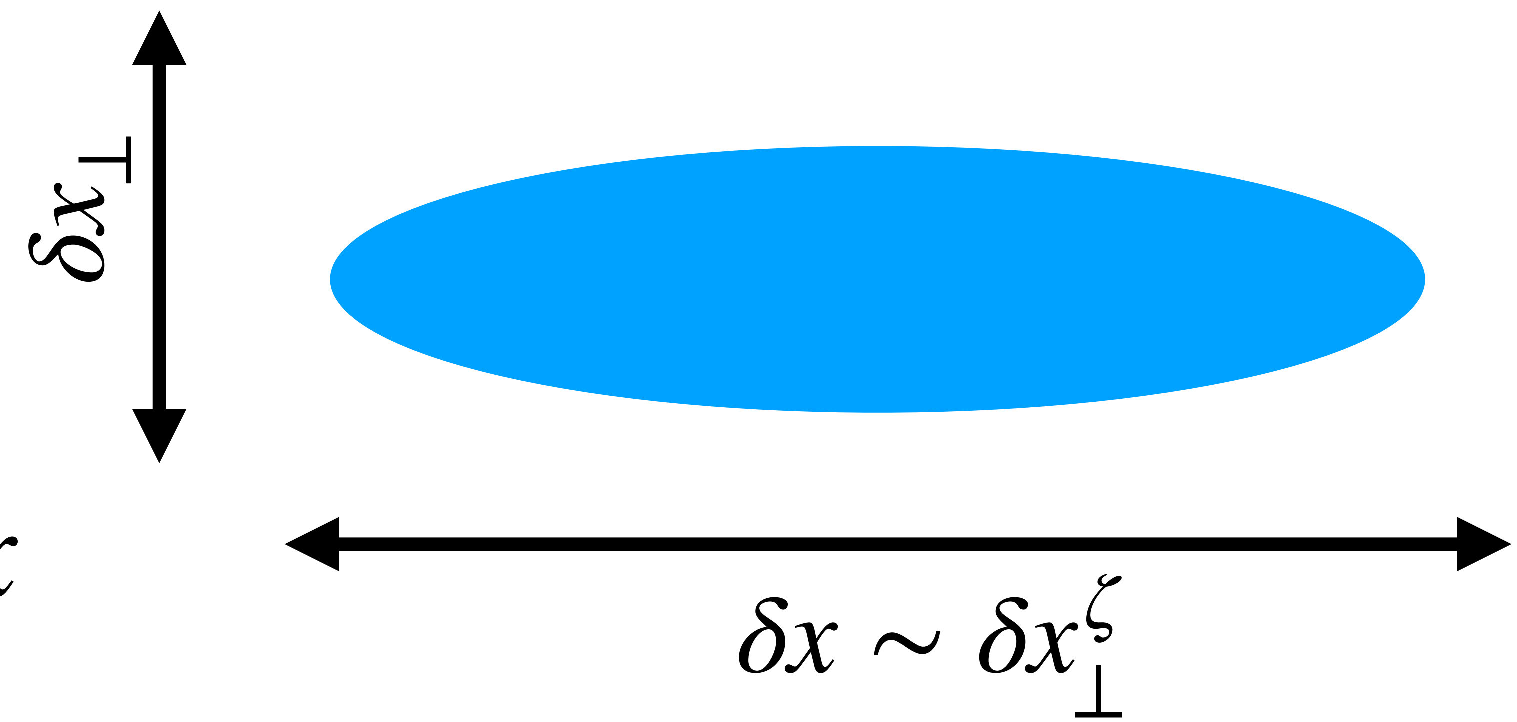
$$\frac{\partial \phi(\boldsymbol{x},t)}{\partial t} \to \frac{\partial \phi(\boldsymbol{x},t)}{\partial t} + \boxed{\dot{\gamma} y \frac{\partial \phi(\boldsymbol{x},t)}{\partial x}}$$

Advection term

Scaling argument

Due to shear, critical fluctuations are anisotropic





anisotropy exponent
ζ=1 → isotropic
ζ≠1→ anisotropic

Anisotropic scaling Ansatz

$$x = b^\zeta x',\ \boldsymbol{x}_\perp = b\boldsymbol{x}'_\perp,\ t = b^z t',\ \phi(x, \boldsymbol{x}_\perp, t) = b^\chi \phi'(x', \boldsymbol{x}'_\perp, t')$$

Parallel to shear

$\boldsymbol{x}_\perp = \{y, z, \cdots\}$: Perpendicular coordinates

Theory

Scaling argument

$$\frac{\partial \phi}{\partial t} + \dot{\gamma} y \frac{\partial \phi}{\partial x} = \cdots$$

Advection by simple shear

Scale transformation

$$b^{\chi - z} \frac{\partial \phi}{\partial t} + b^{\chi + 1 - \zeta} \dot{\gamma} y \frac{\partial \phi}{\partial x} = \cdots$$

Scaling relation

$\zeta > 1 \rightarrow$ anisotropic fluctuations

$$\chi - z = \chi + 1 - \zeta \rightarrow \zeta = 1 + z$$

In equilibrium

$$\int d\boldsymbol{x} \sim b^d$$

In shear

$$\int d\boldsymbol{x} \sim b^{d-1+\zeta} \sim b^{d_{\mathrm{eff}}}$$

$$d_{\mathrm{eff}} = d + z$$

The shared system has the effective dimension *d*+*z*, which reduces the upper critical dimension by *z*

$$d_{\mathrm{up}} \rightarrow \max[d_{\mathrm{up}} - z,\ 2]$$

Shear is defined for $d \geq 2$

Theory

Upper critical dimension

	z	d_{up} **in equilibrium**	d_{up} **in shear**
Model-A	2	4	2
Model-B	4	4	2
A+B→C	2	4	2

Mean-field behaviors are expected in d=2 for various critical phenomena.

Result for model A

Scaling argument

$$\frac{\partial \phi}{\partial t} + \dot{\gamma} y \frac{\partial \phi}{\partial x} = k \partial_x^2 \phi + k \nabla_\perp^2 \phi - \varepsilon \phi - u \phi^3 + \sqrt{2T} \xi$$

For concreteness, hereafter we consider model A, and provide more detailed arguments for the derivation of the scaling exponents.

Result for model A

Scaling argument

$$\frac{\partial \phi}{\partial t} + \dot{\gamma} y \frac{\partial \phi}{\partial x} = k \partial_x^2 \phi + k \nabla_\perp^2 \phi - \varepsilon \phi - u \phi^3 + \sqrt{2T}\xi$$

Anisotropic scaling Ansatz

$$x = b^\zeta x', \ \boldsymbol{x}_\perp = b \boldsymbol{x}'_\perp, \ t = b^z t', \ \phi(x, \boldsymbol{x}_\perp, t) = b^\chi \phi'(x', \boldsymbol{x}'_\perp, t')$$

$$b^{\chi - z} \frac{\partial \phi'}{\partial t'} + b^{1-\zeta+\chi} \dot{\gamma} y' \frac{\phi'}{x'} = b^{\chi - 2\zeta} k \partial_{x'}^2 \phi' + b^{\chi - 2} k \partial_{x'}^2 \phi' - b^\chi \varepsilon \phi' - b^{3\chi} u \phi'^3 + b^{-\frac{\zeta + d - 1 + z}{2}} \sqrt{2T}\xi'$$

$$\frac{\partial \phi'}{\partial t'} + \dot{\gamma}' y' \frac{\partial \phi'}{\partial x'} = k'_\parallel \partial_{x'}^2 \phi' + k'_\perp \nabla'^2_\perp \phi' - \varepsilon' \phi' - u' \phi'^3 + \sqrt{2T'}\xi'$$

$T' = b^{\frac{z - 2\chi - (d-1) - \zeta}{2}} T$

$\dot{\gamma}' = b^{z - \zeta + 1} \dot{\gamma}$

$k'_\parallel = b^{z - 2\zeta} k$

$k'_\perp = b^{z-2} k$

$\varepsilon' = b^z \varepsilon$

$u' = b^{2\chi + z} u$

Nonlinear term

Result for model A

Gaussian fixed point with shear

Advection **Diffusion**

$$\dot{\gamma} y \frac{\partial \phi}{\partial x} \gg k_{\parallel} \partial_x^2 \phi$$

$k_{\parallel}$ is irrelevant!

We require the scale invariance of $\dot{\gamma}, \ \ k_{\perp}, \ T$

$$\dot{\gamma}' = b^{z-\zeta+1}\dot{\gamma}, \ k_{\perp} = b^{z-2}k, \ T' = b^{\frac{z-2\chi-(d-1)-\zeta}{2}}T$$

Scaling relations

$$z-\zeta+1=0, \ z-2=0, \ \frac{z-2\chi-(d-1)-\zeta}{2}=0$$

Critical exponents with shear

$$z=2, \chi=-\frac{d}{2}, \zeta=3$$

Anisotropic

Self-consistency check

$$k_{\parallel}' = b^{z-2\zeta}k = b^{-4}k \xrightarrow{b\to\infty} 0$$

Result for model A

Static structure factor

Two point correlation function

$$C(x_{\|}, \boldsymbol{x}_{\perp}, \varepsilon) \equiv \langle \phi(\boldsymbol{x}, t)\phi(\boldsymbol{0},0)\rangle$$

Fourier transformation

$$S(q_{\|}, q_{\perp}, \varepsilon) = b^{z} S(b^{\zeta} q_{\|}, b q_{\perp}, b^{z}\varepsilon)$$

- $S(0,0,\varepsilon) = b^{z} S(0,0,b^{z}\varepsilon) \xrightarrow{b=\varepsilon^{-1/z}} \varepsilon^{-1} S(0,0,1) \sim \varepsilon^{-1}$
- $S(q_{\|},0,0) = b^{z} S(b^{\zeta} q_{\|},0,0) \xrightarrow{b=q_{\|}^{-1/\zeta}} q_{\|}^{-z/\zeta} S(1,0,0) \sim q_{\|}^{-2/3}$
- $S(0,q_{\perp},0) = b^{z} S(0,b q_{\perp},0) \xrightarrow{b=q_{\perp}^{-1}} q_{\perp}^{-z} S(0,1,0) \sim q_{\perp}^{-2}$

$$S(q_{\|}, q_{\perp}, \varepsilon) = \left(c_1 \varepsilon + c_2 q_{\|}^{2/3} + c_3 q_{\perp}^{2} + \cdots\right)^{-1}$$

Anisotropic correlation function

DeGennes 1976

Result for model A

Summary

For $d \geq d_{\rm up} = 2$, the scaling properties can be obtained by the mean-field approximation

φ4 model

shear

$$\frac{\partial \phi(\boldsymbol{x},t)}{\partial t} + \boxed{\dot{\gamma} y \frac{\partial \phi(\boldsymbol{x},t)}{\partial x}} = -\frac{\delta F[\phi(\boldsymbol{x},t)]}{\delta \phi(\boldsymbol{x},t)} + \sqrt{2T}\xi(\boldsymbol{x},t)$$

Mean-field approximation (linear analysis)

Critical exponents

Anisotropic

$$z = 2, \chi = -\frac{d}{2}, \zeta = 3$$

Static structure factor

$$S(\boldsymbol{q}) \propto \begin{cases} q_\perp^{-z} & \text{parallel} \\ q_\parallel^{-z/\zeta} & \text{perpendicular} \end{cases}$$

Minami, Nakano and Hidaka (2021)

Numerical simulation of Φ4 model in d=2

Finite size scaling for order parameter

$$\phi(\varepsilon, L_x, L_y) = b^{\chi}\phi(b^{z}\varepsilon, b^{-\zeta}L_x, b^{-1}L_y) = L_x^{\chi/\zeta}\phi(L_x^{z/\zeta}\varepsilon, 1, L_x^{-1/\zeta}L_y)$$

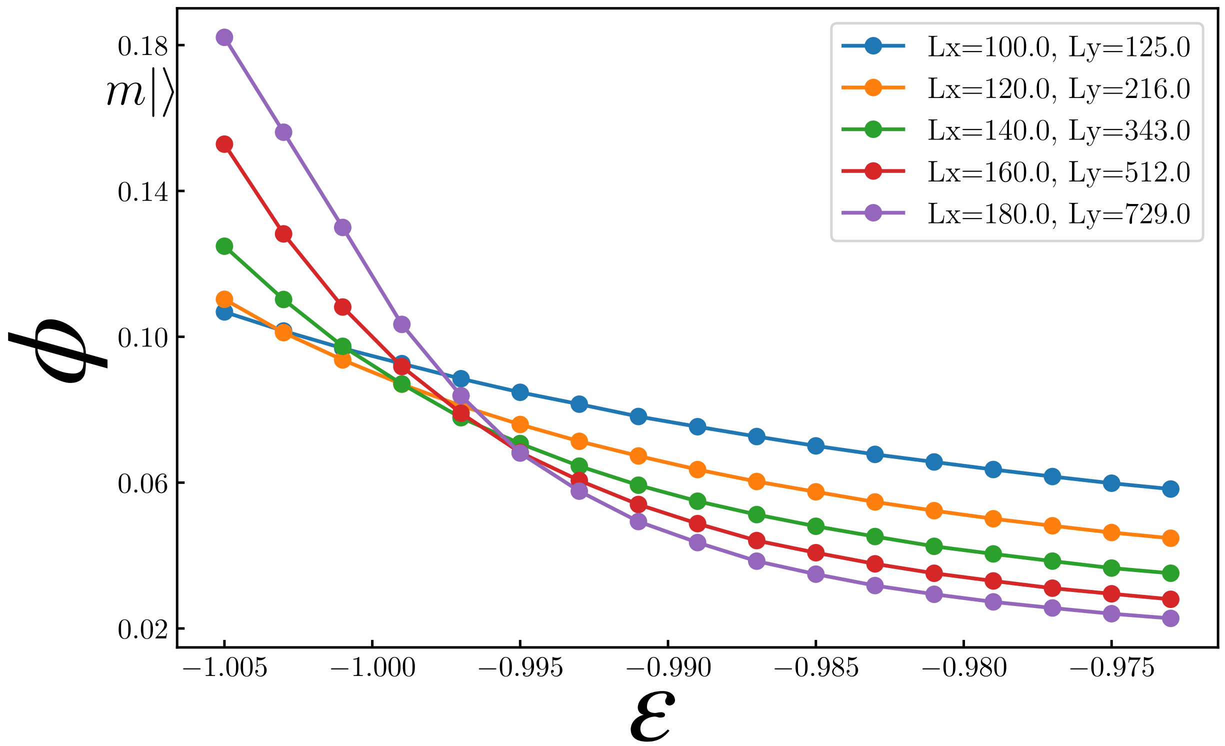
Scaling exponents

$$z = 2,\ \zeta = 3,\ \chi = -1$$

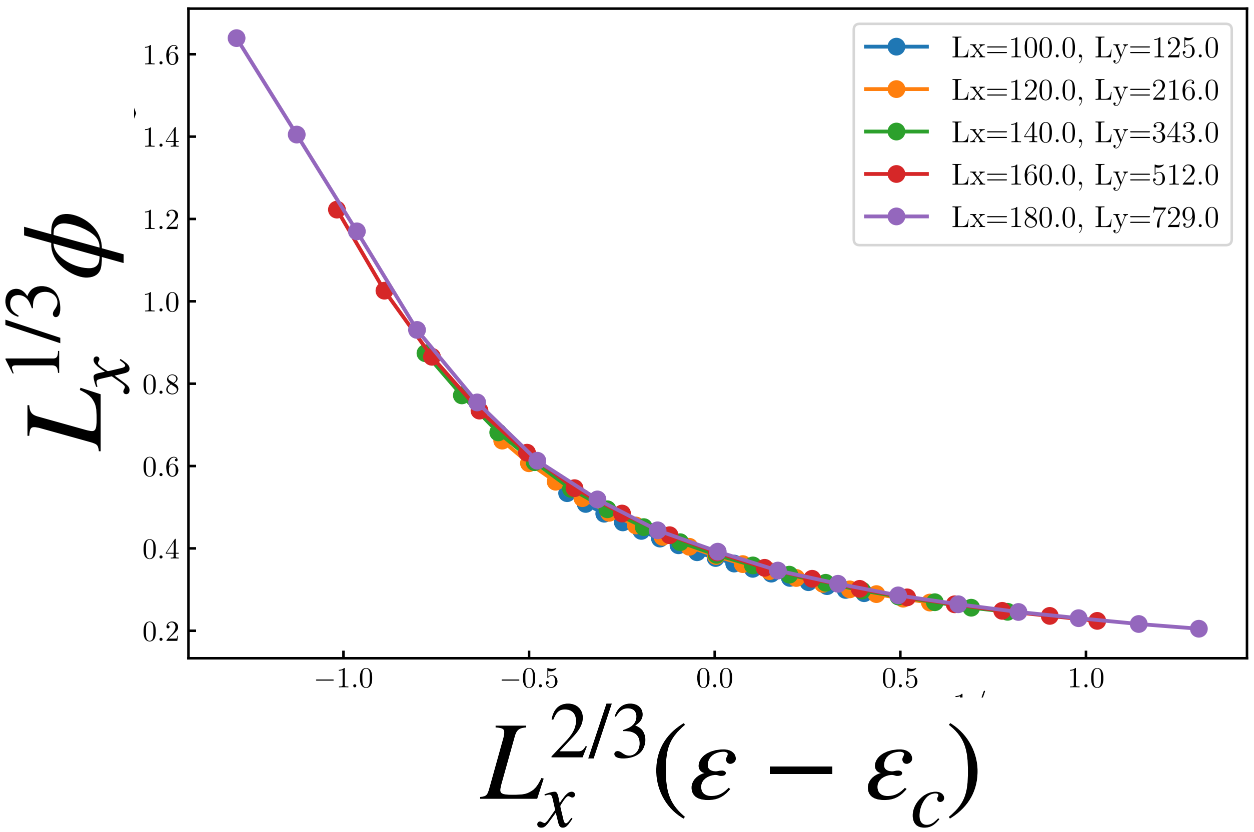
Scaling function

$$\phi(\varepsilon, L_x, L_y) = L_x^{-1/3}\Phi(L_x^{2/3}\varepsilon, L_x^{-1/3}L_y)$$

Order parameter for $\dot{\gamma} = 5.0$



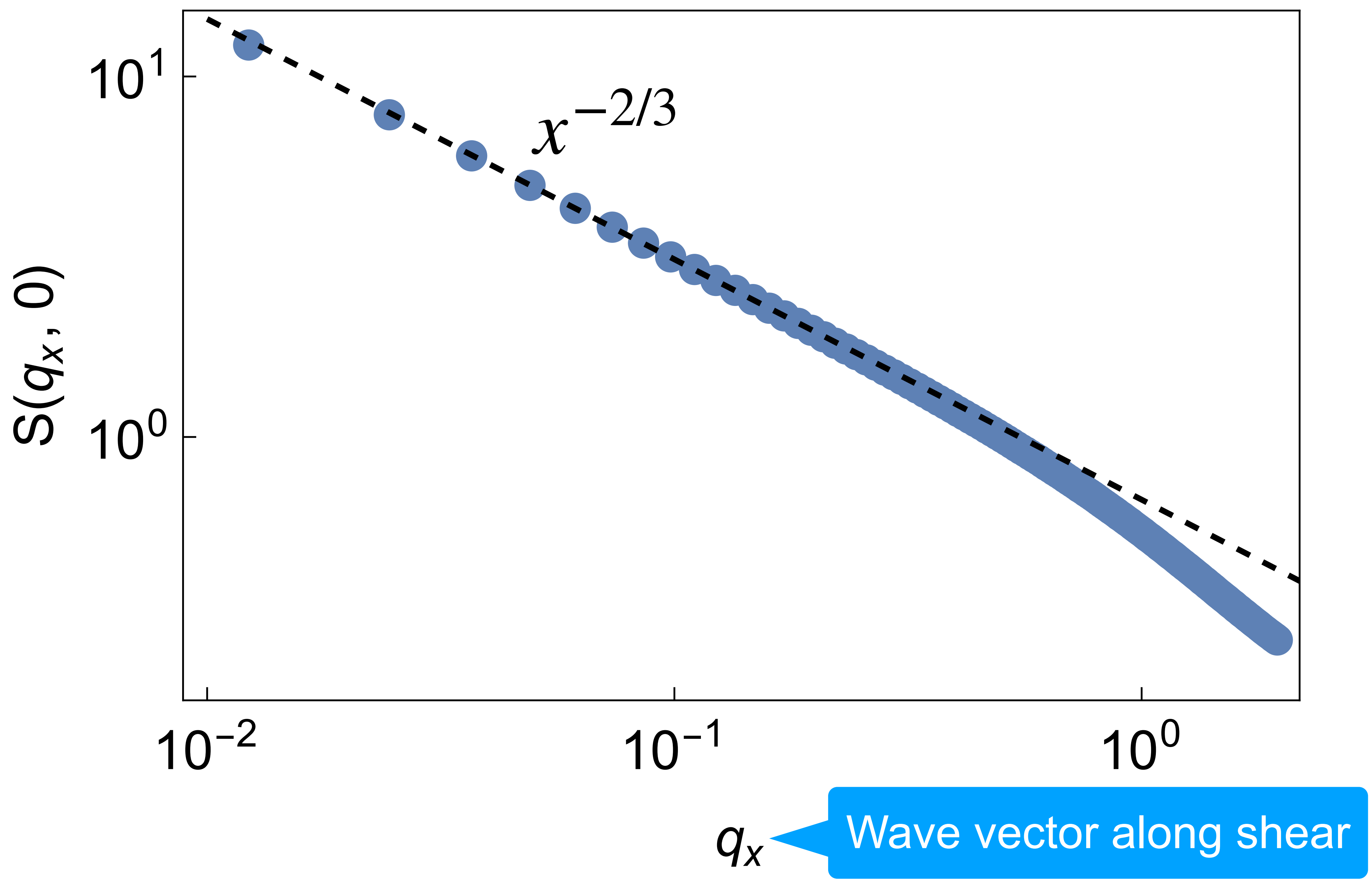
Scaling plot for $\dot{\gamma} = 5.0$



Numerical simulation of Φ4 model in d=2

Static structure factor

Static structure factor at critical point for $\dot{\gamma} = 5.0$



$S(q_x,0) \sim q_x^{-2/3}$ **is indeed observed in numerics.**

Numerical simulation of Φ4 model in d=2

Crossover scaling

Scaling behavior for crossover phenomenon

$$S(q_x, 0) = \dot{\gamma}^{-a} \mathcal{S}(\dot{\gamma}^{-b} q_x)$$

$\dot{\gamma} \to 0$ → **Equilibrium Ising model (2D)**

$$S(q_x, 0) \sim q_x^{-2+\eta}, \ \eta = 1/4$$

$q_x \ll \dot{\gamma}^b$ → **Anisotropic scaling**

$$S(q_x, 0) \sim \dot{\gamma}^{-2/3} q_x^{-2/3}$$

Scaling exponents

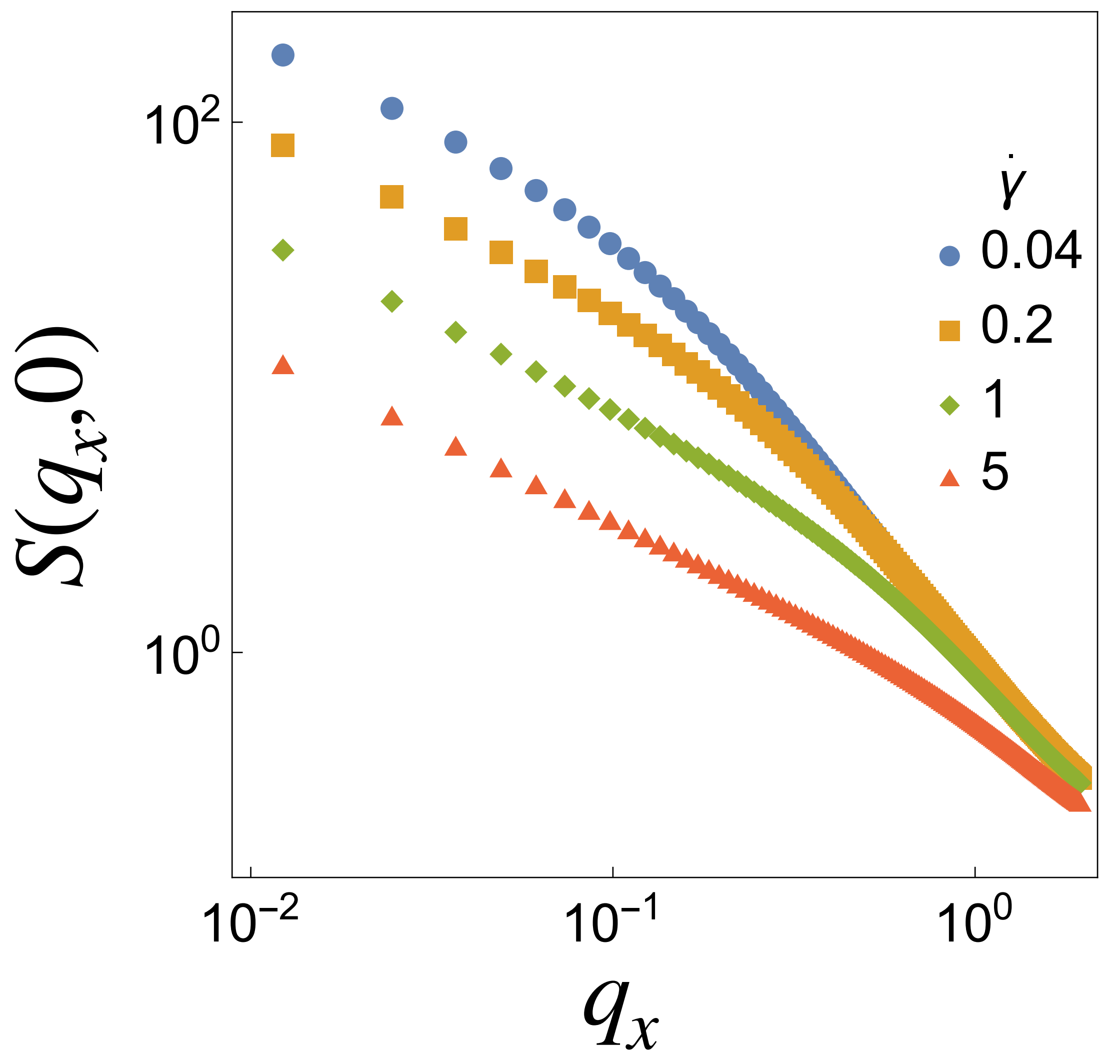
$$a = \frac{14}{13}, \quad b = \frac{8}{13}$$

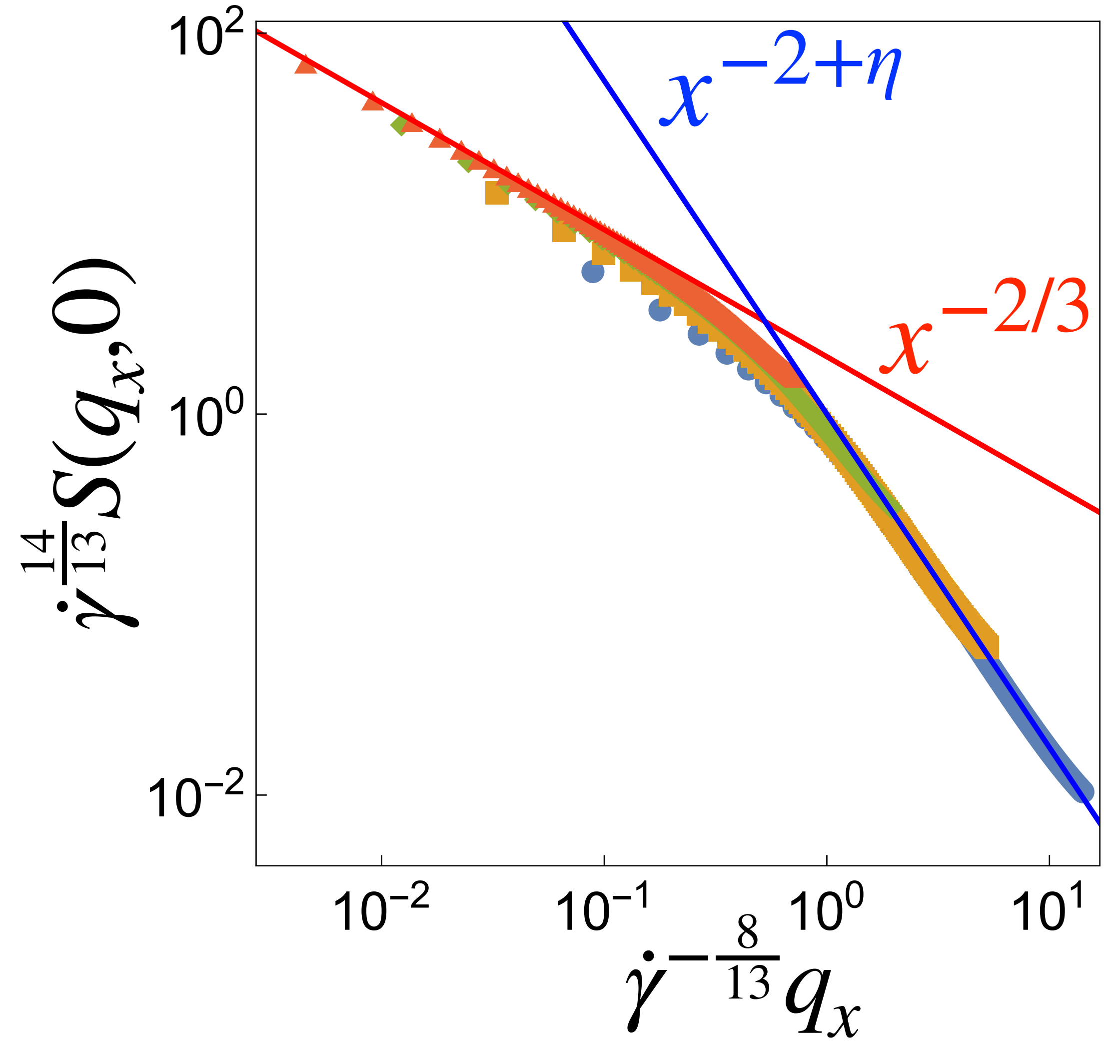
Numerical simulation of Φ4 model in d=2

Crossover scaling

Scaling function

$$S(q_x,0) = \dot{\gamma}^{-\frac{14}{13}} \mathcal{S}(\dot{\gamma}^{-\frac{8}{13}} q_x)$$





The mean-field scaling is always observed for $q_x \ll \dot{\gamma}^{\frac{8}{13}}$

→ **Even infinitesimal shear rate changes the universality class!**

Summary

- The simple shear reduces the upper critical dimension as $d_{\rm up} \to d_{\rm up}^{\rm eq} - z$
- For $d \geq d_{\rm up}$, the scaling behaviors are investigated by the mean-field (linear) theory.
- Our theory for the phi-4 model agrees well with the numerical results
- Even infinitesimally small shear rate can change the universality class

Thank you very much for your attention!!!