Quantum many-body scars and symmetry-associated topological properties

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Outline

- Non-thermalization and algebraic structure
- rSGA-based QMBS and symmetry-invariant scar subspaces
- Symmetry-associated topological properties of AKLT bimagnon scar subspace
- Beyond AKLT: generalities of symmetry-associated topological properties
- Summary

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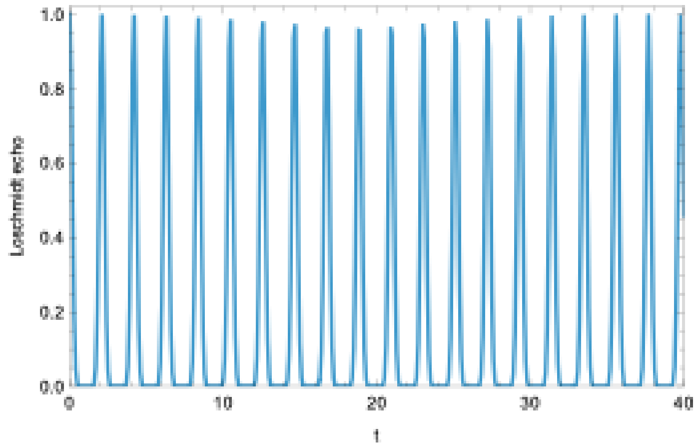
Non-thermalization and conserved quantities

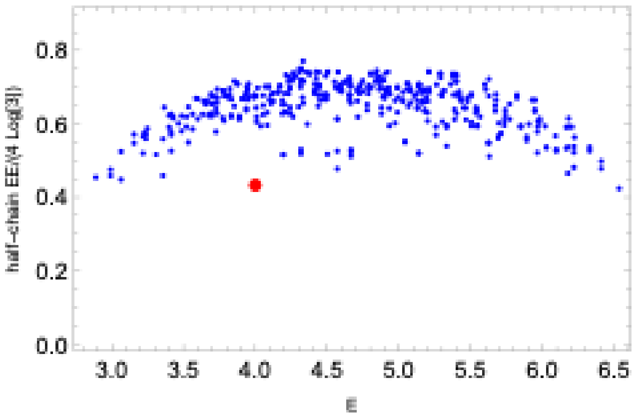
- Thermalization

Any macroscopic variable relaxes to the thermal average given by the Gibbs ensemble .

No persistent oscillations



No exceptional behaviors
such as low entanglement entropy



There are the signatures of non-thermalization.

Non-thermalization and conserved quantities

- Thermalization

  Any macroscopic variable relaxes to the thermal average given by the Gibbs ensemble .

  - Persistent oscillations

    Degenerate energy eigenstates ⇒ Commuting operator with the Hamiltonian (Conserved quantities)

    Equally-spaced energy spectrum ⇒ Spectrum-generating algebra (Dynamical symmetry)

  - Low entanglement entropy

    Matrix product states with a small bond dimension ⇒ Exactly solvable states,
    States inheriting the ground-state properties

Ground-state properties may show up in non-thermal states.

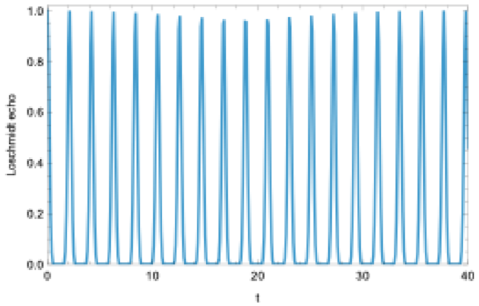
More experimentally realizable than the GS.

Non-thermalization and conserved quantities

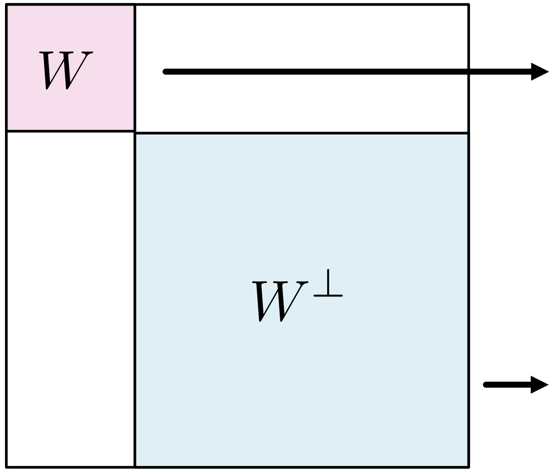
- Partial non-thermalization

A quantum system that thermalizes from a generic initial state but non-thermalizes from a special initial state.

"Quantum many-body scars (QMBS)" [Turner et al. (2017)]

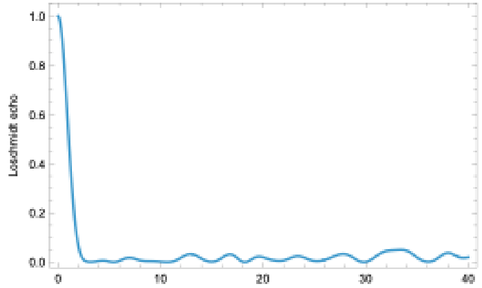


$H =$



A non-thermal subspace.
An initial state prepared in the non-thermal subspace does not thermalize.

(Extra algebraic relation is "embedded".)



A thermal subspace.
An initial state prepared in the thermal subspace thermalizes.

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Embedded spectrum-generating algebra

- Spectrum-generating algebra [Arno et al. (1988), Yang (1989)]

$[H, Q^\dagger] - \mathcal{E}Q^\dagger = 0$ holds in the entire Hilbert space.

⇒ Once an energy eigenstate $|\Psi_0\rangle$ is found,
the other eigenstates are constructed as $Q^\dagger|\Psi_0\rangle, (Q^\dagger)^2|\Psi_0\rangle, \ldots, (Q^\dagger)^N|\Psi_0\rangle$.

⇒ Equally-spaced spectrum, leading to persistent oscillation

$$H(Q^\dagger)^n|\Psi_0\rangle = (E_0 + n\mathcal{E})(Q^\dagger)^n|\Psi_0\rangle$$

- Restricted spectrum-generating algebra (rSGA) [Moudgalya et al. (2018)]

$[H, Q^\dagger] - \mathcal{E}Q^\dagger\big|_W = 0$ holds in the subspace W of the Hilbert space. $W = \mathrm{span}\,\{Q^n|\psi_0\rangle\}_n$

⇒ Once an energy eigenstate $|\Psi_0\rangle$ is found,
the other eigenstates are constructed as $Q^\dagger|\Psi_0\rangle, (Q^\dagger)^2|\Psi_0\rangle, \ldots, (Q^\dagger)^N|\Psi_0\rangle$.

⇒ Equally-spaced spectrum is embedded in the full energy spectrum.

Embedded spectrum-generating algebra

- Example: AKLT model [Affleck et al. (1987), Moudgalya et al. (2018)]

$$H = \sum_{j=1}^{L} \mathbf{1} \otimes \cdots \otimes \underset{j,j+1}{h} \otimes \cdots \otimes \mathbf{1} \in \mathrm{End}((\mathbb{C}^3)^L), \quad \mathbb{C}^3 = \mathrm{span}\{|0\rangle, |1\rangle, |2\rangle\}$$

$$h_{j,j+1} = P^{(2)}_{j,j+1} = \vec{S}_j \cdot \vec{S}_{j+1} + \tfrac{1}{3}(\vec{S}_j \cdot \vec{S}_{j+1})^2$$: Non-integrable spin chain

  - MPS ground state [Affleck et al. (1987)]

$$H|\Psi_0\rangle = 0, \ |\Psi_0\rangle = \mathrm{tr}_a(\vec{A} \otimes_p \vec{A} \otimes_p \cdots \otimes_p \vec{A})$$

$$= \textstyle\sum_{\{m_1,\ldots,m_N\}\in\{0,1,2\}^N} \mathrm{tr}_a(A_{m_1} A_{m_2} \cdots A_{m_N}) \, |m_1, m_2, \ldots, m_N\rangle$$

$$A_0 = \sqrt{\tfrac{2}{3}}\sigma_a^+, \ A_1 = -\sqrt{\tfrac{1}{3}}\sigma_a^z, \ A_2 = -\sqrt{\tfrac{2}{3}}\sigma_a^-$$: MPS with the 2-dim bond space

$$h_{j,j+1}\vec{A} \otimes_p \vec{A} = 0$$: Frustration-free condition

Embedded spectrum-generating algebra

- Example: AKLT model [Affleck et al. (1987), Moudgalya et al. (2018)]

$$H = \sum_{j=1}^{L} \mathbf{1} \otimes \cdots \otimes \underset{j,j+1}{h} \otimes \cdots \otimes \mathbf{1} \ \in \mathrm{End}((\mathbb{C}^3)^L), \quad \mathbb{C}^3 = \mathrm{span}\{|0\rangle, |1\rangle, |2\rangle\}$$

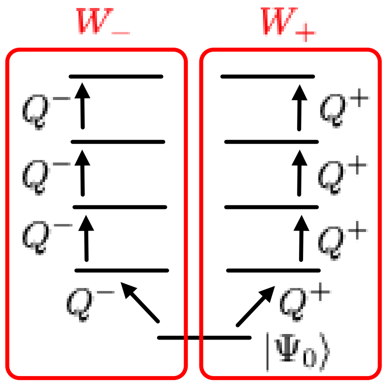
$$h_{j,j+1} = P^{(2)}_{j,j+1} = \vec{S}_j \cdot \vec{S}_{j+1} + \tfrac{1}{3}(\vec{S}_j \cdot \vec{S}_{j+1})^2$$

  - Exact quasiparticle excitations [Moudgalya et al. (2018)]

Each QMBS does not respect the H's symmetry.

$$H(Q^{\pm})^n|\Psi_0\rangle = n\mathcal{E}(Q^{\pm})^n|\Psi_0\rangle, \quad Q^{\pm} = \textstyle\sum_{j=1}^{L}(-1)^j(S_j^{\pm})^2$$

$$= \textstyle\sum_{j_1,\ldots,j_n}(-1)^{\sum_{k=1}^{n} j_k}\mathrm{tr}_a(\vec{A} \otimes_p \cdots \otimes_p \underset{j_1}{\vec{B}} \otimes_p \cdots \otimes_p \underset{j_n}{\vec{B}} \otimes_p \cdots \otimes_p \vec{A})$$



$(h_{j,j+1} - \frac{\mathcal{E}}{2})(\vec{B} \otimes_p \vec{A} - \vec{A} \otimes_p \vec{B}) = 0$: Local eigenstate condition

$\vec{B} \otimes_p \vec{B} = 0, \quad (S^{\pm})^2\vec{B} = (S^{\pm})^4\vec{A} = 0$: Double / adjacent occupations forbidden in W.

$W_{\pm} = \mathrm{span}\{|\Psi_n\rangle = (Q^{\pm})^n|\Psi_0\rangle\}$ are two solvable subspaces with $[H, Q^{\pm}] - \mathcal{E}Q^{\pm}|_{W_{\pm}} = 0$.

Symmetry-invariant scar subspace

- Symmetry-invariant scar subspace [CM-Quella-Tsuji (2025)]

  $[H, U_g] = 0, \quad \forall g \in G$: G-symmetric Hamiltonian

  $HW \subset W \Rightarrow HW_g \subset W_g = U_g W U_g^{-1}$: g-transformed subspace W_g is also invariant under H!

  The composite scar subspace $\mathrm{span} \bigcup_g W_g$ is G-symmetric, inheriting the same (on-site) symmetry of H.

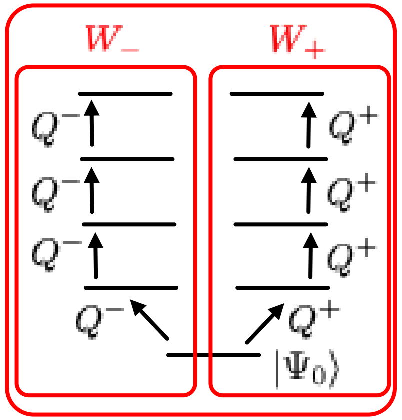
  - AKLT model : Inversion, time-reversal, $Z_2 \times Z_2$-symmetric [Affleck et al. (1987)] $\Rightarrow \mathrm{span}\, W_+ \cup W_-$ is symmetric under inversion, time-reversal, & $Z_2 \times Z_2$

    $U_{\mathcal{P}} |\Psi_n^{\pm}\rangle = |\Psi_n^{\pm}\rangle$: Inversion symmetric

    $U_{\mathcal{T}} |\Psi_n^{\pm}\rangle = (-1)^L |\Psi_n^{\mp}\rangle$: Time-reversal exchanges the species

    $(e^{i\pi S^z})^{\otimes L} |\Psi_n^{\pm}\rangle = |\Psi_n^{\pm}\rangle$: Invariant under the π-rotation w.r.t. the z-axis.

    $(e^{i\pi S^x})^{\otimes L} |\Psi_n^{\pm}\rangle = (-1)^L |\Psi_n^{\mp}\rangle$ The π-rotation w.r.t. the x-axis exchanges the species.



$\Rightarrow$ Can the composite scar subspace serve as the platform for SPT properties?

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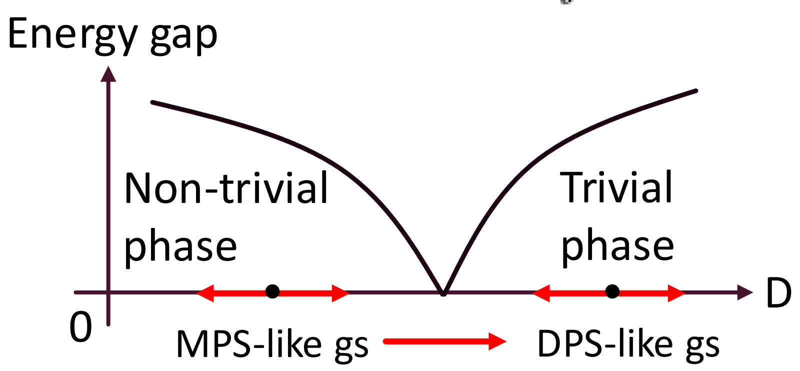
Symmetry-associated topological properties

- Symmetry protected topological (SPT) phase
  - A quantum phase that is defined for short-range entangled states with an (on-site) symmetry at zero temperature. i.e. the ground state
  - No local order parameter.

    ⇒ Characterized by hidden long-rage order (String order), Topological response, Projective rep. on the bond space, Robustness under perturbations, and etc.

cf.) Spin-1 model $H = J\sum_j(\vec{S}_j \cdot \vec{S}_{j+1} + D(S_j^z)^2)$ [Haldane (1983), Pollmann et al. (2010)]



Inversion, Time-reversal, & $Z_2 \times Z_2$-symmetric

The phases are protected by symmetries + the energy gap.

Different phases cannot be adiabatically connected without closing the energy gap or violating the on-site symmetry!

[Gu-Wen (2009), Pollmann et al. (2010), Chen et al. (2013)]

Symmetry-associated topological properties

- MPS GS vs DPS GS
  - MPS GS (spin-1 model)

$|\Psi_0\rangle = \mathrm{tr}_a(\vec{A} \otimes_p \cdots \otimes_p \vec{A}), \ \vec{A} = (A_0 \ A_1 \ A_2) = (\sigma^+ \ \sigma^z \ -\sigma^-)$: 2-dimensional bond

G ($Z_2 \times Z_2$)-invariant (abelian group)

On-site symmetry + Injectivity

$$\Rightarrow u_g \begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix} = e^{i\theta_g} \begin{pmatrix} V_g A_0 V_g^\dagger \\ V_g A_1 V_g^\dagger \\ V_g A_2 V_g^\dagger \end{pmatrix}$$

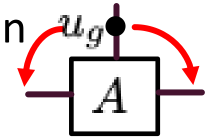
$g, h \in G \Rightarrow gh = hg$: Abelian group symmetry

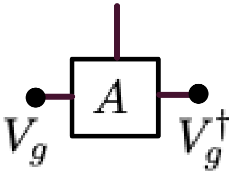
$u_g u_h = u_h u_g$: Unitary (linear) rep.

$V_g V_h = (-1) V_h V_g$ $(g \neq h)$: Projective rep. (Emergent nontrivial phase which cannot be removed by Gauge transformation.)

[Pollmann et al. (2010), Chen et al. (2012)]

Phys. action





Bond action

⇒ Symmetry appears on the bond space!

  - DPS GS

$|\Psi_0\rangle = |11\ldots1\rangle$: 1-dimensional bond. G-invariant (G: abelian group)

$V_g \propto \mathbf{1}, \forall g \in G \Rightarrow V_g V_h = V_h V_g \ (g, h \in G)$: Linear (genuine) rep.

Topological properties appear through the symmetry rep. in the bond space.

Symmetry-associated topological properties

- Topological properties of the GS
  - (On-site) symmetry invariance is required.
  - Appears as the symmetry rep. in the bond space of MPS.
- Topological properties of the scar subspace
  - (On-site) symmetry invariance is required.
  - May appear through the symmetry rep. in the bond space of MPS.

The scar subspace inherits the symmetry of the Hamiltonian, but each QMBS is not symmetry invariant.

Are the topological properties well-defined for the scar subspace??

Topological properties of the AKLT GS

- AKLT model [Affleck-Kennedy-Lieb-Tasaki (1987)]

$$H = J\sum_j \left(\vec{S}_j\cdot\vec{S}_{j+1} + \frac{1}{3}(\vec{S}_j\cdot\vec{S}_{j+1})^2\right)$$: Inversion, time-reversal, & $Z_2\times Z_2$-symmetric

  - Matrix product ground state

$$|\Psi_0\rangle = \mathrm{tr}_a(\vec{A}\otimes_p\cdots\otimes_p\vec{A}),\ \ A_0=\sqrt{\tfrac{2}{3}}\sigma^+,\ A_1=-\tfrac{1}{\sqrt{3}}\sigma^z,\ A_2=-\sqrt{\tfrac{2}{3}}\sigma^-$$: 2-dimensional bond

Inversion, time-reversal, and $Z_2\times Z_2$-symmetric, injective

⇒ Symmetry appears on the bond space.

$Z_2\times Z_2$ $\{\mathbf{1}, g_x, g_z, g_xg_z\},\ g_xg_z=g_zg_x$

$$(u_{g_z},u_{g_x})=(e^{i\pi S^z}e^{i\pi S^x})\Rightarrow(V_{g_z},V_{g_x})=(\sigma^z,\sigma^x),\ \ V_{g_x}V_{g_z}=\boxed{(-1)}V_{g_z}V_{g_x}$$

Inversion $\mathcal{P},\ \mathcal{P}^2=\mathbf{1}$

$$\mathcal{P}(\vec{A})=-{}^{t_a}\vec{A}\Rightarrow V_{\mathcal{P}}=i\sigma^y,\ \ V_{\mathcal{P}}V_{\mathcal{P}}^*=\boxed{(-1)}\mathbf{1}$$

Time-reversal $\mathcal{T},\ \mathcal{T}^2=\mathbf{1}$

$$\mathcal{T}(\vec{A})=u_{g_y}\vec{A}^*\Rightarrow V_{\mathcal{T}}=i\sigma^y,\ \ V_{\mathcal{T}}V_{\mathcal{T}}^*=\boxed{(-1)}\mathbf{1}$$

The AKLT GS carries nontrivial topology under inversion, time-reversal, & $Z_2\times Z_2$-sym.

[Pollmann et al. (2010), Chen et al. (2012)]

Topological properties of the AKLT QMBS

- Bimagnon excitations [Moudgalya (2018)]

$|\Psi_n^{\pm}\rangle = (Q^{\pm})^n|\Psi_0\rangle, \quad |\Psi_0\rangle = \mathrm{tr}_a(\vec{A}\otimes_p \cdots \otimes_p \vec{A})$: MPS GS

Symmetry-invariant subspace
+ MPS expression of QMBS

⇒ Symmetry-associated topological properties through bond rep.?

- MPS form of bimagnon excitations

$$Q^{\pm} = \sum_{j=1}^{L}(-1)^j(S_j^{\pm})^2 = \underbrace{(-\mathbf{1})\times\cdots\times(-\mathbf{1})}_{j-1}\times(-S^{\pm})^2\times\underbrace{\mathbf{1}\times\cdots\times\mathbf{1}}_{L-j}$$

$$= {}_b\langle\uparrow|M^{\pm}\otimes_p\cdots\otimes_p M^{\pm}|\downarrow\rangle_b,\ M^{\pm} = \begin{pmatrix} -\mathbf{1} & -(S^{\pm})^2 \\ 0 & \mathbf{1}\end{pmatrix}_b$$: MPO creation operator with 2-dim bond

$$(Q^{\pm})^n = {}_{\boldsymbol{b}}\langle\Uparrow|\mathrm{M}^{\pm}\otimes_p\cdots\otimes_p\mathrm{M}^{\pm}|\Downarrow\rangle_{\boldsymbol{b}},\ \mathrm{M}^{\pm} = \otimes_b^n M^{\pm}, \quad |\Uparrow\rangle_{\boldsymbol{b}} = |\underbrace{\uparrow\cdots\uparrow}_{n}\rangle_b, \quad |\Downarrow\rangle_{\boldsymbol{b}} = |\underbrace{\downarrow\cdots\downarrow}_{n}\rangle_b$$

$$\Rightarrow |\Psi_n^{\pm}\rangle = {}_{\boldsymbol{b}}\langle\Uparrow|\mathrm{tr}_a(\underline{\vec{A}\mathrm{M}^{\pm}}\otimes_p\cdots\otimes_p\underline{\vec{A}\mathrm{M}^{\pm}})|\Downarrow\rangle_{\boldsymbol{b}}$$

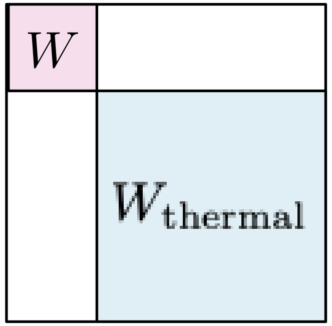
An n-bimagnon state is written as the MPS with 2^{n+1}-dimensional bond.

Topological properties of the AKLT QMBS

- Bond representations of symmetries [CM-Quella-Tsuji (2025); cf. Moudgalya (2018)]

Inversion, time-reversal, Z2×Z2-sym.

$\mathrm{span}\{|\Psi_n^+\rangle, |\Psi_n^-\rangle\} =$



The scar subspace $W = \mathrm{span}\, W_+ \cup W_-$ is inversion, time-reversal, & $Z_2 \times Z_2$-sym.

Consisting of two-species QMBS $|\Psi_n^\pm\rangle = {}_b\langle \Uparrow | \mathrm{tr}_a(\vec{A}M^\pm \otimes_p \cdots \otimes_p \vec{A}M^\pm) | \Downarrow \rangle_b$

Each QMBS is not invariant!

Do bimagnon creation MPO of different species belong to the same cohomological class?

⇒ Yes! [CM-Quella-Tsuji (2025)]

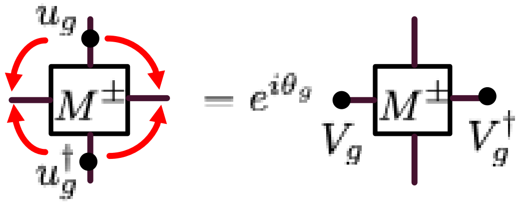
$u_{\mathcal{P}} M^\pm u_{\mathcal{P}}^\dagger = {}^{t_b}(M^\pm) = V'_{\mathcal{P}} M^\pm V'^\dagger_{\mathcal{P}} \Rightarrow V'_{\mathcal{P}} = i\sigma^y$: Inversion

$u_{\mathcal{T}} M^\pm u_{\mathcal{T}}^\dagger = e^{i\pi S^y} M^\pm e^{-i\pi S^y} = V'_{\mathcal{T}} M^\mp V'^\dagger_{\mathcal{T}} \Rightarrow V'_{\mathcal{T}} = \mathbf{1}$: Time-reversal

$e^{i\pi S^z} M^\pm e^{-i\pi S^z} = V'_{g_z} M^\pm V'^\dagger_{g_z} \Rightarrow V'_{g_z} = \mathbf{1}$

$e^{i\pi S^x} M^\pm e^{-i\pi S^x} = V'_{g_x} M^\mp V'^\dagger_{g_x} \Rightarrow V'_{g_x} = \mathbf{1}$

: $Z_2 \times Z_2$



The two species carry the same cohomological class!

⇒ Symmetry-associated topological properties are well-defined for a composite scar subspace.

Topological properties of the AKLT QMBS

- Topological properties of the bimagnon QMBS [CM-Quella-Tsuji (2025)]

The topological properties of bimagnon QMBS are determined by the combination of the topological properties of the GS MPS & bimagnon MPO. [CM-Quella-Tsuji (2025); cf. Moudgalya et al. (2018)]

Ex.) $Z_2 \times Z_2$ symmetry

Ground state MPS $V_{g_x} V_{g_z} = (-1) V_{g_z} V_{g_x}$

Projective rep. ⇒ Nontrivial topological properties

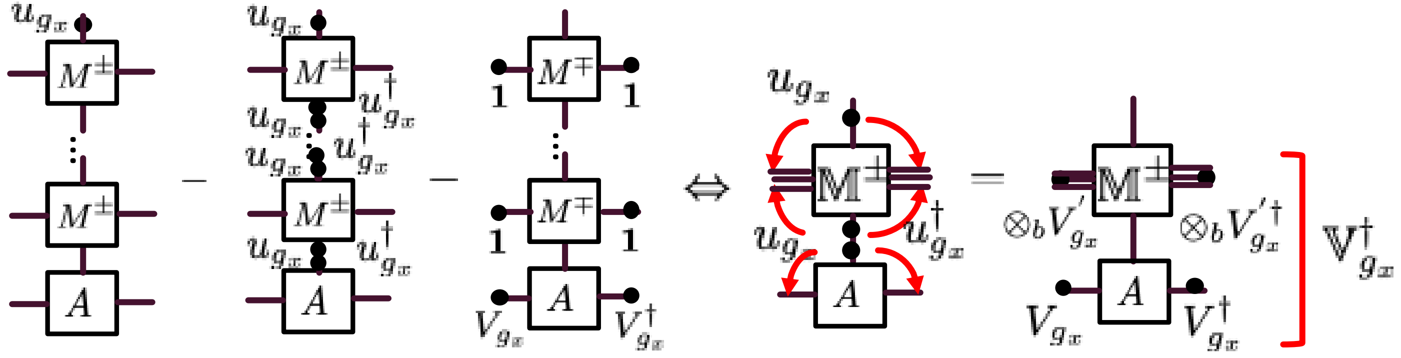
Bimagnon MPS

$$e^{i\pi S^z} M^{\pm} e^{-i\pi S^z} = M^{\pm} \Rightarrow V'_{g_z} = 1$$

$$e^{i\pi S^x} M^{\pm} e^{-i\pi S^x} = M^{\mp} \Rightarrow V'_{g_x} = 1$$

$$\Rightarrow \mathbb{V}_{g_x} \mathbb{V}_{g_z} = - \mathbb{V}_{g_z} \mathbb{V}_{g_x}$$

Projective rep.



Bimagnon MPS directly inherits the GS topological properties associated with $Z_2 \times Z_2$ sym.

Topological properties of the AKLT QMBS

- Topological properties of the bimagnon QMBS [CM-Quella-Tsuji (2025)]

The topological properties of bimagnon QMBS are determined by the combination of the topological properties of the GS MPS & bimagnon MPO. [CM-Quella-Tsuji (2025); cf. Moudgalya et al. (2018)]

Ex.) Inversion symmetry

Ground state MPS $V_{\mathcal{P}} V_{\mathcal{P}}^* = (-1) \mathbf{1}$

Projective rep. ⇒ Nontrivial topological properties

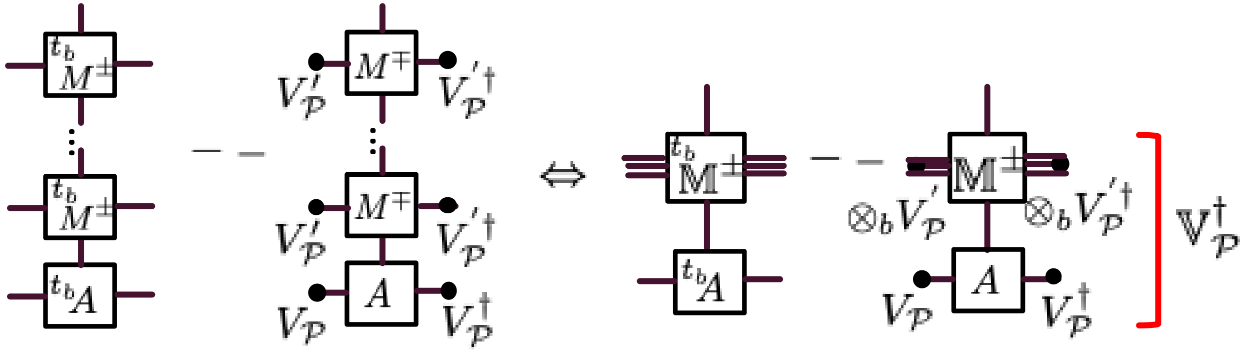
Bimagnon MPS

$\mathcal{P}(M^{\pm}) = -{}^{t_b}M^{\pm} \Rightarrow V'_{\mathcal{P}} = i\sigma^y$

$\Rightarrow \mathbb{V}_{\mathcal{P}} \mathbb{V}_{\mathcal{P}}^* = (-1)^{n+1} \mathbf{1}$

Projective rep. for even n

Linear rep. for odd n



Bimagnon MPS systematically modifies the GS topological properties associated with inversion sym.

Topological properties of the AKLT QMBS

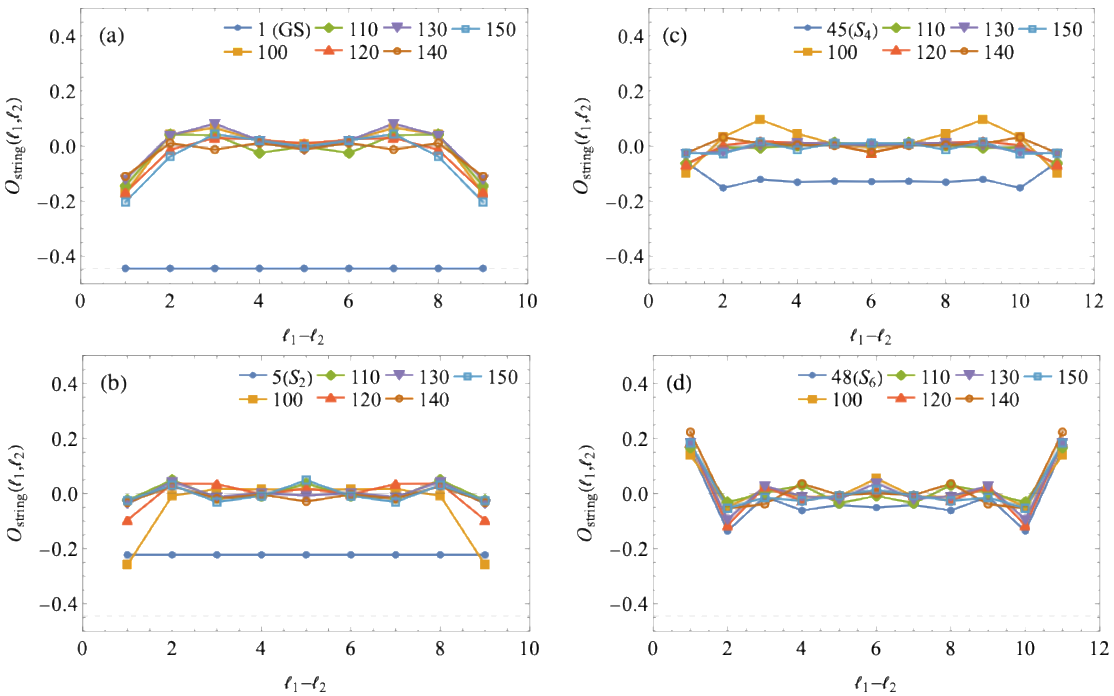
- Non-vanishing string order parameter [CM-Quella-Tsuji (2025)]

$$O_{\text{string}}^{(n,\pm)}(\ell_1,\ell_2) = \langle\Psi_n^{\pm}|S_{\ell_1}^z \cdot \prod_{j=\ell_1+1}^{\ell_2-1} e^{i\pi S_j^z} \cdot S_{\ell_2}^z|\Psi_n^{\pm}\rangle / \langle\Psi_n^{\pm}|\Psi_n^{\pm}\rangle = -\frac{4}{9} + O(L^{-1})$$

Large L & large distance analysis through the MPS expression

⇒ Nonzero constant value in the dilute regime!

$(n/L \to 0)$



(a) L=10, S^z_{tot}=0 (b) L=10, S^z_{tot}=2
(c) L=12, S^z_{tot}=4 (d) L=12, S^z_{tot}=6

The finite-size effect is very large for QMBS (S_2, S_4, S_6), but QMBS behave differently compared to the other states (a signature of finite values).

Long-range string order survives in the scar states.

cf.) Asymptotic calculation tells that the normal S^z-S^z correlation decays exponentially for small n.

Topological properties of the AKLT QMBS

- The two-species bimagnon excitations for the AKLT model
  - form the symmetry-invariant scar subspace.
  - always belong to the same cohomological class.
  - provide the platform for defining symmetry-associated topological properties.
  - exhibit non-zero string order parameter, which coincides with the GS value in the dilute regime.
- Is emergence of topological properties generically observed for scar subspaces of the other models?

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Beyond AKLT

- Deformed AKLT model [Mark et al. (2020)]

$H_{\text{pert}} = \sum_{j=1}^{L} h_{j,j+1}$

$$\begin{aligned} h_{j,j+1} = \; & \mathcal{E}(|00\rangle\langle 00| + |22\rangle\langle 22|) \\ & + \tfrac{\mathcal{E}}{2}\textstyle\sum_{a=0,2}(|a1\rangle\langle a1| + |1a\rangle\langle 1a| + |a1\rangle\langle 1a| + |1a\rangle\langle a1|) \\ & + \alpha(|02\rangle\langle 02| + |20\rangle\langle 20| + |02\rangle\langle 20| + |20\rangle\langle 02|) \\ & + \alpha\gamma(|02\rangle\langle 11| + |11\rangle\langle 02| + |20\rangle\langle 11| + |11\rangle\langle 20|) \\ & + \alpha\gamma^2|11\rangle\langle 11| \end{aligned}$$

Determine spacing of energy spectrum (first two lines)

Determine topological properties (last three lines)

$$|\Phi_0\rangle = \sum_{m_1,\ldots,m_L} \operatorname{tr}(A_{m_1}A_{m_2}\cdots A_{m_L})|m_1 m_2 \ldots m_L\rangle$$

$$A_0 = \frac{\gamma}{\sqrt{1+\gamma^2}}\sigma_a^+, \quad A_1 = -\frac{1}{\sqrt{1+\gamma^2}}\sigma_a^z, \quad A_2 = -\frac{\gamma}{\sqrt{1+\gamma^2}}\sigma_a^- \qquad (\gamma \neq 0)$$

Bimagnon scar subspace [Mark et al. (2020)]

$$|\Phi_n^\pm\rangle = (Q^\pm)^n|\Phi_0\rangle, \quad Q^\pm = \sum_{j=1}^{L}(-1)^j(S_j^\pm)^2$$

$$\Rightarrow [H_{\text{pert}}, Q^\pm] - \mathcal{E}Q^\pm\big|_W = 0, \quad W = \operatorname{span}\{|\Phi_n^+\rangle, |\Phi_n^-\rangle\}$$: Inversion, time-reversal, & Z2×Z2-sym.

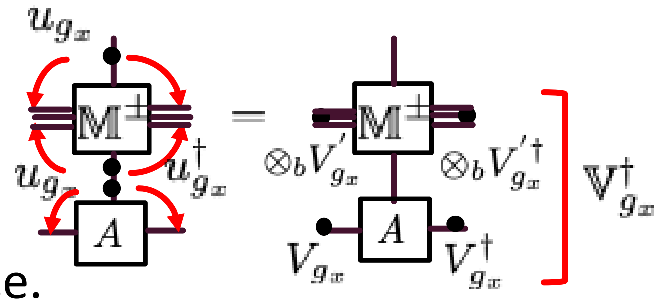
Beyond AKLT

- Deformed AKLT model [Mark et al. (2020)]
  - Symmetry representation in the bond space [CM-Quella-Tsuji (2025)]

    $V_{\mathcal{P}} = i\sigma^y$: Inversion

    $V_{\mathcal{T}} = i\sigma^y$: Time-reversal

    $V_{g_z} = \sigma^z, V_{g_x} = \sigma^x$: $Z_2 \times Z_2$



    Without modifying the symmetry action in the MPO bond space.

    ⇒ The scar subspace W shows the same topology as the AKLT case!

  - String order parameter [CM-Quella-Tsuji (2025)]

$$\widetilde{O}^{(N,\pm)}_{\text{string}}(\ell_1, \ell_2; S^z, g_z; \gamma) = \langle \Phi_N^\pm | S^z_{\ell_1} \cdot \prod_{\ell=\ell_1+1}^{\ell_2-1} R^z_\ell \cdot S^z_{\ell_2} | \Phi_N^\pm \rangle / \langle \Phi_N^\pm | \Phi_N^\pm \rangle$$

$$= -\left(\frac{\gamma^2}{1+\gamma^2}\right)^2 + O(L^{-1})$$

: Takes a fixed (non-zero) value in the dilute regime.

Beyond AKLT

- Higher-spin AKLT model [Moudgalya et al. (2018)]

$$H = \sum_{j=1}^{L} \sum_{J=s+1}^{2s} \alpha_J P_{j,j+1}^{(J,s)}, \quad \alpha_J > 0, \quad P_{j,j+1}^{(J,s)}$$

: The projector onto the total-spin-J sector of two neighboring spin-s sites

Scar subspace [Moudgalya et al. (2018)]

$$|\Phi_0\rangle = \sum_{\substack{m_1,\ldots,m_L \\ \in\{0,1,\ldots,2s\}}} \mathrm{tr}(A_{m_1} \cdots A_{m_L}) |m_1 \cdots m_L\rangle$$

: MPS with (s+1)-dim bond sp

$$|\Phi_n^{\pm}\rangle = (Q^{\pm})^n |\Phi_0\rangle, \; Q^{\pm} = \sum_j (-1)^j (S_j^{\pm})^{2s} \Rightarrow M_{\pm} = \begin{pmatrix} -1 & -(S^{\pm})^{2s} \\ 0 & 1 \end{pmatrix}_b$$

$W = \mathrm{span}\{|\Phi_n^{+}\rangle, |\Phi_n^{-}\rangle\}$

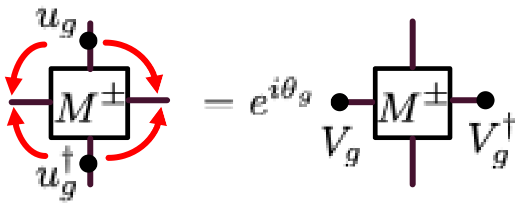
: Inversion, time-reversal, & Z2×Z2-sym.

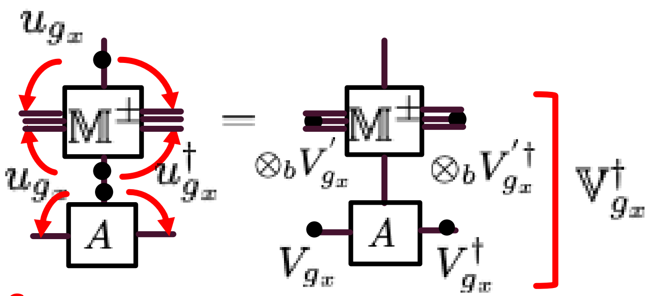
Symmetry rep. in the bond space [CM-Quella-Tsuji (2025)]

$V'_{\mathcal{P}} = i\sigma^y$: Inversion

$V'_{\mathcal{T}} = 1$: Time-reversal

$V'_{g_x} = V'_{g_z} = 1$: $Z_2 \times Z_2$





: Spin-s/2 projective rep. for odd s
Spin-s/2 linear rep. for even s

Bimagnon MPS directly inherits or systematically modifies the GS topological properties associated with inversion sym.

Summary

- We showed that symmetry-associated topological properties are well-defined for scar subspaces constructed on an SPT ground state.
  - A composite scar subspace inherits the (on-site) symmetries of the Hamiltonian.
  - QMBS with different species belong to the same cohomological class if they are connected by a symmetry transformation.
  - The scar subspace inherits or systematically modifies the symmetry-associated topological properties of the SPT ground state.
  - As a physical signature, the string order parameter takes a nonzero value that remains constant in the dilute regime.
- Can such SPT-like scar subspaces undergo phase transitions?
  - How robust are symmetry-associated topological properties under perturbations?
  - Are these topological properties well-defined for scar subspaces constructed on an unsolvable reference ground state? [cf. CM (2026)]