Quantum signatures of chaos from free probability[1]

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[1] Based on 2503.20338 and 2506.04520 in collaboration with Viktor Jahnke (ITP, Brazil), Pratik Nandy (VUB, Belgium) and Yichao Fu, Kuntal Pal, & Keun-Young Kim (GIST, Korea) .

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Classical and Quantum Signatures of Chaos

- We have an intuitive understanding of what "chaotic" means in our day-to-day life.

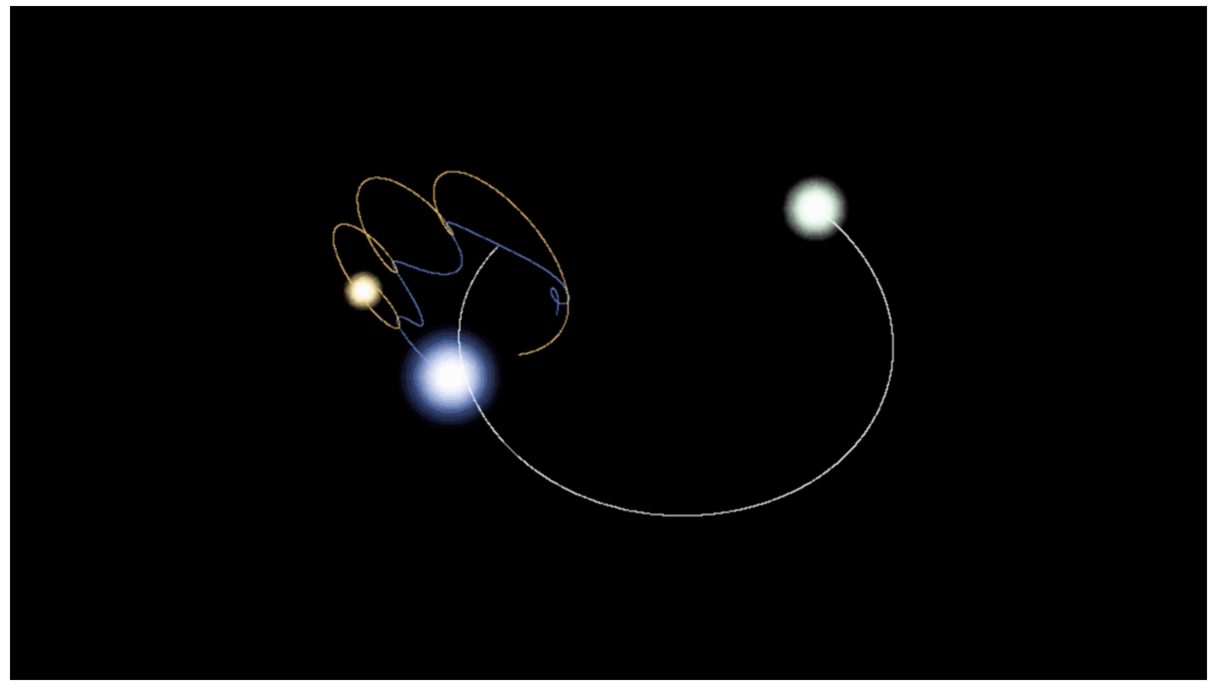
"Disordered"

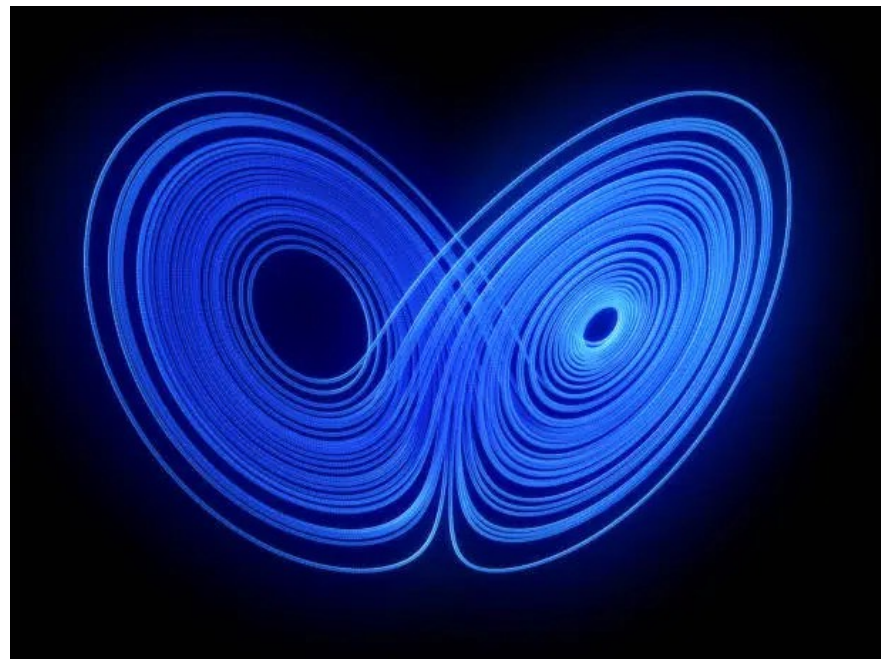
"Random"

"Unpredictable"

- In the context of **classical** dynamical systems, chaos (mixing) has been studied using classical ergodic theory

"Sensitivity to initial conditions" (H. Poincaré, 19th C.) → "The butterfly effect" [E. Lorenz, (1960's)]



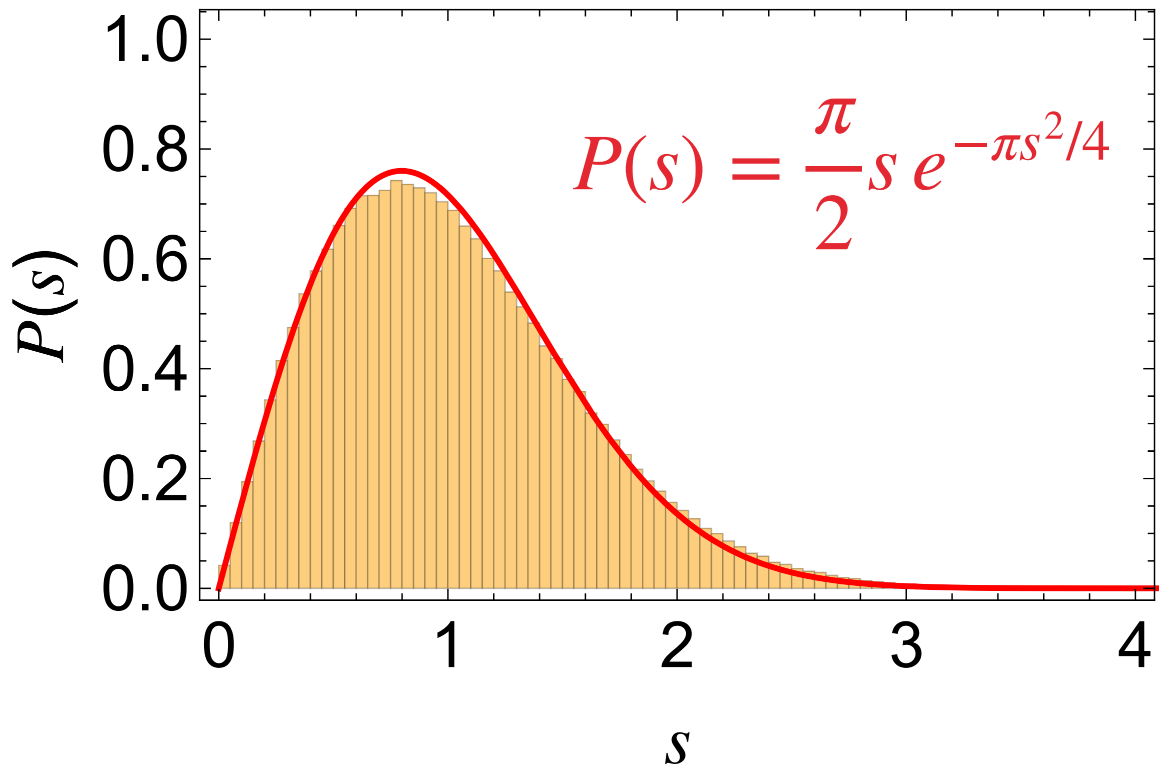


Classical and Quantum Signatures of Chaos

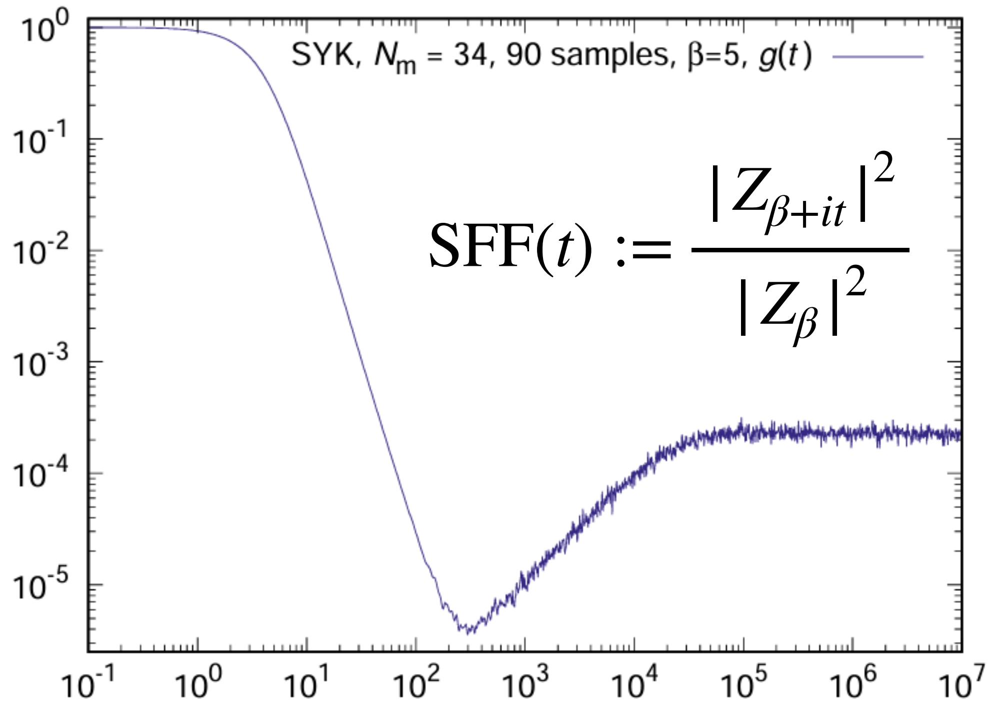
- In **quantum** systems, there are different *signatures* of chaotic behavior: “Quantum Chaology” [Michael Berry (1987)].

Spectral Statistics and RMT

- [Bohigas–Giannoni–Schmit (BGS) conjecture (1984)].



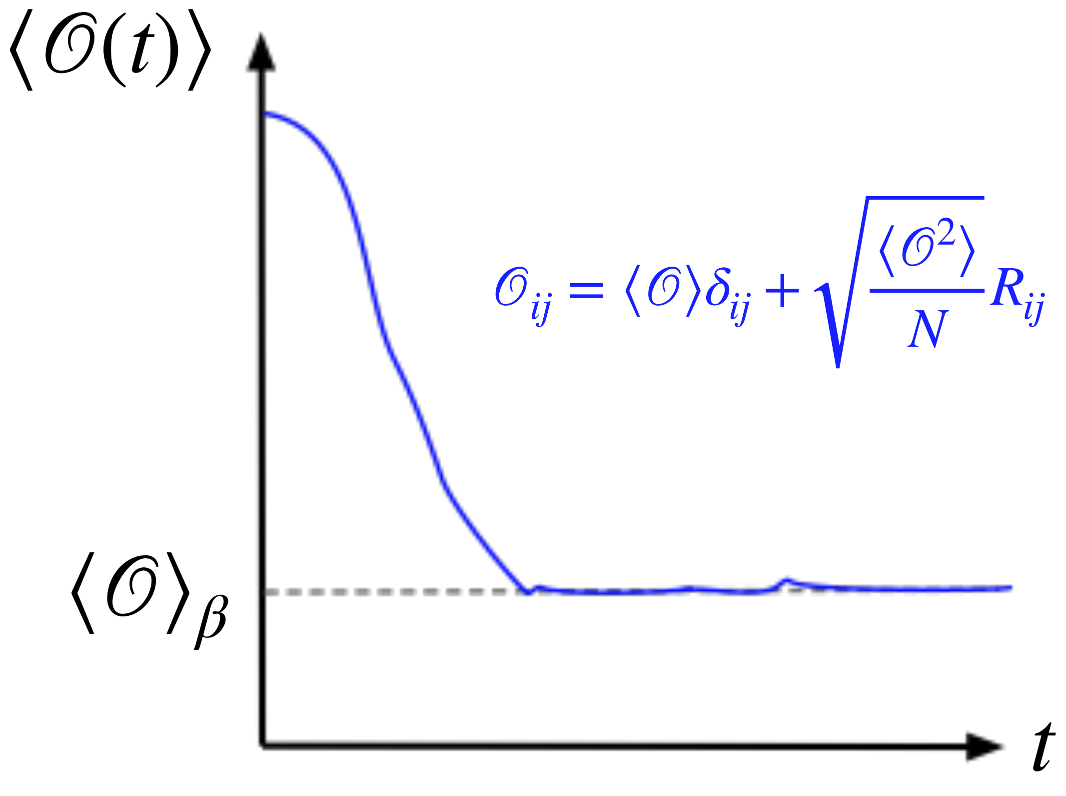
Spectral Form Factor



- [Cotler, Gur-Ari, Hanada, Polchinski, Saad, Shenker, Stanford, Streicher & Tezuka (2016), Saad, Shenker & Stanford (2018,2019),...]

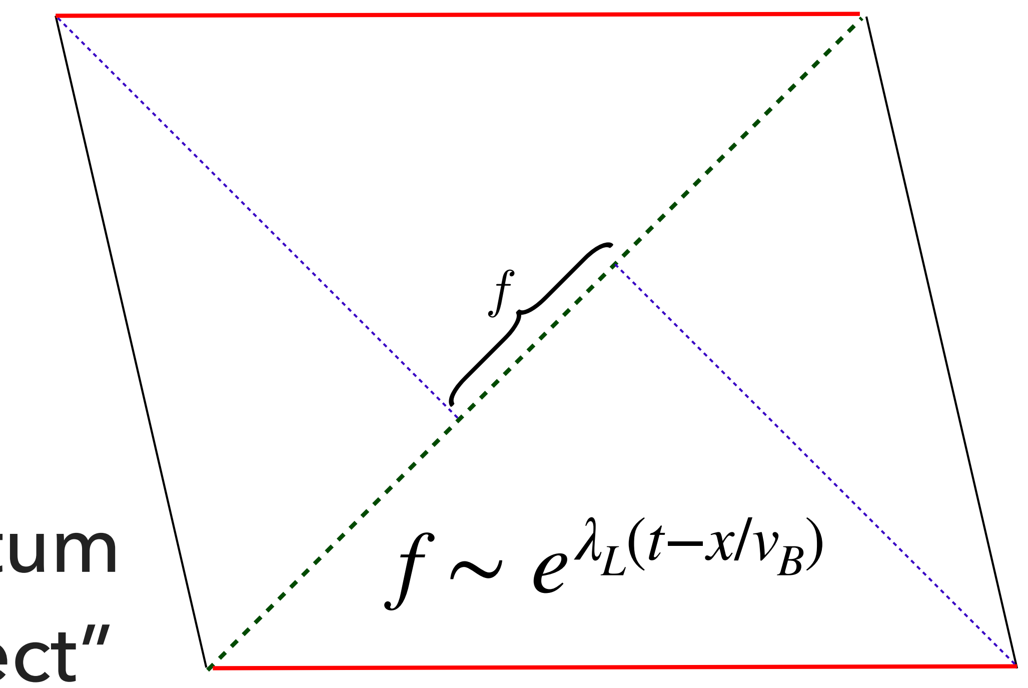
Eigenstate Thermalization Hypothesis

- [Deutsch (1991), Srednicki (1994),...]



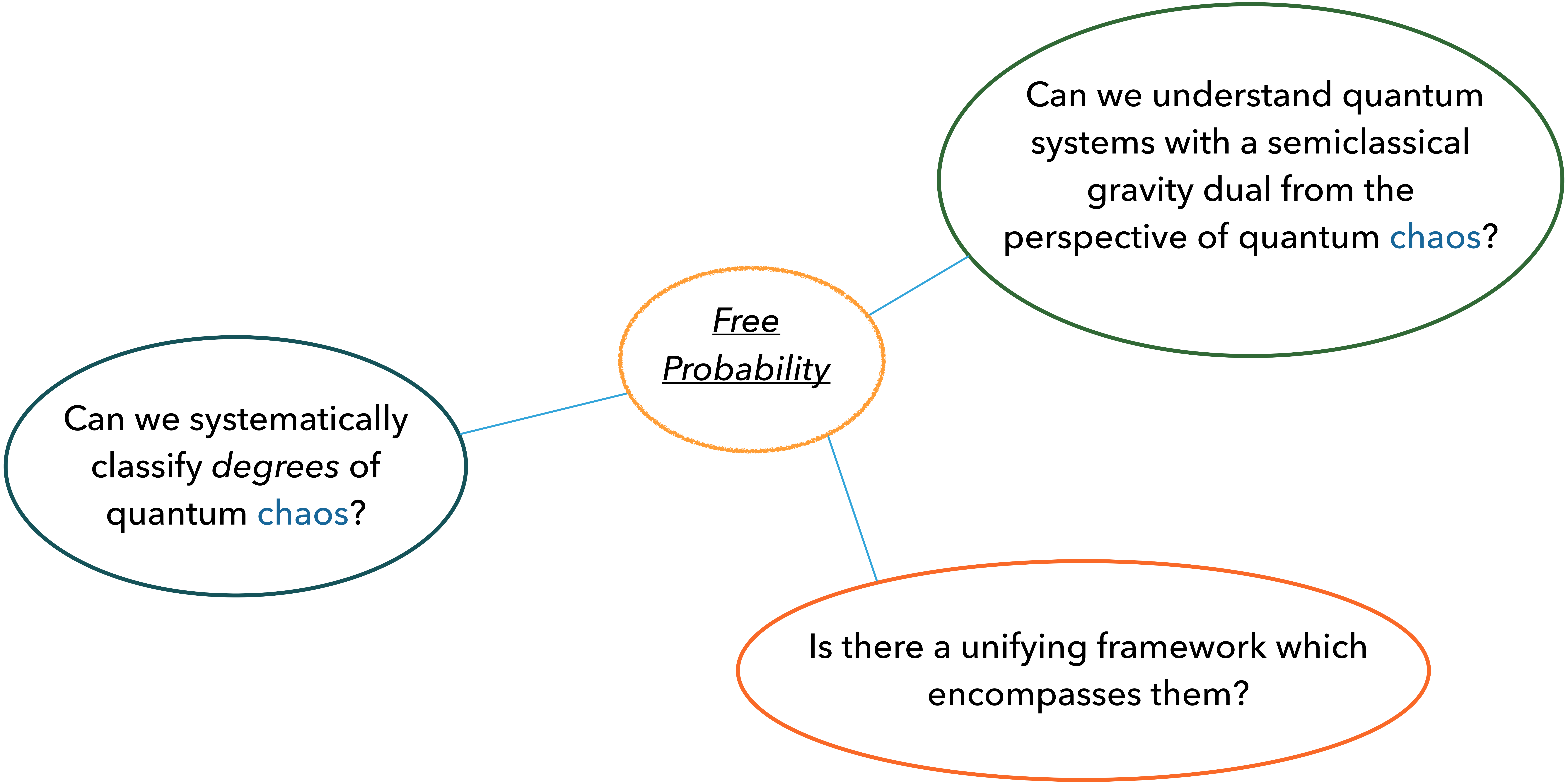
Information Scrambling

“The quantum butterfly effect”



- [Larkin & Ovchinnikov (1969), Hayden & Preskill (2007), Sekino & Susskind (2008), Shenker & Stanford (2014), (MSS) (2016),...]

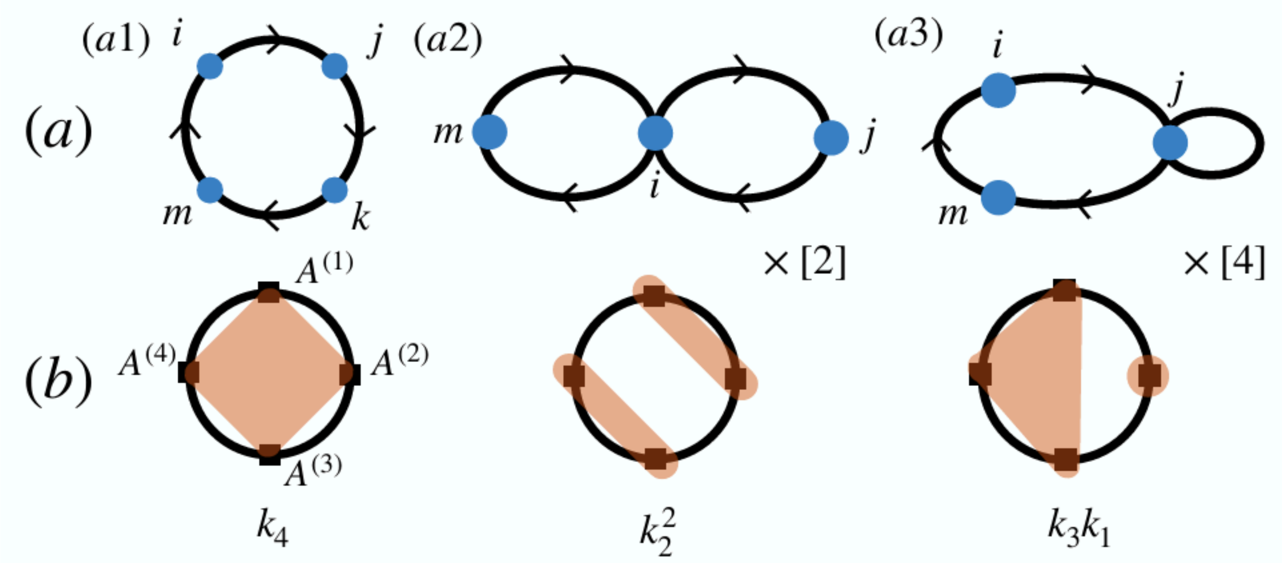
Free Probability Theory and Quantum Chaos



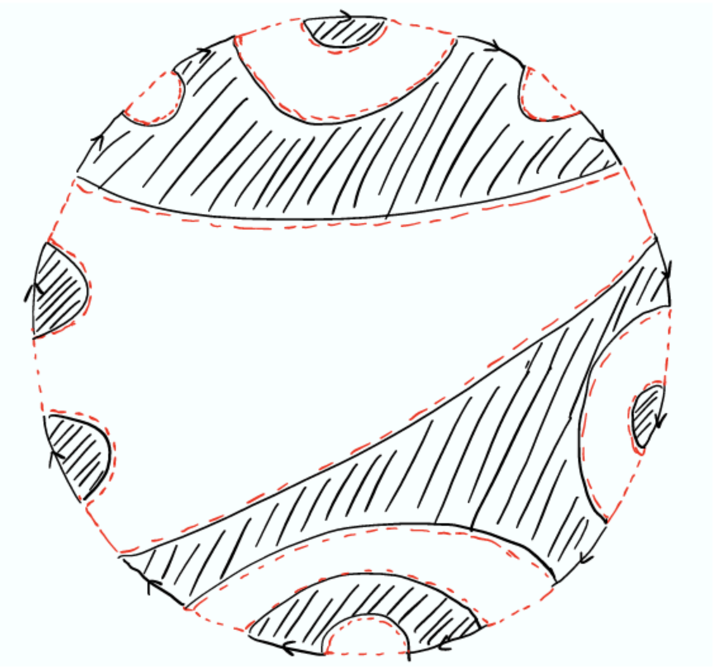
Free Probability Theory and Quantum Chaos

- Free Probability Theory recently been explored as a tool for modeling aspects of **quantum chaos**, thermalization, and *scrambling* in quantum many-body systems and also in holographic settings.

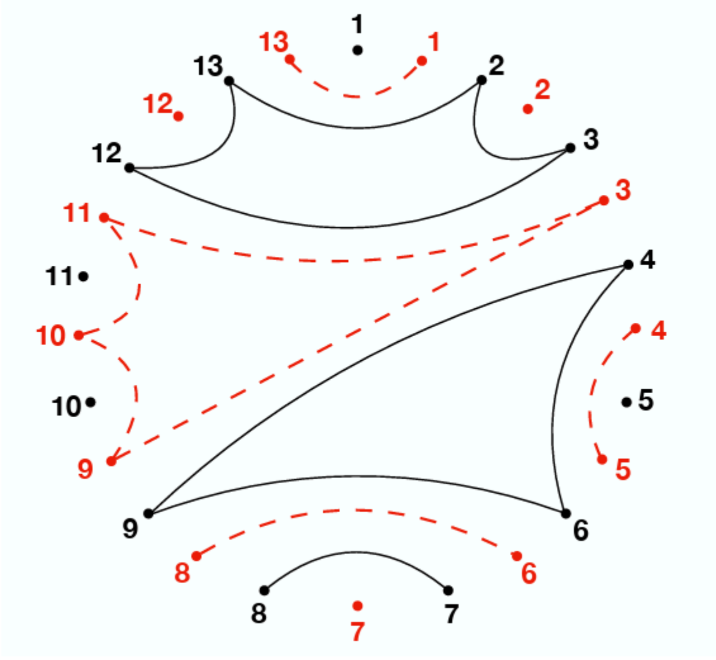
  - In generalizations of the ETH [Pappalardi, Foini & Kurchan (2022)] , [Jindal & Hosur (2024)].



  - As a tool to compute the gravitational replica partition function and fine-grained entropy in AdS/CFT [Wang (2022)].

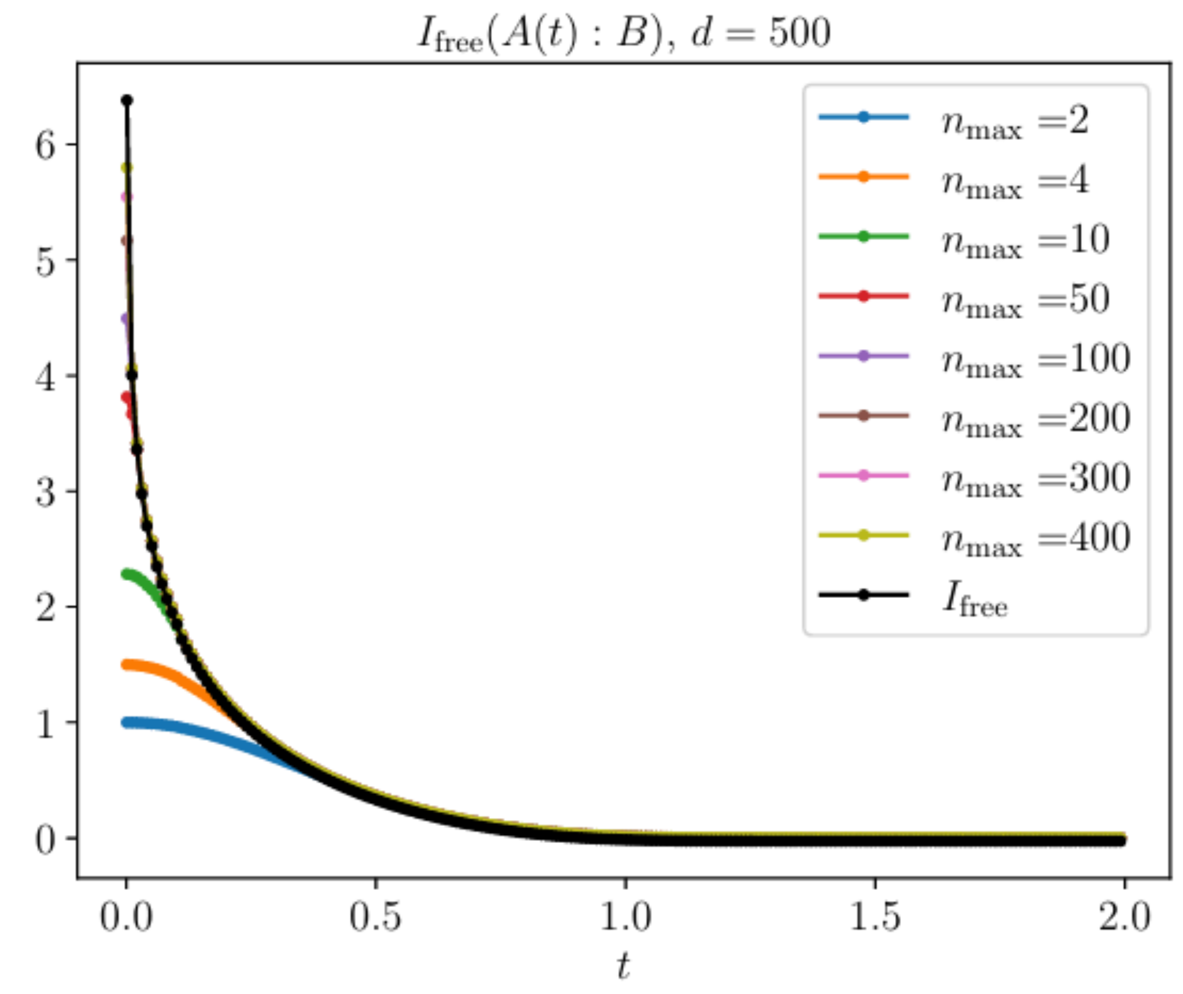


(a) A generic saddle



(b) A NC partition (black) and its Kreweras complement (red)

  - In characterizing operator growth through information-theoretic quantities (Free mutual information) [Vardhan & Wang (2025)].



Goal and Outline

Goal

- Discuss a characterization of quantum chaos based on **free probability theory** and its prediction for finite-dimensional quantum many-body systems through **operator statistics** (asymptotic freeness).

Punchline

- **Operator statistics** provides a robust probe of quantum chaos, connected to the late-time vanishing of mixed free cumulants in the thermodynamic limit, that is consistent with RMT expectations.

Goal and Outline

- **Outline**

1. Motivation & Introduction ✓
2. Classical Dynamical Systems (Classical Ergodic Theory)
3. Quantum Dynamical Systems (Quantum Ergodic Theory)
4. Free Probability Theory and Asymptotic Freeness
5. Operator Statistics in Many-Body Models
6. Conclusions

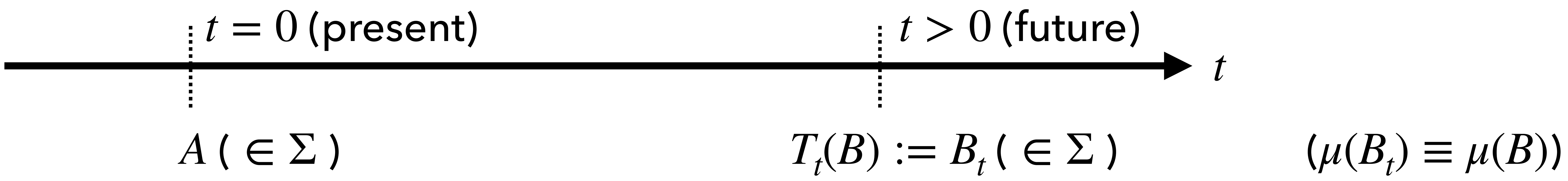
Classical Dynamical Systems

- The study of chaos (mixing) in classical dynamical systems is phrased in the language of classical probability theory

Classical Dynamical System $\mathscr{D}$ = Probability space $\mathscr{P} = (\Omega, \Sigma, \mu)$ + Time evolution map $T_t : \Omega \rightarrow \Omega$

- Ω = Space of configurations (states)/Phase space
- $\Sigma \subset$ P(Ω) = σ-algebra of <u>measurable</u> subsets of Ω (all possible events in Ω)
- $\mu : \Sigma \rightarrow [0,1]$ = Probability <u>measure</u> ($\mu(\Omega) = 1$)
- T_t = Measure-preserving and invertible flow (automorphism)

- Instead of chaos, one typically talks about <u>mixing</u> in classical dynamical systems.



- E.g. two states (events) $A, B \in \Sigma$ are **independent** if $F_2(A, B) := \mu(A \cap B) - \mu(A)\mu(B) = 0$.

Classical Dynamical Systems

- Mixing makes events in the future statistically independent from events in the past. One can classify **degrees of mixing** by how quickly a system decorrelates events, or how quickly it "forgets".

- A classical dynamical system $\mathscr{D} = (\Omega, \Sigma, \mu, T_t)$ is said to be
  - **Ergodic**, if $\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{d}t\, F_2(A, B_t) = 0\,,\ \forall A, B \in \Sigma\,.$ (Time averages = Phase space (ensemble) averages)
  - **Strongly 2-Mixing**, if $\lim_{t\to\infty} F_2(A, B_t) = 0\,,\ \forall A, B \in \Sigma\,.$ (Generalizes to n-Mixing using F_n)
  - **K-Mixing**, if $\lim_{t\to\infty} \big(\sup_{C\in\Sigma(B_t)} |F_2(A, C)| \big) = 0,\ \forall A, B \in \Sigma.$ (Positive Kolmogorov-Sinai entropy $\Longrightarrow$ positive Lyapunov exponents)

Classical "chaos" starts here. [Berkovitz, Frigg & Kronz (2006)]

(Bernoulli $\Longrightarrow$) **K-mixing** $\Longrightarrow$ **Strongly 2-Mixing** $\Longrightarrow$ **Ergodic**

Classical Ergodic Hierarchy

← Degree of mixing increases ←

Algebraic Perspective

- Given a classical probability space $\mathscr{P} = (\Omega, \Sigma, \mu)$, one can define real-valued **random variables** as measurable functions $\{f : \Omega \to \mathbb{R}\}$. This forms an algebra $\mathscr{A}$. An example is the Banach algebra $L^\infty(\Omega, \Sigma, \mu)$ of bounded measurable functions. This space is equipped with a trace functional (expectation value) $\varphi : L^\infty(\Omega, \Sigma, \mu) \to \mathbb{R}$ given by

$$\varphi(f) = \int_{X\in\Omega} \mathrm{d}\mu\, f(X) \equiv \langle f \rangle$$

- For example, given $f, g, \in L^\infty(\Omega, \Sigma, \mu)$, and $T_t : \Omega \to \Omega$

$$\kappa_2(f, g; t) = \int_{X\in\Omega} \mathrm{d}\mu\, f(X) g(X_t) - \left(\int_{X\in\Omega} \mathrm{d}\mu\, f(X)\right)\left(\int_{X\in\Omega} \mathrm{d}\mu\, g(X)\right)$$

Ergodicity : $\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{d}t\, \kappa_2(f, g; t) = 0$

Strongly 2-mixing : $\lim_{t\to\infty} \kappa_2(f, g; t) = 0\,.$

Quantum Dynamical Systems

- The quantum analogue of classical ergodic theory is an active area of research (see e.g. [Gesteau (2023), Ouseph, Furuya, Lashkari, Leung & Moosa (2023)]).

Quantum Dynamical System $\mathscr{Q}$ = Algebraic Probability Space $\mathscr{P}_{\mathscr{A}} = (\mathscr{A}, \varphi)$ + Time evolution map T_t

- $\mathscr{A}$ = (von Neumann) algebra of observables of a quantum system.
- $T_t : \mathscr{A} \to \mathscr{A}$ = one-parameter group of automorphisms in $\mathscr{A}$.
- $\varphi : \mathscr{A} \to \mathbb{C}$ = normal state (positive and faithful, unital linear map).

- The algebra $\mathscr{A}$ of observables of a quantum system is a **non-commutative** probability space.
- Quantum mixing can be characterized by the decay of connected correlation functions (**free cumulants**):

$$F_2(A, B; t) := \langle AB_t \rangle - \langle A \rangle \langle B_t \rangle$$

Does quantum "chaos" start here?

Q. K-mixing $\Longrightarrow$ Q. Strongly 2-Mixing $\Longrightarrow$ Q. Ergodic

← Degree of quantum mixing increases ←

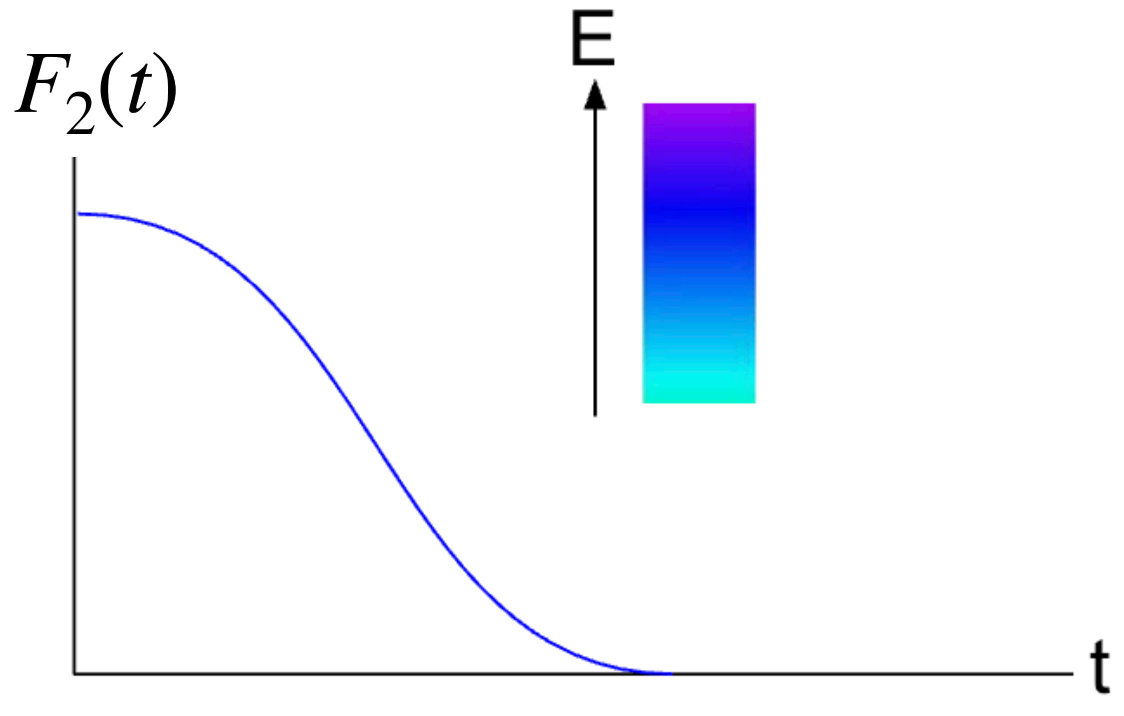
Quantum Ergodic Hierarchy

- **Remarks:**

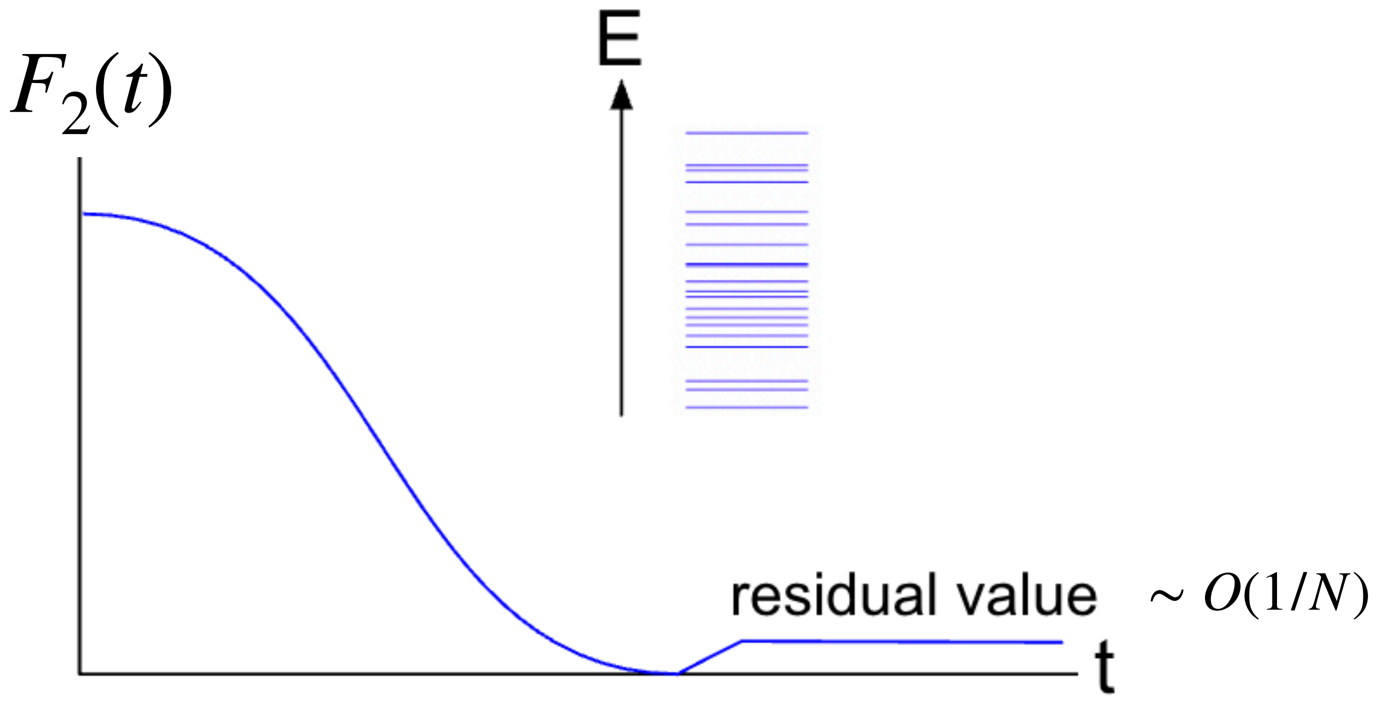
Continuous energy spectrum

$$\lim_{t\to\infty} F_2(t) = 0$$

$$(N \to \infty)$$



mixing



no mixing

Discrete energy spectrum

$$\lim_{t\to\infty} F_2(t) \propto O(1/N)$$

$$(N \gg 1)$$

- Strictly speaking, the vanishing of cumulants (mixing) can only occur in type II or III von Neumann algebras.

- Isolated, finite-dimensional quantum systems do not exhibit true quantum mixing, even if they exhibit chaos à la BGS.

- Connected correlation functions (**free cumulants**) quantify the **free independence** between observables measured in the future from those measured in the past.

Free Probability Theory

- **Free Probability Theory (FPT)** [Voiculescu (1985), Speicher (1994)] is the probability theory of non-commuting (random) variables.

- Two non-commuting variables $A, B \in (\mathscr{A}, \varphi)$ are mutually **free (freely independent)** if all **mixed free cumulants** of all orders vanish

$$A, B \in (\mathscr{A}, \varphi) \quad \textbf{free} \quad \Longleftrightarrow \quad \kappa_n(A, B, \ldots, A, B) = 0 \quad \text{for all } n \geq 1$$

where $\langle A_1 \cdots A_n \rangle = \sum_{\pi \in \mathrm{NC}(n)} \prod_{b \in \pi} \kappa_{|b|}(A_{b(1)}, \cdots, A_{b(n)})$ and $\mathrm{NC}(n)$ is the lattice of **non-crossing** partitions of the set $S_n = \{1, \ldots, n\}$.

- If $A, B \in (\mathscr{A}, \varphi)$ are **free** then, the n-th order mixed moments $F_n(A, B, \ldots, A, B) = \langle AB \cdots AB \rangle$ **factorize** into sums or products of individual lower-order moments $a_k = \langle A^k \rangle$, $b_k = \langle B^k \rangle$.

(A.S. 1-3)

Free Probability Theory

- **Remarks:**

- If $A, B \in (\mathscr{A}, \varphi)$ are **free** (and if $\langle A \rangle = \langle B \rangle = 0$) then $F_{2n}(A,B) := \langle (AB)^n \rangle = 0$!

- For $N \times N$ (Hermitian) RMT, freeness arises only asymptotically in the $N \to \infty$ limit, where

$$\varphi(\,\cdot\,) := \lim_{N\to\infty} \varphi_N(\,\cdot\,) = \lim_{N\to\infty} \frac{1}{N}\mathbb{E}[\mathrm{tr}(\,\cdot\,)] = \lim_{N\to\infty} \frac{1}{N}\int_{\mathbf{X}\in R(\mathbf{X})} \mathrm{d}\mu(\mathbf{X})\,\mathrm{tr}(\,\cdot\,)$$

- Consider two sequences of definite (deterministic) $N \times N$ matrices $\{A_N, B_N\}$ whose spectra $\{\rho_{A_N}, \rho_{B_N}\}$ are well-defined in the $N \to \infty$ limit $\{\rho_{A_N} \to \rho_A, \rho_{B_N} \to \rho_B\}$. Then, for a Haar random unitary $U_N \in \mathrm{Haar}(\mathrm{U}(N))$

A_N and $U_N^\dagger B_N U_N$ will become **asymptotically free** for $N \to \infty$

(A.S. 4)

Asymptotic Freeness and Quantum Chaos

- **Q**: Does time evolution generated by a chaotic Hamiltonian H induce asymptotic freeness?

Do A and $U^\dagger B U$ with $U = e^{-itH}$ become **asymptotically free** after some t? ***(A.S. 5)***

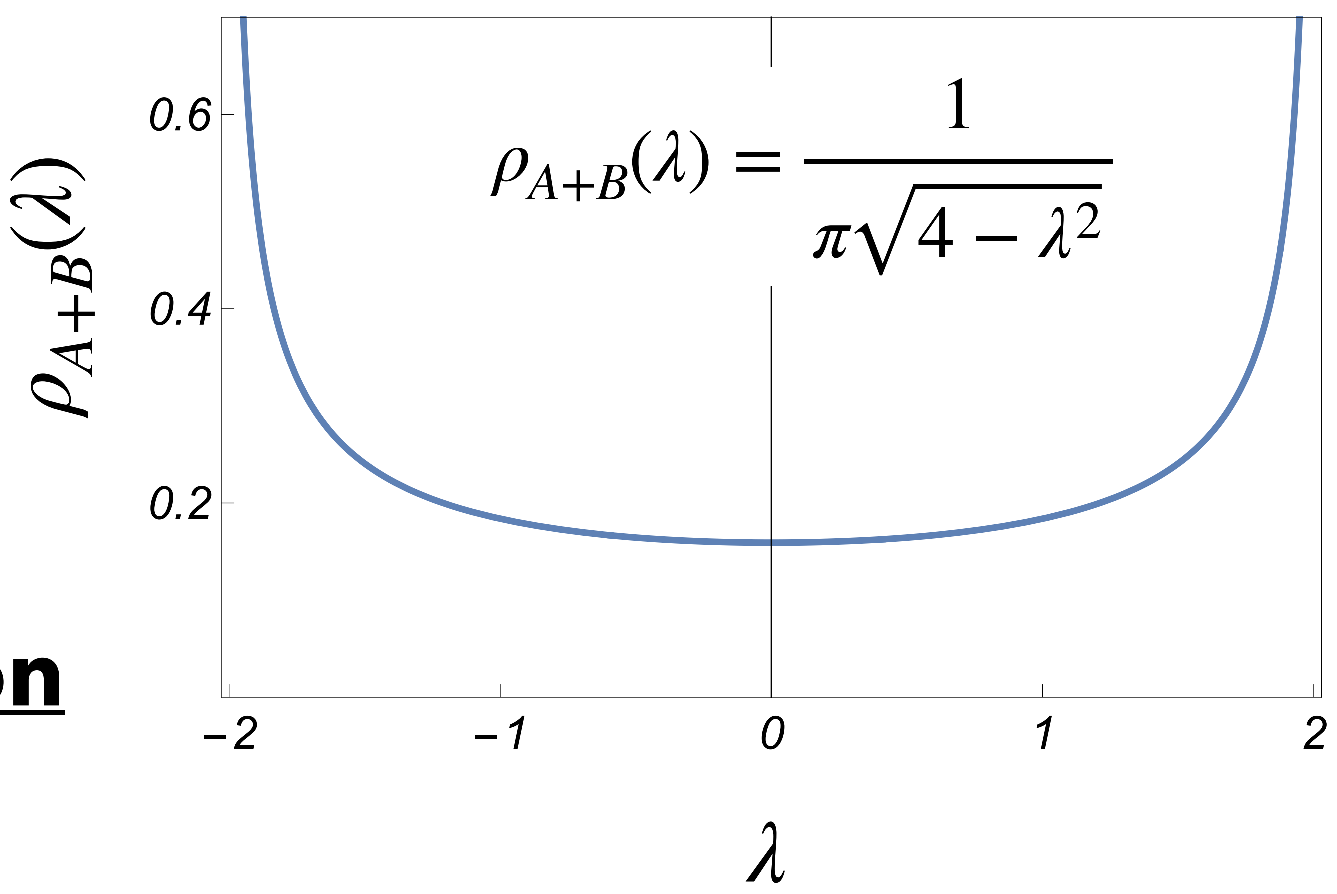
- **Idea**: Given two free variables A, B with well-defined spectra ρ_A, ρ_B FPT provides a prediction of the spectrum (eigenvalue density) of $A + B$. It is given by the **free additive convolution** $\rho_{A+B} = \rho_A \boxplus \rho_B$!

- Consider two deterministic Hermitian $N \times N$ matrices A, B with well-defined spectra ρ_A, ρ_B. The spectrum of $A + B$ takes a specific form, depending on the number of distinct eigenvalues of A, B. For example:

spin-1/2

$$\rho_A(\lambda) = \frac{1}{2}(\delta(\lambda - 1) + \delta(\lambda + 1)) = \rho_B(\lambda)$$

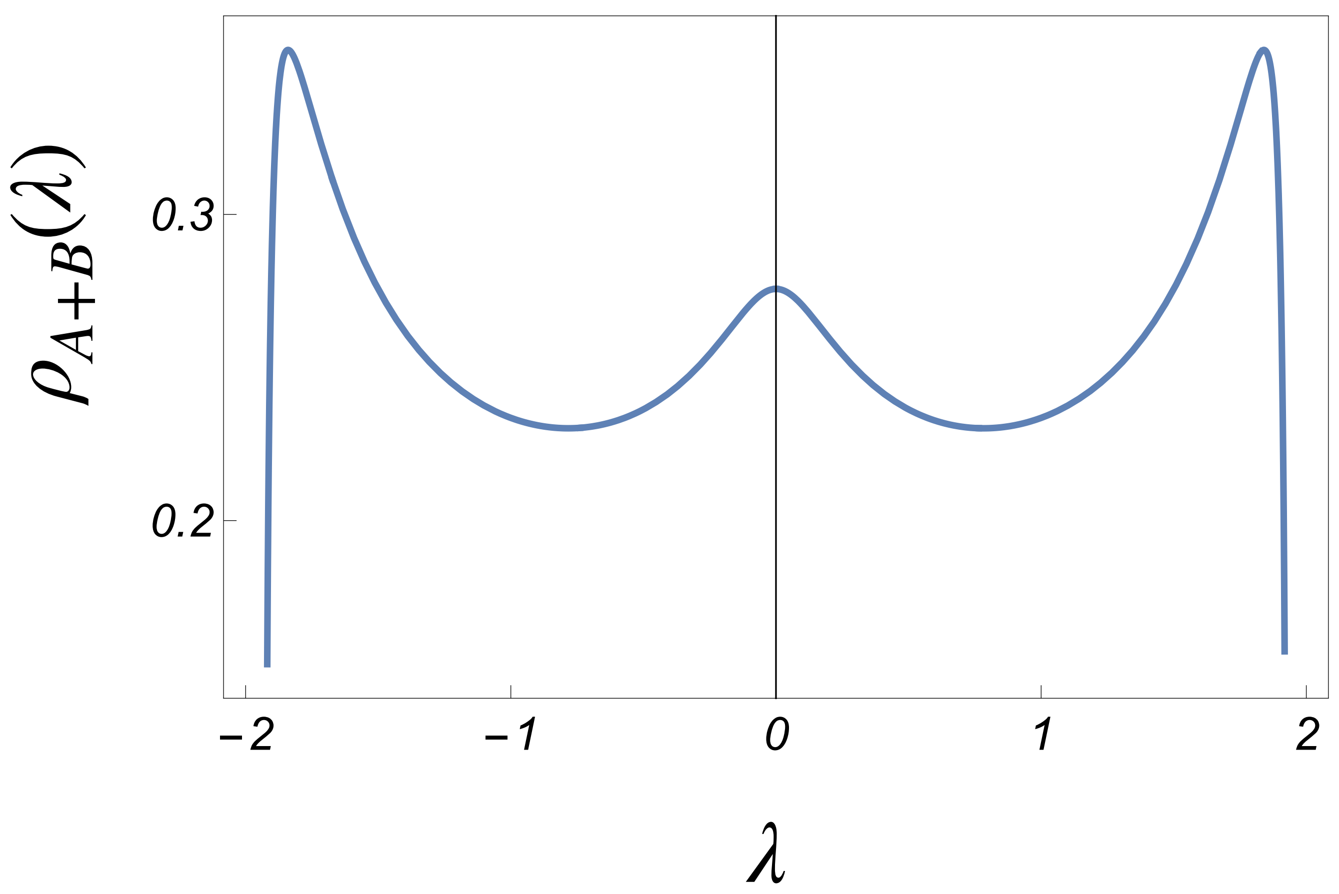
[Mingo & Speicher (2017), Chen & Kudler-Flam (2024)]



arcsine distribution

spin-1

$$\rho_A(\lambda) = \frac{1}{3}(\delta(\lambda - 1) + \delta(\lambda) + \delta(\lambda + 1)) = \rho_B(\lambda)$$

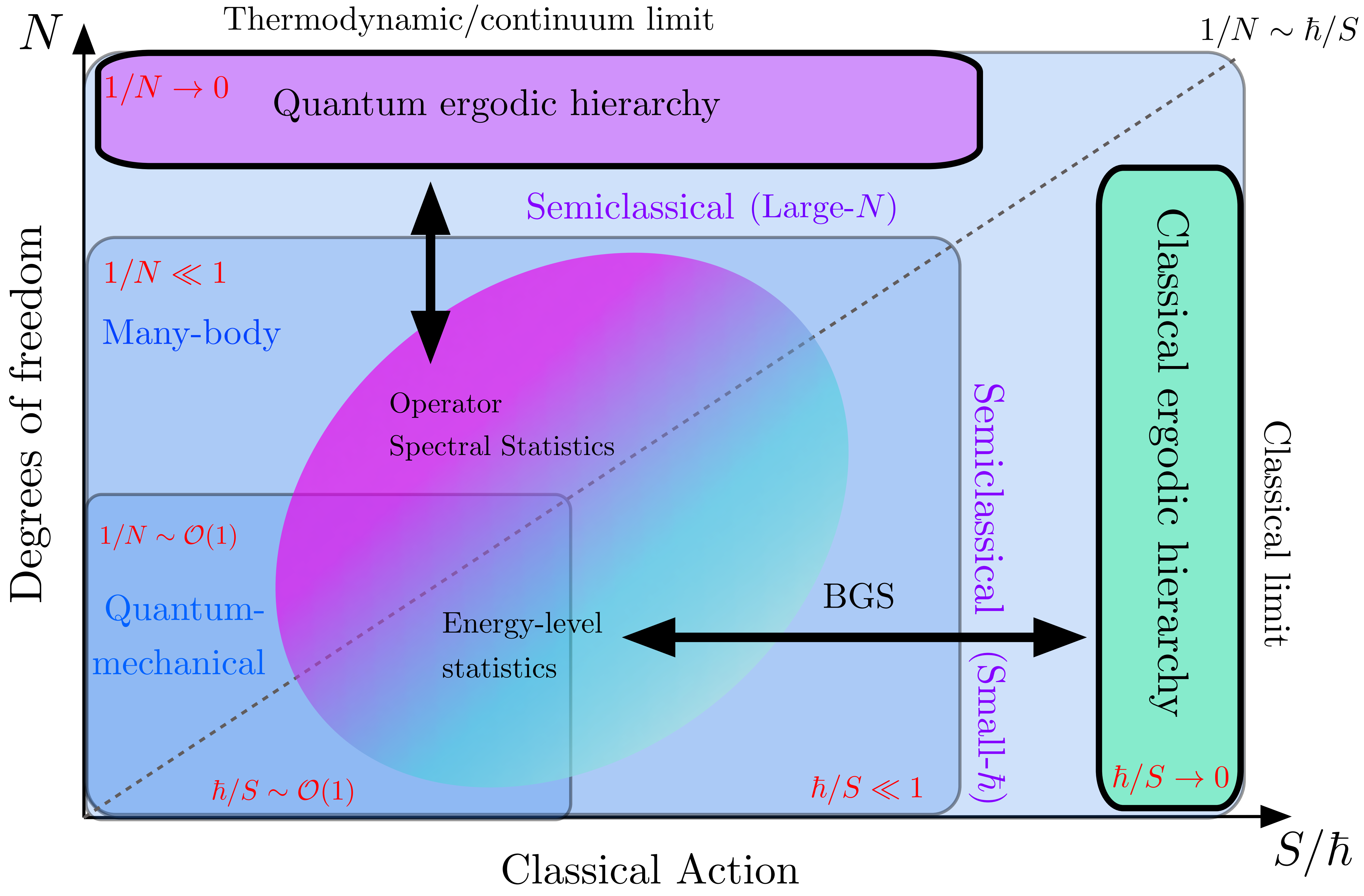


[Fu, Jahnke, Kim, Pal, **HC** (2025)]

(A.S. 6-7)

Operator Statistics

- **Proposal:** Use **operator statistics** as a smoking gun to probe the decay of free cumulants in quantum systems with chaos (à la BGS).
- **Operator statistics** is the finite-N probe reflecting the large-N dynamics, similar to the way in which spectral statistics encodes finite-$\hbar$ information from the small-$\hbar/S$ classical dynamics.



Freeness in the Mixed-Field Ising

- The Hamiltonian of the mixed-field Ising(with open boundary conditions) is

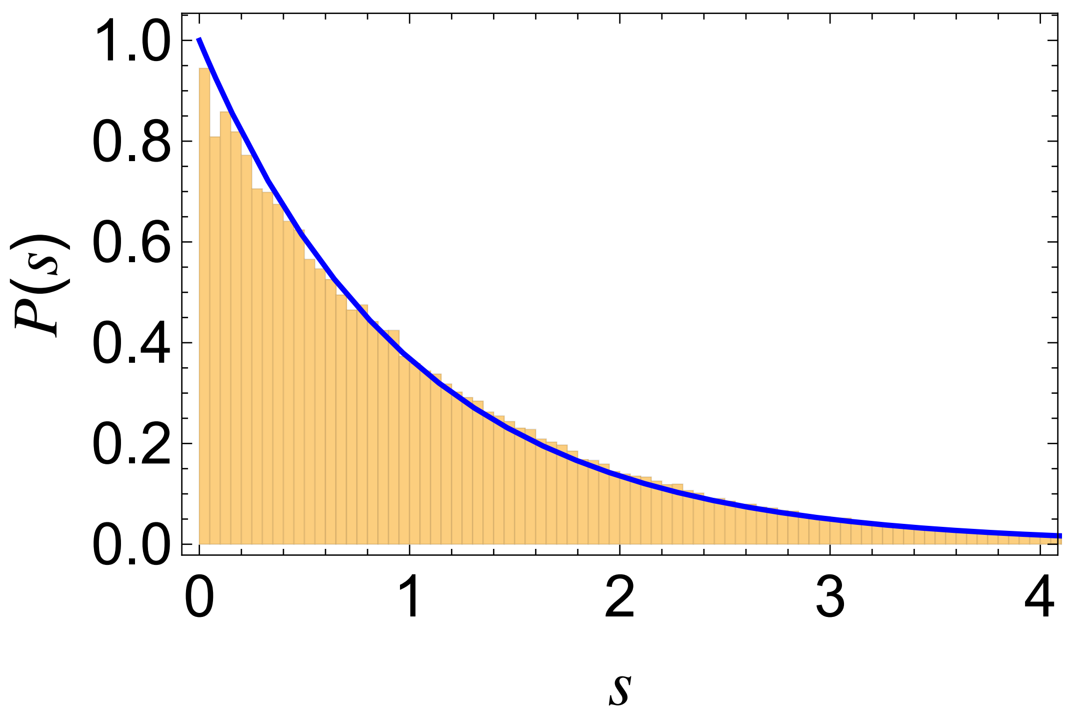
$$H = -\sum_{i=1}^{L-1} Z_i Z_{i+1} - \sum_{i=1}^{L} \left(h_x X_i + h_z Z_i\right) + g \sum_{i=1}^{L} \epsilon_i X_i$$

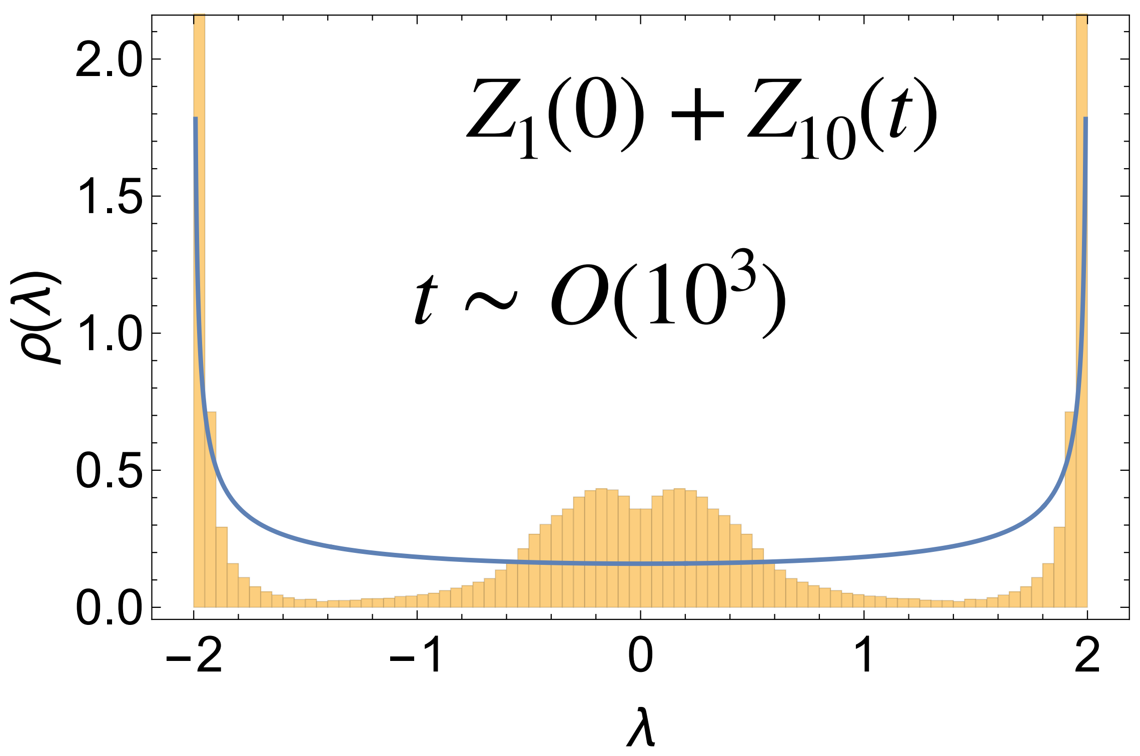
(Z_i, Y_i, X_i spin-1/2 operators)
(SU(2) algebra)

where $\epsilon_i \sim N(0,1)$ are small random numbers used to break leftover symmetries (after resolving $\mathbb{Z}_2$-symmetry).

$L = 10,\ h_x = -1$
$h_z = 0,\ g = 0.2$

- *Close* to Poissonian statistics





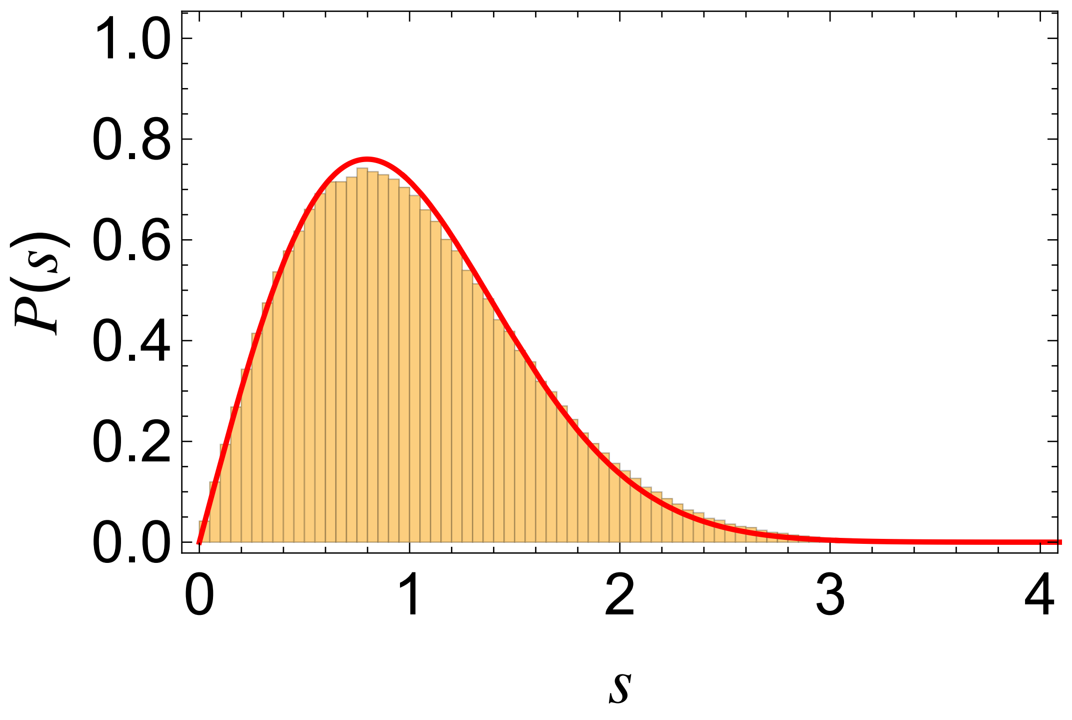
- The arcsine distribution takes much longer to form.

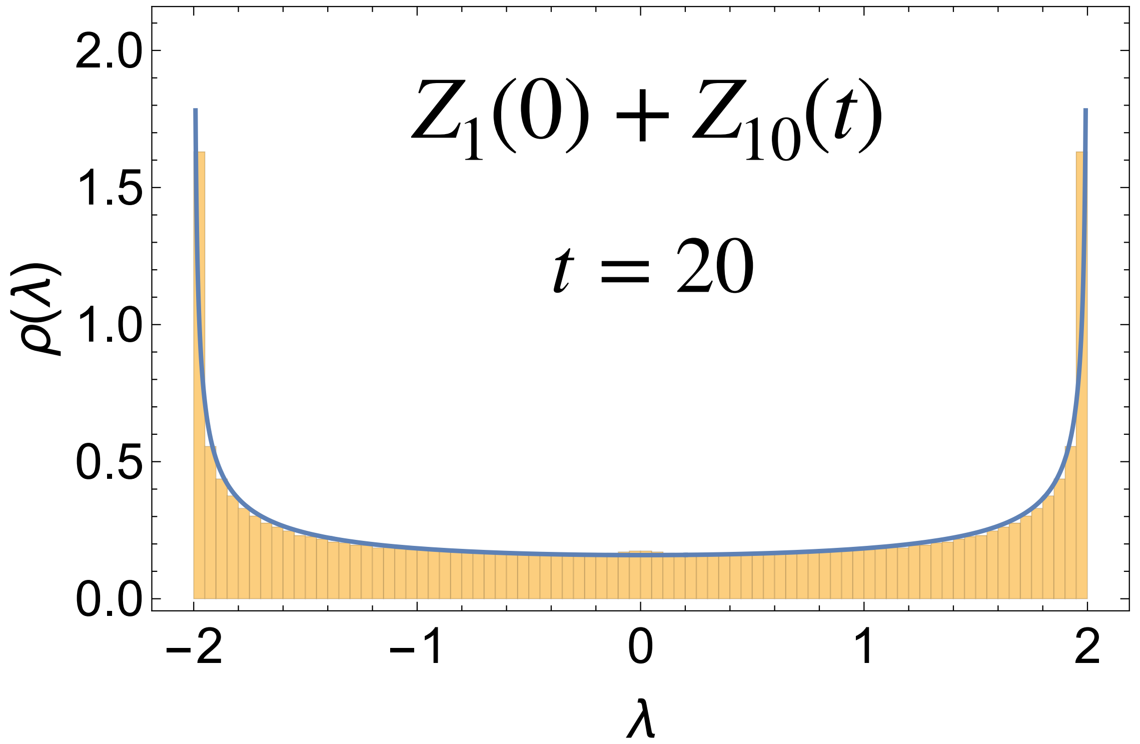
Spectral (level-spacing) statistics

Operator statistics

$L = 10,\ h_x = -1.05$
$h_z = 0.5,\ g = 0.2$

- GOE Wigner–Dyson statistics





- Matches the arcsine distribution.

"Arcsine Time"

- To identify the time at which the arcsine distribution emerges, we employ a least-squares procedure through a quantity quantifying the deviation of the eigenvalue distribution from the arcsine distribution.

$$\chi^2(t) = \sum_i \left(\frac{\rho(\lambda_i, t) - f(\lambda_i)}{f(\lambda_i)} \right)^2 \qquad f(\lambda) = \begin{cases} \frac{1}{\pi\sqrt{4-\lambda^2}}, & \text{if } -2 < \lambda < 2, \\ 0, & \text{otherwise.} \end{cases}$$

where $\rho(\lambda_i, t)$ denotes the eigenvalue density of $A(0) + B(t)$ at time t. Fitting a function $F(t) = a(e^{-(t-t_0)/t_d} + 1)$ to the $\chi^2(t)$ data, we define $t_{arcsine}$ through $F(t_{arcsine}) = a/e + a$, i.e. $t_{arcsine} = t_0 + t_d$.

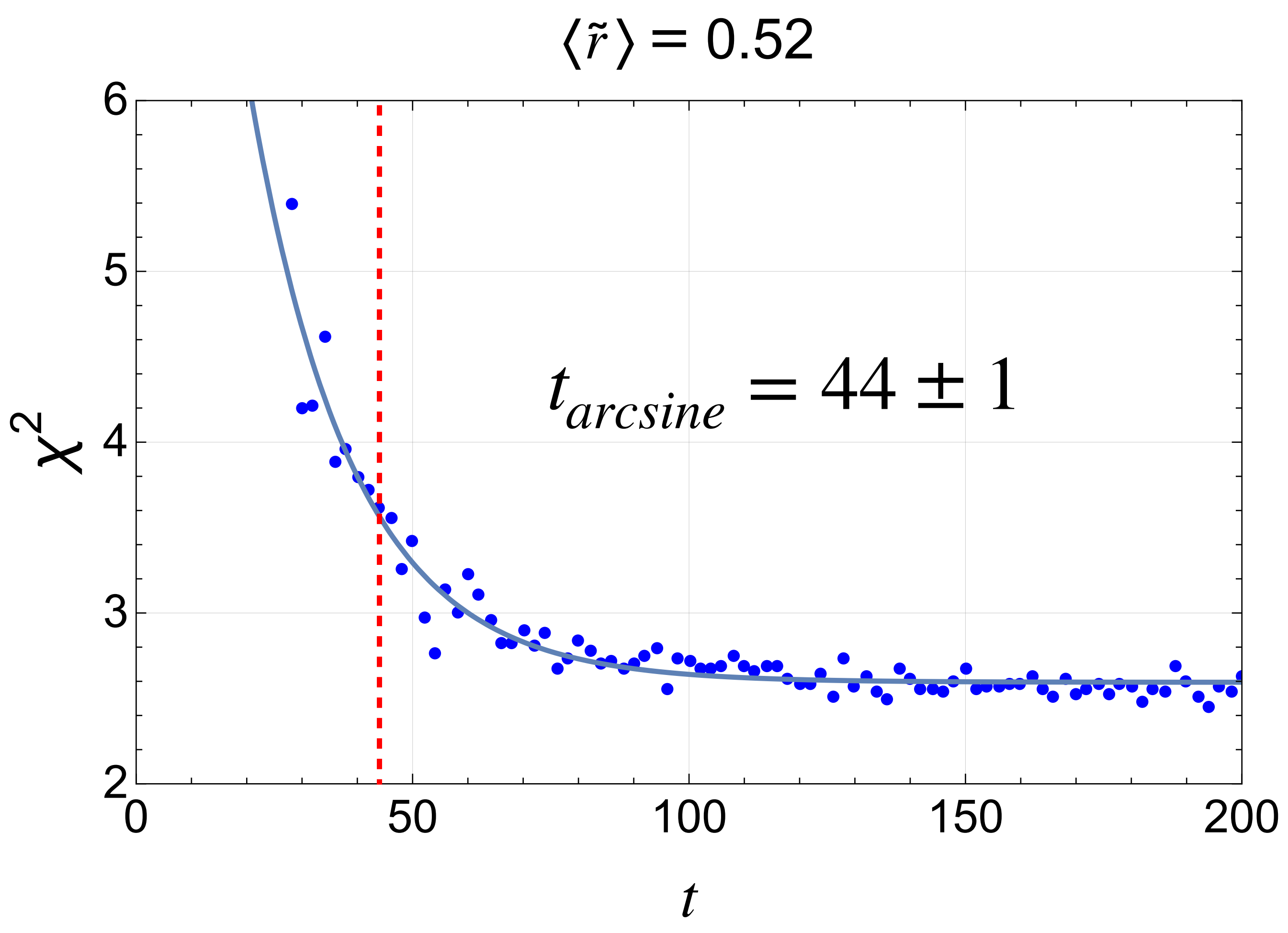
$Z_1(0) + Z_{10}(t)$

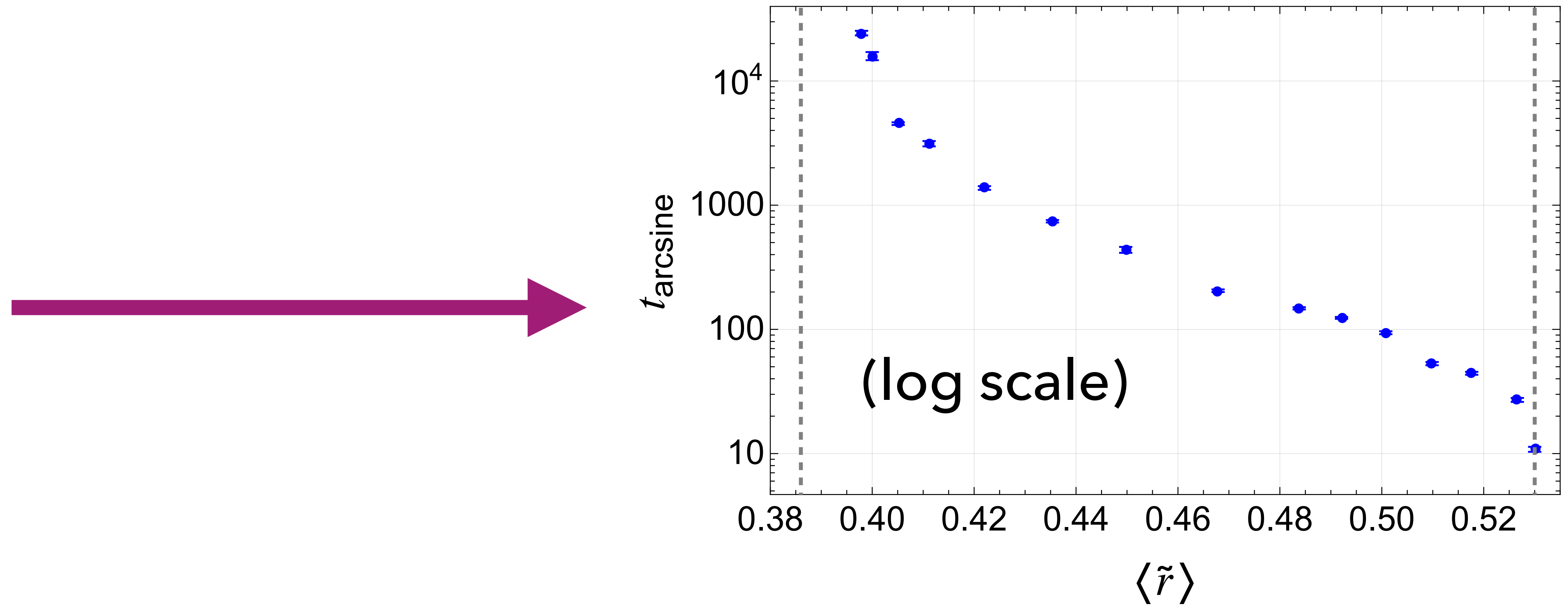
$L = 10$

$g = 0.2$

$h_x = -1$

$h_z = 0.1$





$Z_1(0) + Z_{10}(t)$

$L = 10$

$g = 0.2$

h_z, h_x varied

$t_{arcsine}$ appears to **diverge** as $\langle \bar{r} \rangle$ approaches $\langle \bar{r}_P \rangle \approx 0.387$, while becoming $O(10)$ for $\langle \bar{r}_{GOE} \rangle \approx 0.5307$.

Freeness in the SYK Model

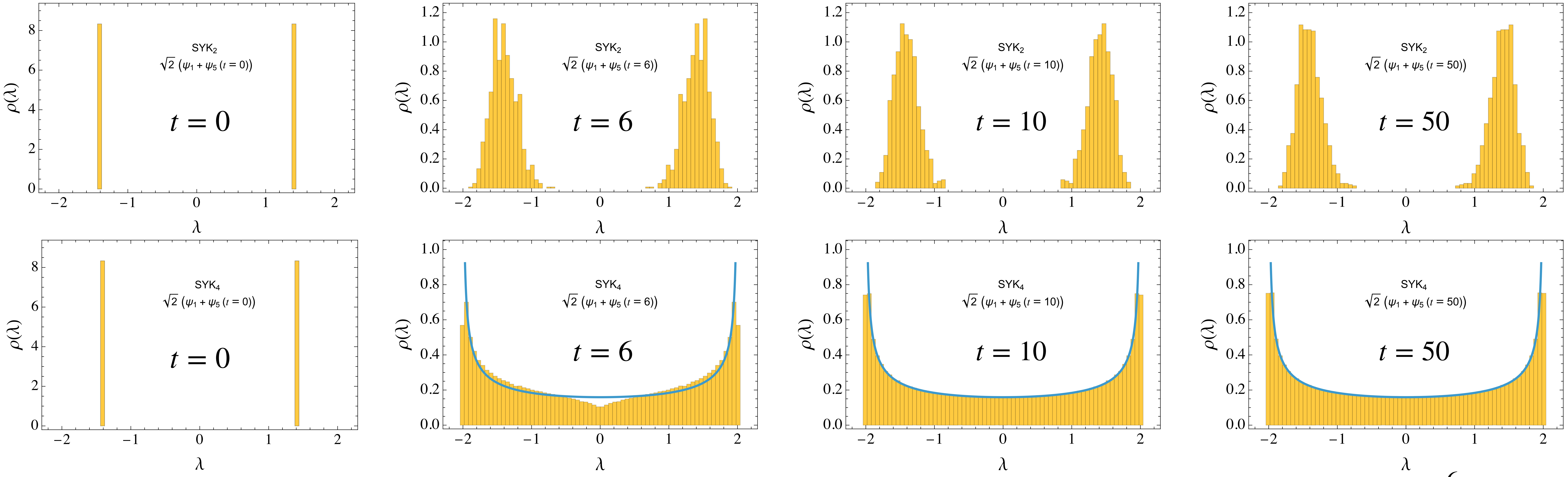
- Other interesting examples are the $q=2{,}4$ Sachdev–Ye–Kitaev (SYK) models for N_f Majorana fermions

$$H_{\mathrm{SYK}_2} = \frac{i}{\sqrt{N_f}} \sum_{j<k} K_{jk}\psi_j\psi_k \qquad H_{\mathrm{SYK}_4} = \sqrt{\frac{6}{N_f^3}} \sum_{j<k<l<m} J_{jklm}\psi_j\psi_k\psi_l\psi_m \qquad \text{(Clifford algebra)}$$

where $\{\psi_i, \psi_j\} = \delta_{ij}$, K_{jk} and J_{jklm} are random couplings drawn from a Gaussian distribution with zero mean and standard deviations K, J , respectively. For example, the operator statistics for $\psi_1 + \psi_5(t)$: ($N_f = 16$, 10^3 realiz.)

(SYK$_2$)

(SYK$_4$)



In the SYK$_2$ case, the acrsine distribution does not form, at least within timescales up to order $t \sim O(10^6)$.

Freeness in the RP Model

- The **operator statistics** can also detect interesting intermediate phases, such as fractal phases. For example, in the Rosenzweig–Porter (RP) model

$$H_{\text{RP}}(\gamma) = D + \frac{1}{N^{\gamma/2}} R$$

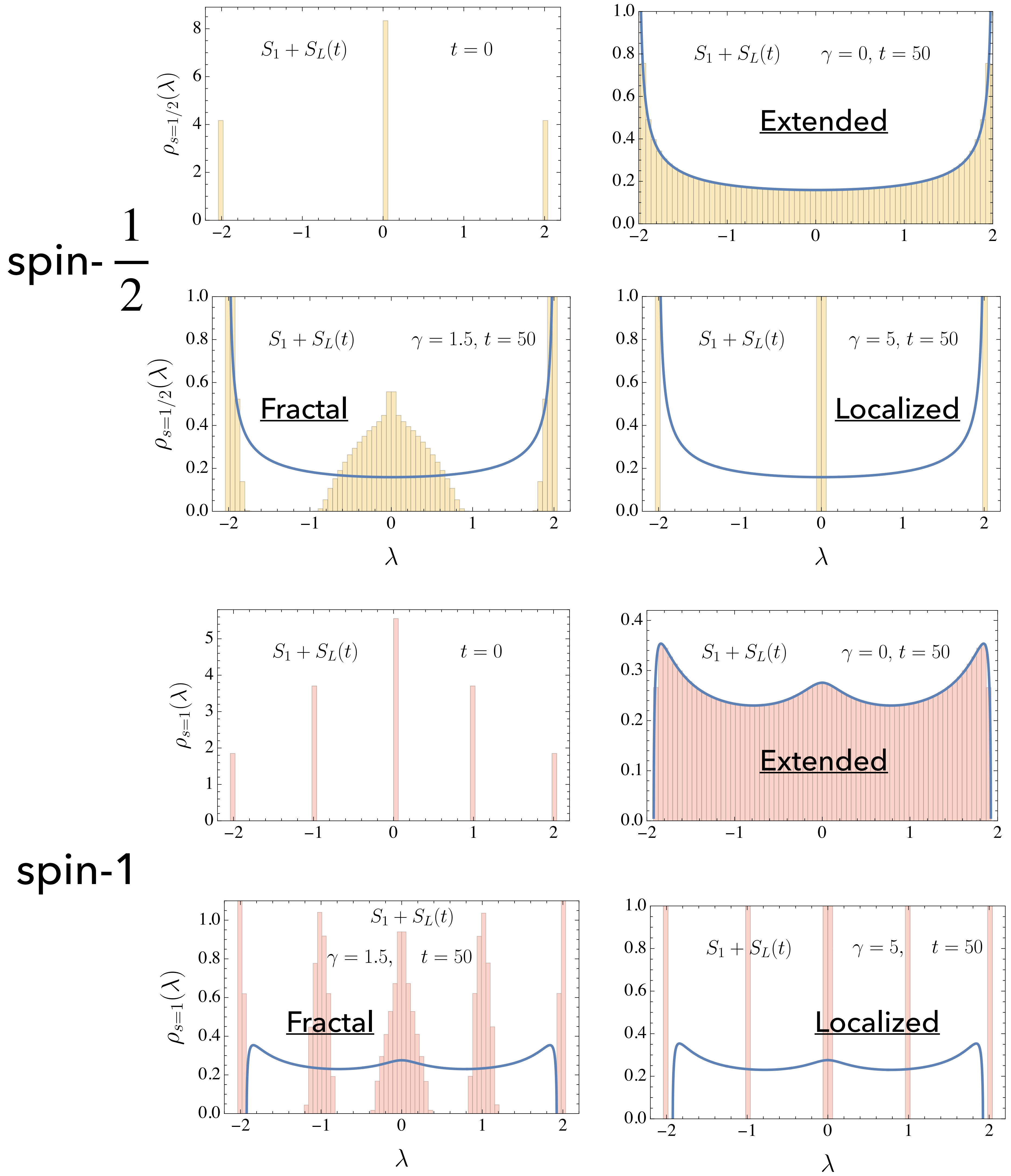
$D = N \times N$ diagonal.

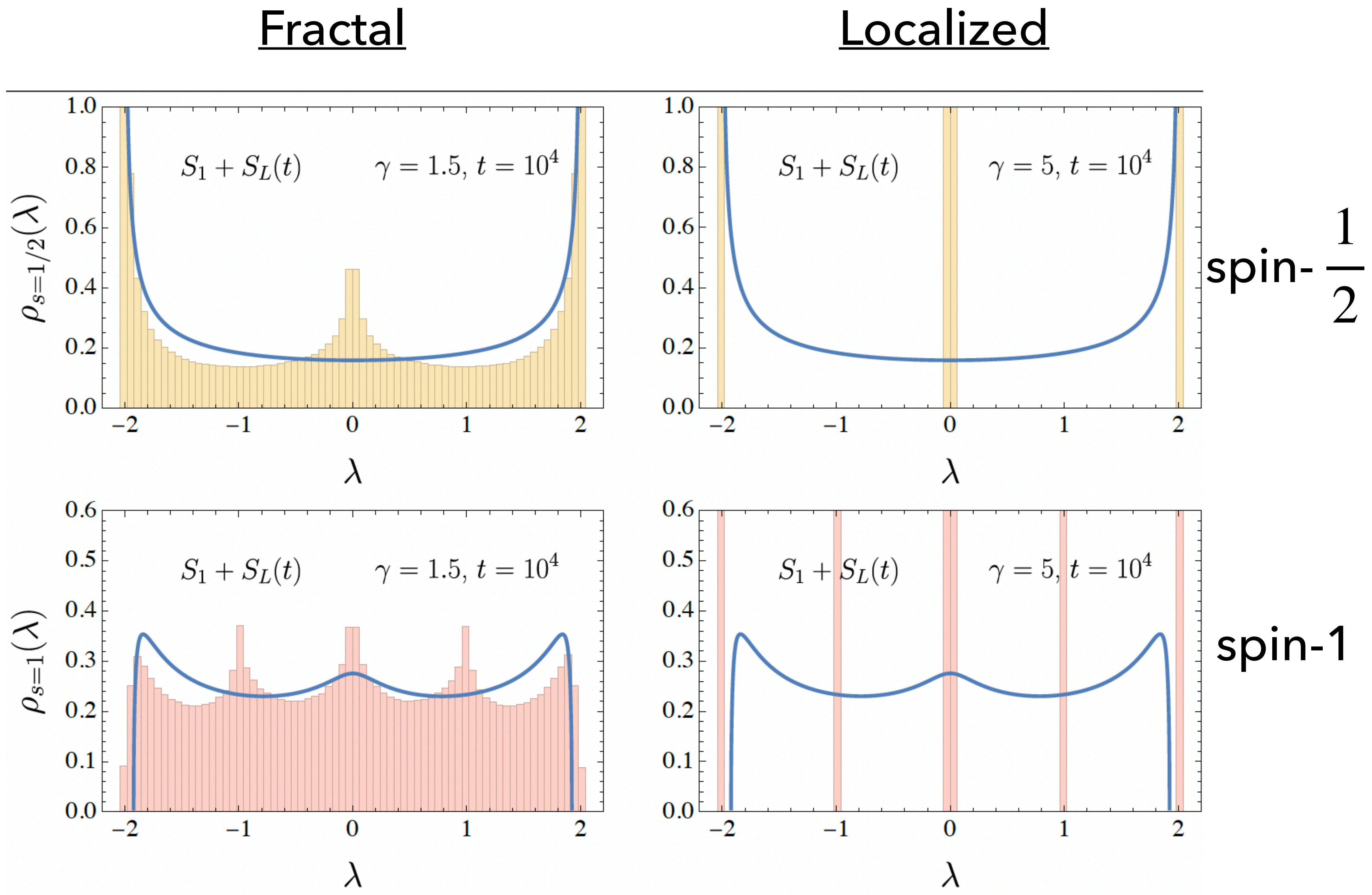
$R = N \times N$ GOE

$$\gamma \in \begin{cases} [0,1], & \text{extended} \\ [1,2], & \text{fractal} \\ [2,\infty), & \text{localized} \end{cases}$$

$\text{IPR}(\gamma) = N^{-d_\gamma}$

$d_\gamma =$ fractal dim.





Conclusions

- Mixing in classical and quantum dynamical systems can be phrased in terms of statistical independence, either classical or **free**, respectively.

- In quantum dynamical systems described by a non-commutative probability space, Free Probability Theory (FPT) is the natural language to study the decay of connected correlation functions (cumulants).

- Chaos à la BGS drives initially non-free operators toward **asymptotic freeness.**

- **Operator statistics** serves as a benchmark for testing the emergence of approximate asymptotic freeness in finite-dimensional quantum many-body systems with spectral chaos under time evolution.

- In finite-dimensional quantum many-body systems with (near) Poissonian spectral statistics, freeness emerges at much later timescales. This suggests that in integrable systems, at $N \to \infty$ freeness does not emerge.

(A.S. 8)

Future Directions

- **Finite temperature:** Can these predictions be generalized to other states? How to incorporate finite temperature effects in the operator statistics?

- **Locality and Choice of Operators:** How does locality affect the emergence of the free probability prediction?

- **Lyapunov exponents:** Freeness requires the late-time vanishing of all free cumulants, but does not specify their decay rate. Can freeness be used to determine Lyapunov exponents? What is the relation to the operator statistics of out-of-time-order "operators" [Rozenbaum, Ganeshan and Galitski (2016)] ?

- **Fortuity and BPS Chaos:** For generic choices of couplings in the $\mathcal{N} = 2$ supersymmetric SYK model, *all* BPS states are "fortuitous": R-charge concentration and RMT behavior near BPS states [Chang & Lin (2024) and Chang, Chen, Sia & Yang (2024)]. Can FPT shed (more) light on this phenomenon?

⋮

Thank you!

Additional Slides

Classical Cumulants

- Given a (classical/commutative) random variable $\mathbf{X} \in \Omega$ with moment (m_n)-generating function

$$M_{\mathbf{X}}(t) = \mathbb{E}[e^{t\mathbf{X}}] = \sum_{n=0}^{\infty} \frac{m_n}{n!} t^n \qquad \mathbb{E}[(\,\cdot\,)] := \int_{\Omega} \mathrm{d}\mu(\mathbf{X})\,(\,\cdot\,)$$

"Classical expectation value"

the classical cumulant (c_n)-generating function is

$$C_{\mathbf{X}}(t) = \log(M_{\mathbf{X}}(t)) = \sum_{n=1}^{\infty} \frac{c_n}{n!} t^n \quad .$$

- The cumulants $\{c_n\}$ are related to the moments $\{m_n = \mathbb{E}[\mathbf{X}^n]\}$ through the relation

$$m_n = \sum_{\pi \in P(n)} \prod_{b \in \pi} c_{|b|}$$

where $P(n)$ is the lattice of **all partitions** π of the set $S_n = \{1,\ldots,n\}$. (Tensor-product independence)

- For example: $c_1 = m_1 := \mathbb{E}[\mathbf{X}]$ and $c_2 = m_2 - m_1^2 = \mathbb{E}[\mathbf{X}^2] - \mathbb{E}[\mathbf{X}]^2 =:$ Var$[\mathbf{X}]$.

- If $\mathbf{X}$ and $\mathbf{Y}$ are **classically independent**, then $c_n(\mathbf{X}+\mathbf{Y}) = c_n(\mathbf{X}) + c_n(\mathbf{Y})\ \forall n \geq 1$.

Free Cumulants

- Given a **non-commutative** probability space $\mathscr{P}_{\mathscr{A}}$, the free cumulants $\{\kappa_n\}$ are defined through the cumulant-moment relation

$$m_n = \sum_{\pi \in NC(n)} \prod_{b \in \pi} \kappa_{|b|} \qquad\qquad m_n = \varphi(\mathbf{X}^n)$$

where $NC(n)$ is the lattice of **non-crossing** partitions π of the set $S_n = \{1,\ldots,n\}$. (Free independence)

- **R-transform** [Speicher (1994)]: Define the moment m_n and (free) cumulant κ_n generating functions

$$G(z) = \sum_{n\geq 0} m_n z^n \quad , \qquad R(z) = \sum_{n \geq 1} \kappa_n z^n \quad .$$

Speicher's fundamental equation is $G(z) = R(zG(z))$ which underlies the **R-transform** in free probability theory.

- $\kappa_1 = \varphi(\mathbf{X})$, $\kappa_2 = \varphi(\mathbf{X}^2) - \varphi(\mathbf{X})^2$. In general: $\varphi(\mathbf{X}_1 \cdots \mathbf{X}_n) = \sum_{\pi \in \mathrm{NC}(n)} \prod_{b \in \pi} \kappa_{|b|}(\mathbf{X}_{b(1)}, \ldots, \mathbf{X}_{b(n)})$

- If $\mathbf{X}$ and $\mathbf{Y}$ are **freely independent (free)**, then $\kappa_n(\mathbf{X}+\mathbf{Y}) = \kappa_n(\mathbf{X}) + \kappa_n(\mathbf{Y})\ \forall n \geq 1$.

Relation between Classical and Free Cumulants

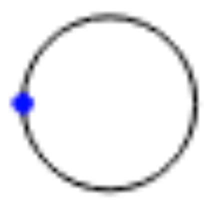
- Partitions of the set $S_n = \{1, \dots, n\}$ can be represented by the following diagrams

Bell numbers: # **all** partitions in S_n:

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$$

Catalan numbers: # **NC** partitions in S_n:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

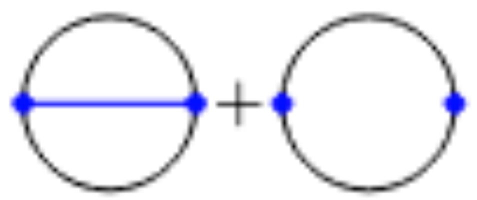


(a) $n=1$

S_1

$m_1 = c_1$

$m_1 = \kappa_1$

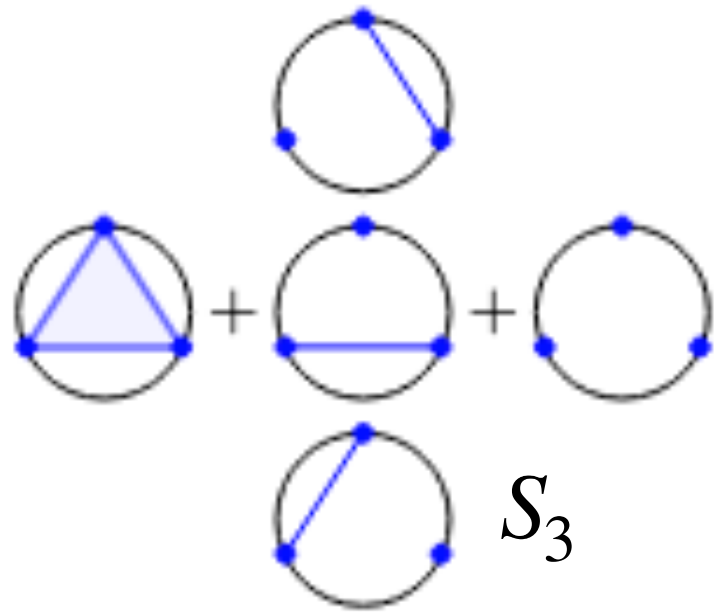


(b) $n=2$

S_2

$m_2 = c_1^2 + c_2$

$m_2 = \kappa_1^2 + \kappa_2$

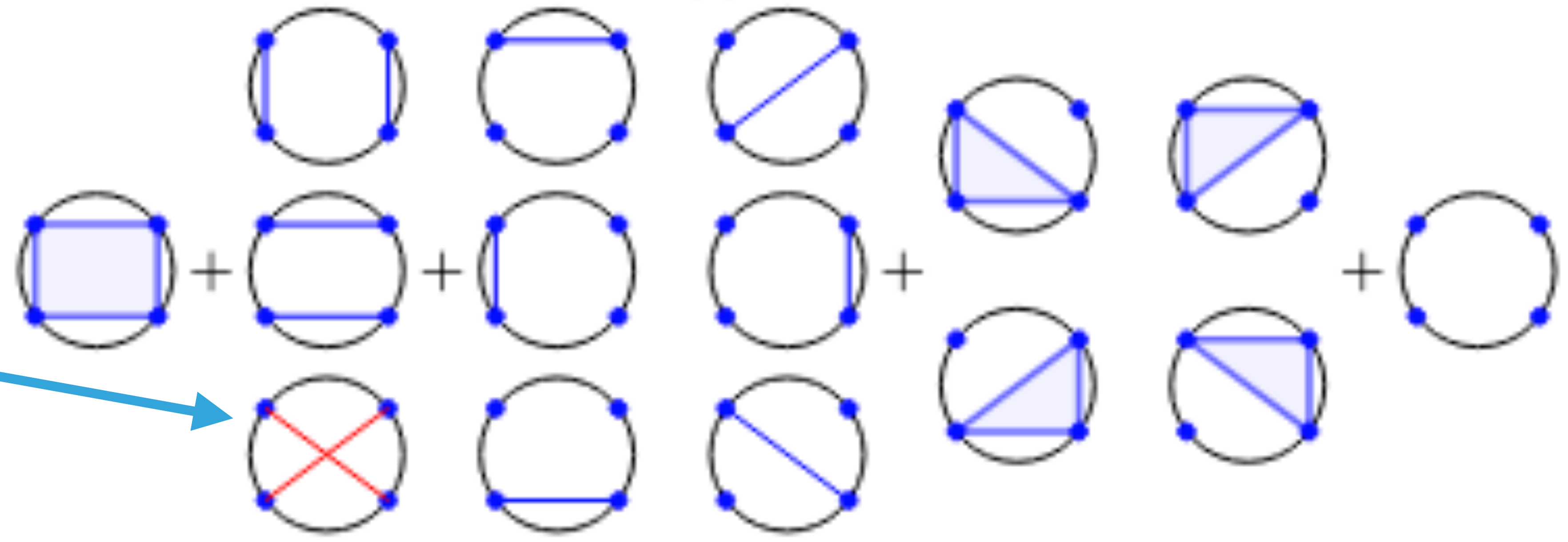


S_3

(c) $n=3$

$m_3 = c_1^3 + 3c_1c_2 + c_3$

$m_3 = \kappa_1^3 + 3\kappa_1\kappa_2 + \kappa_3$



(d) $n=4$

S_4

$m_4 = c_1^4 + 6c_1^2c_2 + 4c_1c_3 + 3c_2^2 + c_4$

$m_4 = \kappa_1^4 + 6\kappa_1^2\kappa_2 + 4\kappa_1\kappa_3 + 2\kappa_2^2 + \kappa_4$

- Crossing /"non-planar" partitions begin to appear from $n = 4$ onward.

Freeness from conjugation by a Haar Random Unitary

- Consider two sequences of definite (deterministic) $N \times N$ matrices $\{A, B\}$ whose spectra (eigenvalue density) $\{\rho_A, \rho_B\}$ are well-defined in the $N \to \infty$ limit $\{\rho_A \to \overline{\rho_A}, \rho_B \to \overline{\rho_B}\}$. Then, for a Haar random unitary $U \in \mathrm{Haar}(\mathrm{U}(N))$

A and $U^\dagger B U$ will become **asymptotically free** for $N \to \infty$

- To see this, consider the mixed correlation functions between A and $B_U := U^\dagger B U$:
  - In general: $\langle A B_U \rangle = \langle A \rangle \langle B \rangle$. This is what we would expect when the operators A and B_U are free.
  - One can also show that (see e.g. [Roberts & Yoshida (2016)] and [Yoshida & Kitaev (2017)])

$$\langle A B_U A B_U \rangle = \frac{N^2}{N^2-1}\left(\langle A^2\rangle\langle B\rangle^2 + \langle A\rangle^2\langle B^2\rangle - \langle A\rangle^2\langle B\rangle^2 - \frac{1}{N^2}\langle A^2\rangle\langle B^2\rangle\right) \xrightarrow[N\to\infty]{} \left(\langle A^2\rangle - \langle A\rangle^2\right)\langle B\rangle^2 + \langle A\rangle^2\langle B^2\rangle$$

Matches the FP prediction in the $N \to \infty$ limit. Also higher order correlation functions.

Time evolution with GUE Hamiltonian and Freeness

- Consider two sequences of definite (deterministic) $N \times N$ matrices $\{A, B\}$. For a unitary constructed from a GUE Hamiltonian H : $\mathscr{U} = e^{-itH}$, one finds the ensemble-averaged two-point correlation function (see e.g. [Cotler, Hunter–Jones, Liu & Yoshida (2017)])

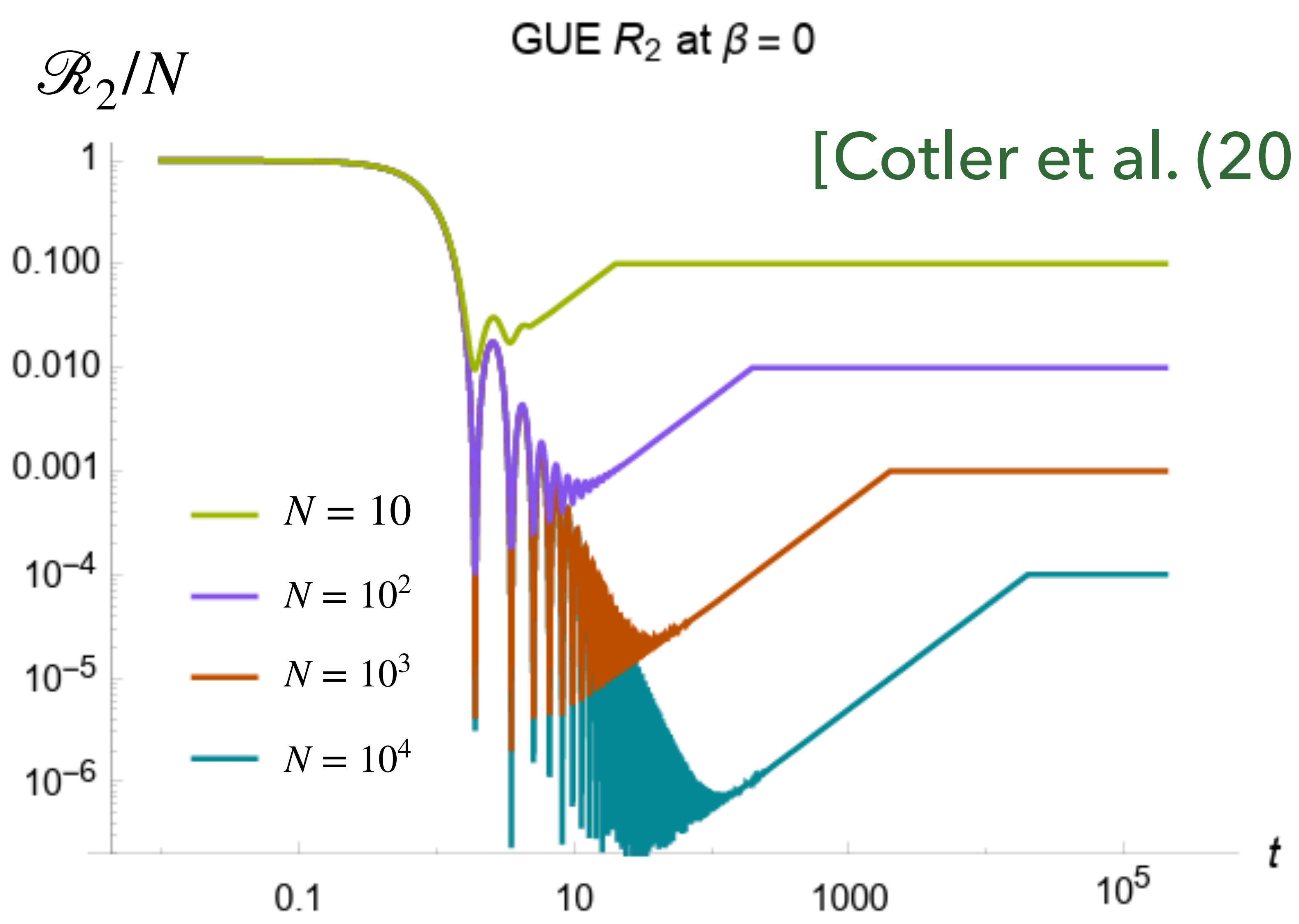
$$\langle AB(t)\rangle_{GUE} = \int \mathrm{d}U \langle A_U e^{itH} B_U e^{-itH}\rangle_{GUE} = \langle AB\rangle \frac{\mathscr{R}_2(t) - 1}{N^2 - 1} + \langle A\rangle\langle B(t)\rangle \frac{N^2 - \mathscr{R}_2(t)}{N^2 - 1}$$

where $\mathscr{R}_2(t) = \langle Z(t)Z^*(t)\rangle = N + N(N-1)\int \mathrm{d}E_a \mathrm{d}E_a \rho(E_a, E_b) e^{(E_a - E_b)t}$ is the spectral form factor (SFF).

- The time at which freeness arises depends on the SFF.

$$\eta(t) := \frac{\mathscr{R}_2(t) - 1}{N^2 - \mathscr{R}_2(t)}$$

$$\mathscr{R}_2(t) \sim \begin{cases} O(N^2) \text{ for } t \sim O(0) \text{ (initial)} \\ O(N^{1/2}) \text{ for } t \sim O(N^{1/2}) \text{ (dip)} \\ O(N) \text{ for } t \sim O(2N) \text{ (plateau)} \end{cases} \qquad \eta(t) \sim \begin{cases} \text{Indet. for } t \sim O(0) \text{ (initial)} \\ O(N^{-3/2}) \text{ for } t \sim O(N^{1/2}) \text{ (dip)} \\ O(N^{-1}) \text{ for } t \sim O(2N) \text{ (plateau)} \end{cases}$$



[Cotler et al. (2017)]

The Free Additive Convolution

- Consider an operator A with well-defined eigenvalue density $\rho_A(\lambda)$ that is compactly supported on $\mathbb{R}$, $\rho_A : I \subset \mathbb{R} \to \mathbb{R}$, the Cauchy transform of $\rho_A(\lambda)$ is defined by

$$G_A(z) := \int_I \mathrm{d}\lambda \frac{\rho_A(\lambda)}{z-\lambda} \quad , z \in \mathbb{C}$$

- The eigenvalue density $\rho_A(\lambda)$ can be recovered from $G_A(z)$ via the Stieltjes inversion formula

$$\rho_A(\lambda) = -\frac{1}{\pi} \lim_{\epsilon \to 0} \left[\Im \left(G_A(\lambda + i\epsilon) \right) \right]$$

- The interval $I \subset \mathbb{R}$ typically contains the origin $\lambda = 0$. Because $\rho_A(\lambda)$ is compactly supported on I, the inverse of the Cauchy transform $G_A^{-1}(z) := B_A(z)$ has a pole at $z = 0$, and can be decomposed as:

$$G_A^{-1}(z) := B_A(z) = \frac{1}{z} + R_A(z) \qquad \text{(choose } R_A(0) = 0\text{)}$$

where $R_A(z)$ is the **R-transform** of $\rho_A(\lambda)$, and corresponds to the regular part of the inverse Cauchy transform.

The Free Additive Convolution

- **Key**: The R-transform $R_A(z)$ is additive for free variables.

- If A, B are **free** variables with corresponding (compactly-supported) eigenvalue densities $\rho_A(\lambda), \rho_B(\lambda)$, then their R-transforms $R_A(z), R_B(z)$ sum:

$$(\rho_{A+B}(\lambda) = \rho_A(\lambda) \boxplus \rho_B(\lambda)) \qquad R_{A+B}(z) = R_A(z) + R_B(z) \qquad (\text{for small } |z| \in \mathbb{C})$$

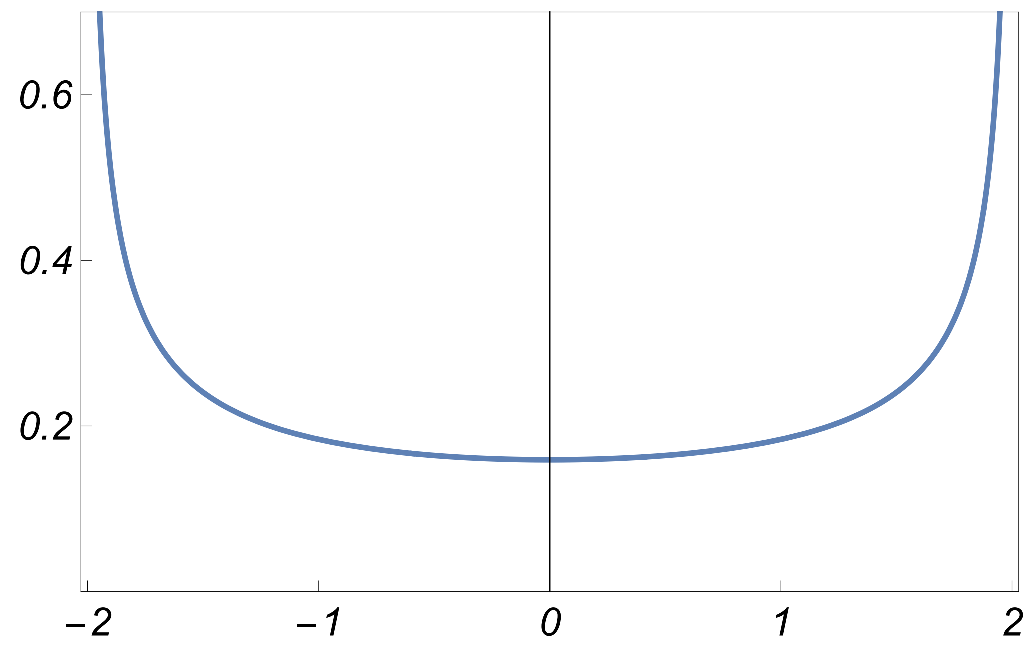
- One then obtains the eigenvalue density of $A + B$ from the Stieltjes inversion formula

$$\rho_{A+B}(\lambda) = -\frac{1}{\pi} \lim_{\epsilon \to 0} \left[\mathfrak{I} \left(G_{A+B}(\lambda + i\epsilon) \right) \right]$$

- **Example Spin-1/2**: The R-transform $R_A(z)$ is additive for free variables.

(1) $\rho_{A,B}(\lambda) = \frac{1}{2}(\delta(\lambda - 1) + \delta(\lambda + 1))$ → (2) $G_{A,B}(z) = \frac{z}{z^2 - 1}$ → (3) $R_{A,B}(z) := B_{A,B}(z) - \frac{1}{z} = \frac{\sqrt{4z^2 + 1} - 1}{2z}$

→ (4) $R_{A+B}(z) = 2R_{A,B}(z) = \frac{\sqrt{1 + 4z^2} - 1}{z}$ → (5) $G_{A+B}(z) = \frac{1}{\sqrt{z^2 - 4}}$ → (6) $\rho_{A+B}(\lambda) = \frac{1}{\pi\sqrt{4 - z^2}}$, $|\lambda| < 1.9277$



Freeness and the Quantum Ergodic Hierarchy