A holographic prescription for dissipative hydrodynamical actions

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Introduction

- Hydrodynamics provides a universal way to describe the long distance, late time dynamics of conserved quantities in many-body quantum systems or quantum field theory.

- Traditionally formulated at the level of equations of motion. E.g. in a relativistic QFT

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P\eta^{\mu\nu} + \dots \qquad\qquad J^\mu = \rho u^\mu + \dots$$

- Energy density, pressure, charge density of functions of local chemical potential and temperature $\mu(t,\vec{x}), T(t,\vec{x})$. Dots indicate expansion in derivatives of $u^\mu(t,x), \mu(t,\vec{x}), T(t,\vec{x})$

- Equations of motion are conservation laws

$$\nabla_\mu T^{\mu\nu} = 0 \qquad\qquad \nabla_\mu J^\mu = 0$$

- Long-standing question had been to obtain an action principle for hydrodynamics. This is non-trivial due to fact hydrodynamics is dissipative beyond ideal order and such action principle must be formulated in Schwinger-Keldysh formalism. Huge progress over last 10-15 years means this is now understood [1].

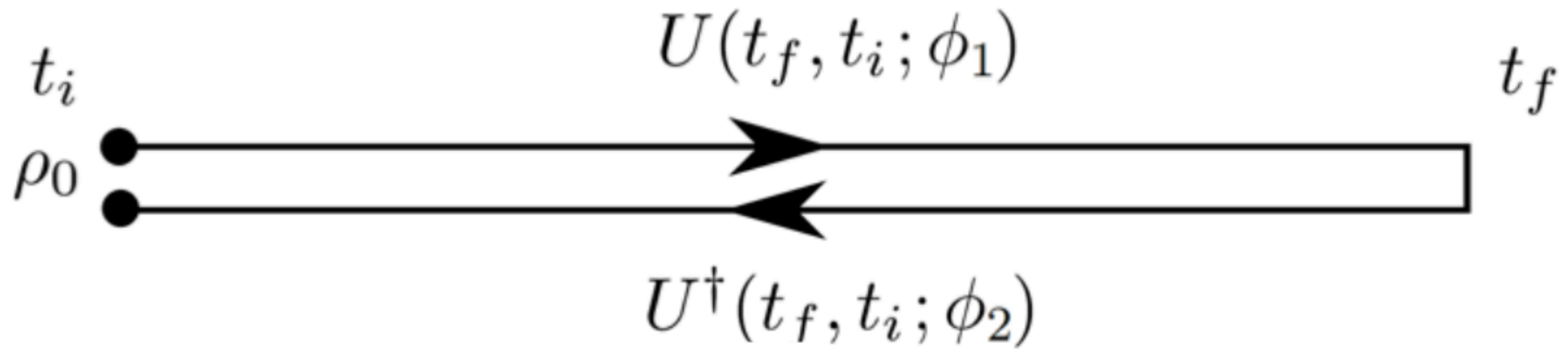
- In case of a single U(1) charge the action can be formulated in terms of fields

$$B^{\sigma}_{\mu}(t,\vec{x}) = A^{\sigma(s)}_{\mu}(t,\vec{x}) + \partial_{\mu}\varphi^{\sigma}(t,\vec{x})$$

- Here $\sigma = 1,2$ is an index referring to the branch of the SK contour, $A^{\sigma(s)}_{\mu}$ are external sources that couple to the hydrodynamic fields φ^{σ}

- For energy-momentum conservation hydrodynamics fields are two copies of diffeomorphism fields X^{σ}_{μ} Couple to external sources $g^{\sigma(s)}_{\mu\nu}$ via pull-back under diffeomorphisms.

SK contour (from Liu & Glorioso, 1805.09331)



[1]. We will be concerned with the formalism of Crossley, Glorioso & Liu (1511.0346 etc). Significant contributions/alternative approaches by others including e.g. Grozdanov & Polonyi, Haehl, Loganayagam & Rangamani.

- These hydrodynamic actions have led to significant new insights including constraints on derivative expansions from first principles and exact loop calculations in hydrodynamics.

- It has been proposed that in maximally chaotic systems such hydrodynamic actions contain exponentially growing modes that are responsible for the exponential growth of OTOCs [1]. Such modes have recently been argued to be related to symmetries of black holes horizons [2].

- The goal of our work is to systematically derive these dissipative hydrodynamic actions from gravity. There is a long history of attempting to do this, but to our knowledge no successful attempts at constructing these actions in the context of dissipative stress tensor dynamics [3].

- Armed with such actions for holographic systems, one can then directly ask whether they possess the conjectured symmetries related to chaos.

[1]. MB, Lee & Liu (1801.0010), MB & Liu (2102.11294)

[2]. Knysh, Liu & Pinzani-Fokeeva (2405.17559).

[3]. An orthogonal approach to SK generating functionals for conserved quantities is described in He, Loganayagam, Rangamani & Virrueta in 2108.03244, 2205.03415 etc.

Review - U(1) charge action

- Long history of attempts to derive hydrodynamic actions for charge and stress tensor hydrodynamics in AdS/CFT [1]. This is done by constructing a partially-on shell action - solving a subset of the bulk equations of motion. Most attempts historically are unable to include the effects of dissipation systematically.

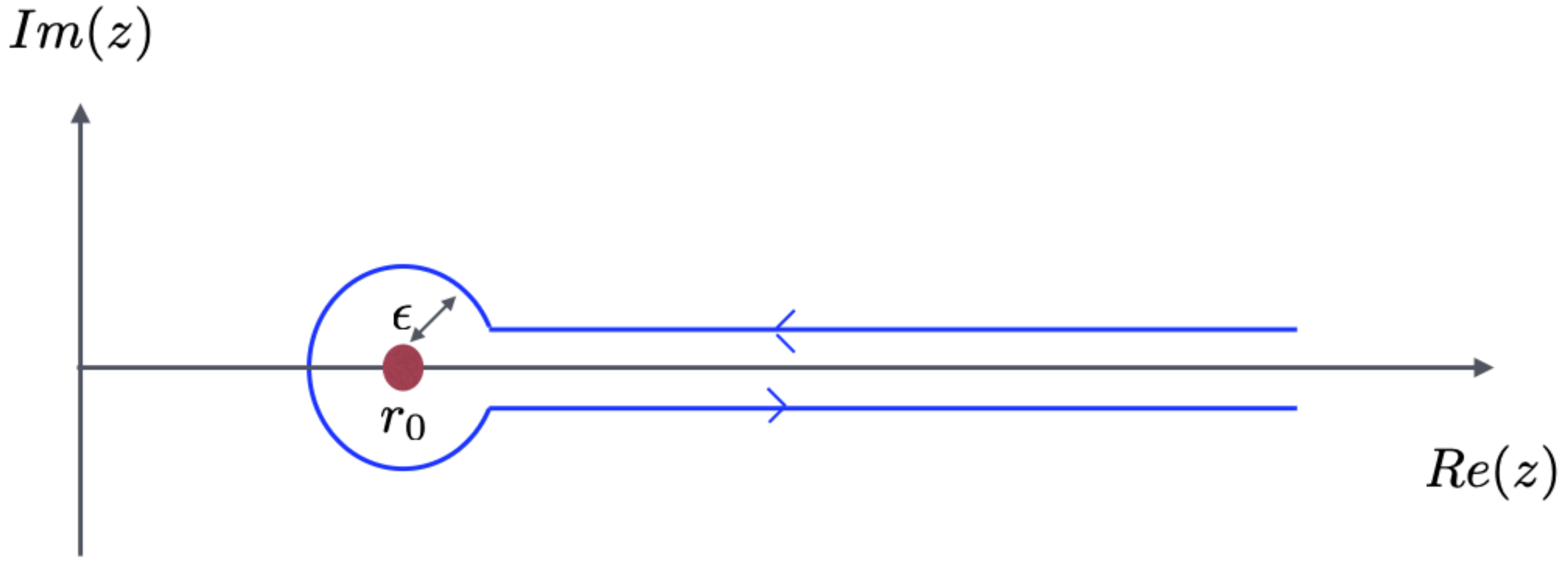
- For a U(1) charge it has been understood how to include dissipation. There are two separate approaches in the literature [2]. We will review the approach of Crossley-Glorioso-Liu which works in terms of an analytically continued black hole geometry.

- Starting point is e.g. AdS4 Schwarzschild black hole

$$ds^2 = -f(z)dv^2 + 2dvdz + z^2(dx^2 + dy^2) \qquad\qquad f(z) = z^2\left(1 - \frac{r_0^3}{z^3}\right)$$

1. E.g. Son & Nickel (1009.3094), Crossley, Glorioso, Liu & Wang (1504.07611), de Boer, Heller & Pinzani-Fokeeva (1504.07616)
2. Glorioso, Crossley & Liu (1812.08785), de Boer, Heller & Pinzani-Fokeeva (1812.06093)

- Consider an analytic continuation of the radial coordinate around the horizon as indicated by the contour



- We want to study fields propagating on this geometry. Conceptually easier to think of this as having two copies of fields for each branch of the contour (now with real r)

$$ds^2 = -f(r)dv^2 + 2dvdr + r^2 d\vec{x}^2 \qquad\qquad f(r) = r^2\left(1 - \frac{r_0^3}{r^3}\right)$$

- For a U(1) gauge field one then considers two gauge fields $A^\sigma_M(v,r,\vec{x})$. It is convenient to decompose these in terms of a piece in radial gauge, and a bulk gauge transformation $\Lambda^\sigma(v,r,\vec{x})$

$$A^\sigma_\mu(v,r,\vec{x}) = C^\sigma_\mu(v,r,x) + \partial_\mu \Lambda^\sigma(v,r,\vec{x}) \qquad\qquad A^\sigma_r(v,r,\vec{x}) = \partial_r \Lambda^\sigma(v,r,\vec{x})$$

$M = v, r, x, y \qquad \mu = v, x, y$

- One now constructs a partially on-shell action by solving a subset of the equations of motion. One solves (in a derivative expansion in ∂_v, ∂_i)

$$\nabla_N F^{N\mu,\sigma} = 0$$

- The remaining Maxwell equation $\nabla_N F^{Nr,\sigma} = 0$ is left unsolved.

- It is also necessary to impose boundary conditions. In the UV you impose standard AdS/CFT boundary conditions $A_\mu^\sigma(v, r \to \infty, \vec{x}) = A_\mu^{\sigma(s)}(v, \vec{x})$

- Near the horizon the partially on-shell solutions for A_i^σ are related in a non-trivial way by analytic continuation as implied by the CGL contour.

- One still needs to specify additional boundary conditions at the horizon for A_v^σ. Conventionally

$$A_v^\sigma(v, r_0 - \epsilon, x) = 0$$

- Computing the Maxwell action $S = -\frac{1}{4}\int d^4x\sqrt{-g}F^{MN}F_{MN}$ for these partially on shell solutions, and taking into account contributions from both branches, gives the SK action

$$S = \int d^3x \left[\frac{r_0}{2}B_t^+ B_t^- + \frac{3ir_0}{4\pi}(B_x^-)^2 + \frac{3ir_0}{4\pi}(B_y^-)^2 + \dots\right]$$

$$B_\mu^\sigma(t,\vec{x}) = A_\mu^{\sigma(s)}(t,\vec{x}) + \partial_\mu \varphi^\sigma(t,\vec{x}) \qquad \varphi^\sigma(v,x) = \Lambda^\sigma(v,\infty,x) - \Lambda^\sigma(v,r_0,x)$$

- The un-imposed equations of motion $\nabla_N F^{Nr,\sigma} = 0$ can be shown to be equivalent to the equations of motion of the hydrodynamic action.

- Summary: Realises a SK action, where hydrodynamic fields are relative gauge transformations between horizon and asymptotic boundaries. There is a non-trivial choice of boundary conditions at the horizon.

Stress-tensor hydrodynamic action

- We have obtained the SK action describing stress-tensor dynamics in an analogous manner. Our approach is valid to quadratic order in perturbations about the thermal state. For AdS4-Schwarzschild we have performed this at zeroth and first order in a derivative expansion [1].

- Starting point is the bulk gravitational action $S = S_{EH} + S_{GH}$ expanded to quadratic order in metric perturbations $\delta g^{\sigma}_{MN}(v,r,x)$

$$S_{EH} = \int d^4x\sqrt{-g}(R+6) \qquad\qquad S_{GH} = \int d^3x\sqrt{-\gamma}(2K-4-R[\gamma])$$

- Metric perturbations on CGL contour $\delta g^{\sigma}_{MN}(v,r,x)$ decomposed in an analogous manner into pieces in radial gauge (in ingoing coordinates) $h^{\sigma}_{\mu\nu}(v,r,x)$ and bulk diffeomorphisms $\zeta^{\sigma}_{M}(v,r,x)$

To simplify equations I am aligning spatial dependence in the x direction WLOG

1. MB, Arpit Das & Richard Davison, work in progress.

- Partially on-shell solutions means (roughly) imposing linearised Einstein equations

$$\delta E^{\sigma}_{xx} = \delta E^{\sigma}_{yy} = \delta E^{\sigma}_{rx} = \delta E^{\sigma}_{rr} = \delta E^{\sigma}_{ry} = \delta E^{\sigma}_{xy} = \delta E^{\sigma}_{vr} = 0$$

- These can be solved order by order in a derivative expansion in ∂_v, ∂_i. The remaining Einstein equations are left unfixed $\delta E^{\sigma}_{vv}, \delta E^{\sigma}_{vx}, \delta E^{\sigma}_{vy}$.

- In the UV (large r) we fix FG gauge and imposes standard boundary conditions

$$\delta g^{\sigma}_{\mu\nu}(v,r,x) \to r^2 \delta g^{\sigma(s)}_{\mu\nu}(v,x) + \dots$$

- Near the horizon we impose radial gauge. The solutions for $\delta g^{\sigma}_{xy}, \delta g^{\sigma}_{m} = \delta g^{\sigma}_{xx} - \delta g^{\sigma}_{yy}$ can be related through non-trivial analytic continuation on the CGL contour.

- One then needs 4 additional boundary conditions (per branch) to fix the remaining fields.

- In analogy with the gauge field case, it is natural to impose

$$\delta g^{\sigma}_{vv}(v, r_0, x) = \delta g^{\sigma}_{vx}(v, r_0, x) = \delta g^{\sigma}_{vy}(v, r_0, x) = 0$$

- To identify a final boundary condition, we consider the variational principle of the quadratic Einstein-Hilbert action. Generically, under a variation there will be a non-trivial boundary term at the horizon. This will vanish if one imposes radial gauge near the horizon and the boundary condition

$$2r_0^2 \partial_r \delta g^{\sigma}_{vv}(v, r_0, x) = \partial_v \delta g^{\sigma}_{xx}(v, r_0, x) + \partial_v \delta g^{\sigma}_{yy}(v, r_0, x)$$

- Note this is a mixed boundary condition [we have not added a Gibbons-Hawking term at the horizon]. This is distinct from previous attempts to compute hydrodynamic actions by imposing Dirichlet boundary conditions at the horizon [1].

[1]. In particular to de Boer, Heller & Pinzani-Fokeeva (1504.07616)

- These boundary conditions completely fix $h^{\sigma}_{\mu\nu}(v,r,x)$. Evaluating the resulting partially on-shell action gives a SK action that depends only on sources $\delta g^{\sigma(s)}_{\mu\nu}(v,x)$ and hydrodynamic fields $\xi^{\sigma}_{\mu}(v,x)$
- The hydrodynamic fields correspond to relative diffeomorphisms between the horizon and asymptotical boundaries].

$$\xi^{\sigma}_{t}(v,x) = [-\zeta^{v,\sigma}(v,r,x)]^{\infty}_{r_0}$$

$$\xi^{\sigma}_{x}(v,x) = [\zeta^{x,\sigma}(v,r,x) - \partial_x \zeta^{v,\sigma}(v,r,x)/r]^{\infty}_{r_0}$$

$$\xi^{\sigma}_{y}(v,x) = [\zeta^{y,\sigma}(v,r,x)]^{\infty}_{r_0}$$

- It is convenient to define the pull-backs of the external metric on each branch under these diffeomorphisms. Suppressing the branch label these are defined by

$$\mathcal{B}_{\mu\nu}(x) = \frac{\partial y^{\alpha}}{\partial x^{\mu}}\frac{\partial y^{\beta}}{\partial x^{\nu}} g^{(s)}_{\alpha\beta}(y(x)) - \eta_{\mu\nu},$$

$$y^{\mu} = x^{\mu} - \xi^{\mu}$$

$$g^{(s)}_{\mu\nu} = \eta_{\mu\nu} + \delta g^{(s)}_{\mu\nu}$$

- The resulting hydrodynamic action can be expressed (to quadratic oder in perturbations, and first order in derivatives) as

$$S_{\text{hydro},0}[\mathcal{B}^{\sigma}_{\mu\nu}] = r_0^3 \int d^3x \Big(\mathcal{B}^-_{tt} + \frac{1}{2}\mathcal{B}^-_p + \mathcal{B}^+_{tx}\mathcal{B}^-_{tx} - \frac{1}{4}\mathcal{B}^+_m\mathcal{B}^-_m + 2\mathcal{B}^+_{tt}\mathcal{B}^-_{tt} + \frac{1}{2}\mathcal{B}^+_{tt}\mathcal{B}^-_p \\ + \frac{1}{2}\mathcal{B}^+_p\mathcal{B}^-_{tt} - \mathcal{B}^+_{xy}\mathcal{B}^-_{xy} + \mathcal{B}^+_{ty}\mathcal{B}^-_{ty} + \frac{3i}{16\pi}(\mathcal{B}^-_m)^2 + \frac{3i}{4\pi}(\mathcal{B}^-_{xy})^2 \Big).$$

$$S_{\text{hydro},1}[\mathcal{B}^{\sigma}_{\mu\nu}] = r_0^2 \int d^3x \left(\mathcal{B}^+_{xy}\partial_v\mathcal{B}^-_{xy} + \frac{3i}{2\pi}\mathcal{B}^-_{ty}\partial_x\mathcal{B}^-_{xy} - \frac{1}{4}\mathcal{B}^-_m\partial_v\mathcal{B}^+_m + \frac{3i}{4\pi}\mathcal{B}^-_{tx}\partial_x\mathcal{B}^-_m \right).$$

- The equations of motion of this action are precisely the un-imposed Einstein equations, and are equivalent to energy momentum conservation.

- Integrating out the hydrodynamics fields gives a generating functional for hydrodynamic response functions. These precisely match form expected from variational hydrodynamics [1].

[1]. Kovtun (1205.5040).

Conclusion

- We derived a SK action for the stress tensor hydrodynamics dual to perturbations of the AdS4 Schwarzschild black hole, to first order in a derivative expansion.

- We believe this is a significant technical breakthrough. To our understanding, this is the first example of a dissipative hydrodynamic action for stress tensor dynamics to have been consistently derived using AdS/CFT [1].

- It should be straightforward to generalise our approach to many other holographic systems (charged black holes etc).

- A key goal is to identify if such SK actions exhibit conjectured symmetries related to chaos [2] and horizon symmetries [3]. Ultimately this requires going beyond a derivative expansion.

[1]. We believe there are errors in the analysis of Bu & Sun (2503.03374).

[2]. MB, Lee & Liu (1801.0010).

[3]. Knysh, Liu & Pinzani-Fokeeva (2405.17559).