Level Rank duality in Quantum Mechanics

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References

- Talk based on: ArXiv 26??.????? D. Radicevic, I. Haldar, L. Janagal, S.M. , C. Patel, N. Prabhakar and P. Ray

Contents

- Quantum field theories are key structures in theoretical physics. The standard model of particle physics is a QFT. In statistical mechanics, the study of phase transitions is set within the framework of QFT. And several condense matter systems are best understood within the framework of QFT.
- For all the reasons mentioned above, QFTs are very well studied - and relatively well understood - mathematical structures. Despite this intensive study, however, some things remain mysterious.
- In particular, while we now have very good evidence that strong weak coupling dualities are ubiquitous among Quantum Field theories, we still - in 2026 - dont understand, in detail, why (and how) this is the case.

Introduction II

- There, however, some exceptions. For example, topological field theories enjoy (relatively well understood) invariance under strong-weak coupling type dualities.
- E.g. consider pure Chern Simons theories. The $SU(N)_k$ Chern Simons theory is defined by the action

$$\mathcal{S}_{SU(N)_k}[y] = \frac{k}{4\pi} \int d^3x\, \epsilon^{\mu\nu\rho}\, \mathrm{tr} \left(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho \right) , \quad (1)$$

  where A is a traceless $N \times N$ matrix.
- This theory (and its $U(N)$ cousin) was famously solved by Witten. The exact solutions of these theories reveal an interesting fact; the $SU(N)_k$ Chern Simons theory listed above is, actually, identical to the $U(|k|)_{-N,-N}$ Chern Simons theory. The topological duality described above is of the 'strong weak' coupling variety, as the level rank flip inverts the t'Hooft coupling $\frac{N}{k}$.

Introduction III

- In this talk, we attempt - in examples - to use our understanding of topological dualities to ‘derive’ a physical duality of propagating QFTs in a particularly simple kinematical limit.
- The conjectured ‘physical’ duality we study takes the form LHS=RHS, where:
- LHS: $U(N)_{k,k}$ CS theory coupled to a fundamental multiplet of (mass deformed) Wilson Fisher bosons.
  RHS: $SU(|k|)_{-N}$ CS theory coupled to (an otherwise free) fundamental multiplet of massive fermions.
- The two theories above are now real (as opposed to topological) field theories, with genuine propagating degrees of freedom. While there is a great deal of evidence for the dualities described above between ‘genuine’ quantum field theories, we have no real understanding of them. E.g. how are bosons dual to fermions?

- On integrating out the massive matter, the two theories above reduce to the topological pure Chern Simons theories, and the duality described above reduces to the well understood topological duality.
- In this talk, we will attempt to learn something about the (poorly understood) QFT duality by starting with the (much better understood) topological duality and 'flowing upwards'.

Introduction V

- In order to to flow a little bit up from the deep IR, we study a modified - but still tractable - low energy limit that retains some propagating degrees of freedom.
- We focus on configurations with a fixed collection of particles, and zoom in to energies that are just larger than the sum of masses of these particles. At such energies, the particles all move slowly compared to the speed of light and are well described by non relativistic quantum mechanics.
- We construct the non relativistic system (Hilbert space, Schrodinger equation) on both sides of the duality, and study the relationship between them.

Review of Quantization of CS with fixed sources

- It is useful to first recall Witten's classic quantization of Chern Simons theory in the presence of fixed sources.
- Witten identified the Hilbert Space of CS theory interacting with a collection of fixed sources - in representations R_1, $R_2 \ldots R_n$ of the gauge group - with the d dimensional vector space of WZW 'conformal blocks' with primary fields at the same location and in the same representations.
- When k is large enough, the space of 'conformal blocks' is just the space of gauge singlets one can construct out of the representations $R_1 \ldots R_n$. At smaller values of k, the space of conformal blocks is a subspace (the Gepner Witten subspace) of this space of singlets. Note that this 'frozen particle' Hilbert space is always finite dimensional. While the dimensionality of the Gepner Witten subspace is independent of $\mathbf{z}$, its 'orientation' changes as we vary particle locations.

Berry's connection

- In other words, Witten's 'frozen particle' Hilbert Space depends on the locations of the insertion particles, $\mathbf{z}$. We need a connection - a Berry's connection - to compare states in Hilbert Spaces at different values of $\mathbf{z}$.
- A useful Berry's connection is obtained from the construction of states, as a path integral on the solid ball in the presence of a tangle of Wilson Lines, with end points in the given representations and at the given locations.
- We define the Berry's connection by the condition that a state defined by a tangle with given end points - and a second state defined by the same tangle, with infinitesimally separated insertion locations (and framing vector rotated by the spin connection with spin h) - are parallel transports of each other. This Berry's connection lies in $GL(d)$, (recall d is the dimensionality of the space of conformal blocks: e.g. when the insertions $FFAA$, $d = 2$ when $k > 1$, and $d = 1$ when $k = 1$.)

Local flatness of the Berry's connection

- It follows from the topological nature of the Chern Simons path integral that the local curvature of this Berry's connection vanishes. It follows that the $GL(d)$ gauge field is locally given by

$$\mathcal{A} = \mathcal{A}^{\rm Berry} - \mathcal{A}^s_{\mu_i} = -V^{-1}(\partial_{z_i} V) \tag{2}$$

  where $V(\mathbf{z})$ is some $GL(d)$ matrix.
- The discussion of the previous para holds only locally, and at generic points. From our path integral definition of the Berry's curvature it that the connection $\mathcal{A}^{\rm Berry}$, hence $\mathcal{A}$, has δ function curvatures at the locations where two insertions collide. It follows that the function V is multivalued as one point is taken around another.
- The full gauge field also includes a $U(1)$ 'spin connection' factor $h_i \omega_i(\mathbf{x}_i)$ for each insertion (this accounts for the parallel transport of the framing vector).

The KZ connection

- A connection with all these properties is familar from the study of CFT. Recall that in the study of correlators of WZW theory, conformal blocks obey all the Gepner Witten constraints. Moreover, their variation with **z** is determined by the KZ equations

$$D_{\mu_i}\mathcal{B}_{\beta'}(\mathbf{z}) = 0,$$

$$D_{\bar{z}_i} = \partial_{\bar{z}_i} \ , \quad D_{z_i} = \partial_{z_i} + \mathcal{A}^T_{z_i\,KZ} \ , \quad \text{with} \quad \mathcal{A}^T_{z_i\,KZ} = \frac{1}{\kappa}\sum_{j\neq i}\frac{T^a_{R_i}T^a_{R_j}}{z_i - z_j} \ ,$$

- This equation can be reinterpreted as the condition of covariant constancy of the conformal blocks, provided we interpret A^T as a $GL(d)$ connection.

Berry's equals KZ

- It is easy to verify that the curvature of $\mathcal{A}^T_{KZ}$ vanishes locally at generic points.
- Moreover, the flux of $\mathcal{A}^T$ (when two insertions collide) matches that of the Berry's connection. Intuitively, this works as follows. The Berry's phase described above is simply a consequence of the flux trapping property of Chern Simons theory. The Chern Simons EOM tells us that Particle j traps a flux proportional to $\frac{T^a_j}{\kappa}$. As particle i rotates around j , it thus picks up a phase proportional to $\frac{T^a_{R_i} T^a_{R_j}}{\kappa}$, in agreement with the KZ connection.
- We conclude that $\mathcal{A}^T$ is thus an algebraic construction of the Berry's connection, that we had earlier described more abstractly. Note that, by definition, the familiar holomorphic blocks, $\mathcal{B}_{\beta'}(\mathbf{z})$, of WZW theory are locally covariantly constant, and so (from a path integral viewpoint) represent continuous variations of tangles.

The Inner Product

- From the path integral viewpoint one computes $\langle\phi|\psi\rangle$ as follows. One reflects the path integral that defines $|\phi\rangle$ about the boundary of the ball, and complex conjugates all representations. One then glues this path interal to the one that defines $|\psi\rangle$.
- Topological invariance guarantees that simultaneous variation of the end points **z** (for the states $\langle\phi|$ and $|\psi\rangle$) , leave the inner product $\langle\phi|\psi\rangle$ unchanged. But we have defined the connection to ensure that $\langle\phi|$ and $|\psi\rangle$ are covariantly constant under such variations. Consquently, the same must be true of the inner products.
- Concretely, let $\mathcal{I}_\alpha(\mathbf{z})$, $\alpha = 1 \ldots d$ represent a basis of blocks. Then the matrix

$$Q_{\alpha^*\beta}(\mathbf{z}) = (\mathcal{I}_\alpha(\mathbf{z}), \mathcal{I}_\beta(\mathbf{z}))$$

is covariantly constant. This is globally (rather than locally) true.

Inner product matrix in the basis of holomorphic blocks

- Locally, we can always choose the our basis blocks $\mathcal{I}_\alpha$ to equal $\mathcal{G}_\alpha$. As $\mathcal{G}_\alpha$ are covariantly constant, the KZ connection vanishes in this choice of basis (gauge). With this choice of basis, it thus follows that the inner product matrix Q is constant.
- In equations

$$\tilde{Q}_{\alpha^*\beta} = (\mathcal{G}_\alpha(\mathbf{z}), \mathcal{G}_\beta(\mathbf{z}))\,,$$

where $\tilde{Q}$ is a constant (i.e. $\mathbf{z}$ independent) matrix.

Dynamical Particles: Wavefunctions

- If we now allow our particles to be dynamical, one might think that our Hilbert space is now the space of conformal block valued functions

$$\psi^\alpha(\mathbf{z})\mathcal{I}_\alpha(\mathbf{z}). \qquad (3)$$

- However, there space of conformal blocks does not have a distinguished basis. We are always free to perform a local change of basis

$$\mathcal{I}' = W\mathcal{I}$$

under which

$$\psi' = \psi W^{-1}$$

More precisely, therefore, our space of wave functions is a field that transforms in the fundamental representation of the $Gl(d)$ gauge group. The base space for this charged scalar field is the set of locations of all particles.

Dynamical particles: covariant derivative

- The covariant derivative on our space uses a linear combination of of the KZ and spin connections, and takes the explicit form

$$D_{\mu_i} = \partial_{\mu_i} + \mathcal{A}^T_{\mu_i} + s^i \omega_{\mu_i}$$

where

$$s_i = s_i^{\rm int} + h_i$$

- s_i^{int}, the intrinsic spin of the i^{th} particle, is external data we need to specify. s_i^{int} is zero for scalar particles, $\frac{m_i}{2}$ for Dirac particles, etc. In contrast, the 'statistical' spin h_i is the spin the particle picks up by virtue of its interation with the Chern Simons gauge field. Value of stat spin clear from 2d viewpoint. Can also be seen directly in 3d from a Noether charge analysis on a spherical lump of charge (contribution somewhat analogous to $\vec{E} \times \vec{B}$ in 4 dimensions).

Quantization of Flux

- The fact that Chern Simons theory renormalizes the spin of all particles is crucial; it makes the covariant derivative

$$D_{\mu_i} = \partial_{\mu_i} + \mathcal{A}^T_{\mu_i} + s^i \omega_{\mu_i}$$

$$s_i = s_i^{\rm int} + h_i$$

well defined on compact manifolds.

- Consider, for example, a two particle configuration, with the first particle in representation R, and the second in representation $\bar{R}$. In this case the gauge field A^T - that appears in the kinetic term for the particle R - includes a flux equal to $4\pi h_R$ delta function localized at the position of particle $\bar{R}$. This cancels against the h_R contribution to $4\pi s_R$ (spin time flux of curvature of the sphere), ensuring that the effective flux seen by this particle is an integer multiple of 2π (recall $s_{\rm int}$ is always a half integer).)

Dynamical particles: an 'equivalence principle'

- Consider a bunch a particles interacting via Chern Simons exchange. The Chern Simons equation of motion $2\pi kF = -*J$ tells us that the $U(N)$ field strength is delta function localized on particle world lines. It follows that particles are locally free. They 'see' the Chern Simons interaction only when they wind around each other. Simple example: motion of a particle in 2d around a point like solenoid at the origin. Local 'freeness' manifest in the (local) $A_\phi = 0$ gauge.
- In our dynamical problem, the analogue of the $A_\phi = 0$ gauge is a gauge in $\mathcal{A}^T = 0$. We have already seen that such a gauge exists; it is simply the gauge in which the basis blocks $\mathcal{I}_\alpha$ are chose to be the conformal blocks that obey the KZ equation. In this gauge, the Hamiltonian of our problem must reduce to that for a collection of locally free particles, and so must take the form $\sum_i \frac{\nabla_i^2}{2m_i}$ (plus possible curvature corrections).

Dynamical particles: Hamiltonian

- We then obtain the Hamiltionian of our system in an arbitrary gauge by covariantizing the free Hamlitonian:

$$H = \sum_i -\frac{1}{2|m_i|}\left(D_i^2\psi + \frac{a_i R}{2}\psi\right) ,$$

- We have allowed for each particle to couple to the curvature in an arbitrary manner because this is a two derivative coupling that is present even when the particles are free. Infact if one takes the non relativistic limit of the Dirac, or massive spin one equation, one finds $a_i = |s_i|$. The chioice $a_i = |s_i|$ may, therefore have some special properties that we have not yet understood. To be safe, for now we leave the couplings a_i arbitrary.

Dynamical particles: inner product for sections

- We need an inner product on sections. Our inner product must be positive definite and gauge covariant. Moreover our ‘equivalence principle’ tells us that it must locally reduce to the product of the free particle inner product and the inner **z** independent inner product governed by the matrix $\tilde{Q}$ on the ‘column space of blocks.
- The covariant version of the statement is

$$\langle\psi|\chi\rangle = \int \prod_i \sqrt{g_i} dz^i d\bar{z}^i \psi^\dagger Q \chi \tag{4}$$

where (4) applies holds in every ‘gauge’ (i.e. for every choice $\mathcal{I}(\mathbf{z})$ of basis vectors in space of blocks).

Hermiticity and Boundary Conditions

- Using the convariant constancy of Q, it is not difficult to formally demonstrate that the Hamiltonian of the previous slide is Hermitian. However there is a subtlety. The proof works if one can integrate by parts and ignore boundary terms. This is potentially problematic when two particles approach each other. We pause to study this point.
- It is convenient to work in a gauge in which A^T vanishes and then later transform back to other gauges. To study the approach of i and j to each other, we choose our basis of blocks to diagonalize i, j fusion. Consider a basis element in which i and j fuse to m. For this element, the monodromy (on taking R_i arond R_j) is given by

$$e^{2\pi i \nu_{ij}^m}, \quad \nu_{ij}^m = h_i + h_j - h_m - [h_i + h_j - h_m] \tag{5}$$

Boundary conditions

- In the limit that r is small, it is easy to check that the most general solution to the wave equation takes the form

$$\psi = \sum_n \psi_n(r) e^{i(n+\nu)\theta}, \qquad \psi_n(r) = a_n r^{n+\nu} + \frac{b_n}{r^{n+\nu}} \qquad (6)$$

- For $n > 1$ the condition of square integrability (well definedness of the norm) sets $b_n = 0$. b_0 is allowed to be zero. However it turns out that our proof of Hermiticity of the Hamiltonian goes through if and only if the ratio $\frac{b_0}{a_0}$ is the same for all wave functions. This ratio - for every choice of i, j, m - thus has to be specified once and for all, and is part of the definition of our Hilbert Space.
- Physically, this ratio contains information about non Chern Simons 'contact type' interactions between particles. Note this ratio has dimension mass$^{-2\nu}$. When our QM is obtained as the low energy limit of a UV QFT, we expect that this ratio vanishes generically. 'RG flow Univerality'.

Level Rank Duality

- Our construction of our multi particle quantum mechanics is now complete. We can construct a quantum mechanics with particles in representations R_i in the $U(k)_{N,N}$ theory. We can separately construct the quantum mechanics of particles in representations $\tilde{R}_i$ (level rank dual reps) in the $SU(N)_k$ theory. Is there a relationship between these distinct quantum systems?
- In order to address this question, we review a standard construction in the study of level rank duality for topological CS theories. Consider a 2d theory of Nk complex chiral fermions. This theory enjoys invariance under $U(Nk)_1$. The $U(Nk)_1$ primaries of this theory are product of (at most Nk) ψ or the product of at most Nk $\bar{\psi}$.

Intuition behind the map

- In what follows, we will use correlators in the theory of NK free chiral fermions to produce a map between wave functions on the two sides of the duality.The rough idea goes as follows.
- Consider a wave function for a collection of fundamental and antifundamental $SU(N)_k$ particles. If there are m such particles, such a wave function carries m $SU(N)$ indices. Now consider Chiral correlators of Nk free fermions in two dimensions. Each fermion can be thought of as ψ^{ab}, where a is an $SU(N)$ index while b is a $U(k)$ index. If one now multiplies this wave function with the appropriate free fermion correlator, and contracts away all $SU(N)$ indices, one is left with an object that carries only $U(k)$ indices. Atleast at the level of index structure, this object has looks like a wave function for m particles in the dual $U(k)$ theory. It turns out also to have all other properties needed to view it as a wave function in the $U(k)$ theory. In more detail:

Embedding in $U(Nk)_1$

- Now $U(Nk)_1$ has a $SU(N)_k \times SU(k)_N \times U(1)_{Nk}$ subgroup. This subgroup is complete, in the sense that the sum of the Sugawra central charges of the subgroup equals the Sugawara central charge of $U(Nk)_1$
- Any primary of $U(Nk)_1$ can be decomposed into a finite sum over $SU(N)_k \times SU(k)_N \times U(1)_{Nk}$ representations. In some cases the $SU(N)_k \times SU(k)_N \times U(1)_{Nk}$ primaries are also $U(Nk)_1$ primaries. In other cases, the $SU(N)_k \times SU(k)_N \times U(1)_{Nk}$ primaries are $U(Nk)_1$ descendents at level n_i.

Level Rank Branching Rules

- It follows that the $U(Nk)_1$ blocks - simply free fermion correlators - can be decomposed into sums of products of $SU(N)_k \times SU(k)_N \times U(1)_{Nk}$ blocks. In equations

$$\mathcal{P}_i\left(\Psi_{\vec{p}}(\mathbf{z})\right) = \Phi_{\vec{p}}(\mathbf{z}) \sum_{\alpha,\beta} C^{\alpha\hat{\beta}} \mathcal{B}_{\alpha}^{R_i}(\mathbf{z}) \hat{\mathcal{B}}_{\hat{\beta}}^{\tilde{R}_i}(\mathbf{z}). \quad (7)$$

  Here $\mathcal{P}_i$ is the projector onto the $SU(N)_k \times SU(k)_N \times U(1)_{Nk}$ representations of interest (the projection works insertion by insertion for each insertion). $C^{\alpha\beta}$ are constants (this follows from the fact that this subgroup is complete).

- In the simplest case, a single $U(Nk)_1$ fermion branches into the product of a $SU(N)_k$ fundamental and a $U(k)_N$ fundamental (no sum over representations), and the projector $\mathcal{P}_i$ in (7) is not needed.

Relation between weights

- We can now choose to absorb the $U(1)$ blocks into the $SU(N)_k$ blocks. This turns these into $U(N)_{k,k}$ blocks.
- Working with the i^{th} particle, let us denote the dimension of the $U(Nk)_1$ primary by H_i. Let the $U(N)_{k,k} \times SU(k)_N$ primary be a $U(Nk)_1$ descendent at level n_i. Let the $U(N)_{k,k}$ and $SU(k)_N$ weights of the branching primaries be denoted by h_i and $\hat{h}_i$. Then it follows by matching weights (see (7)) that
$$H_i + n_i = h_i + \hat{h}_i \tag{8}$$
- In the simplest case, a single $U(Nk)_1$ fermions decomposes into the product of a fundamental $SU(N)_k$ and a fundamental of $U(k)_N$. In this case $H = \frac{1}{2}$ and $n = 0$, so $h + \tilde{h} = \frac{1}{2}$.

The covariantly constant tensor $C^{i\hat{j}}$

- The branching equation on the previous slide may be rewritten as follows. We we insert $\mathcal{B}_\alpha = \mathcal{B}^i_\alpha \mathcal{I}_i$ (and a similar equation for the dual blocks) into this equaton to obtain

$$\mathcal{P}_i\left(\Psi(\mathbf{z})\right) = \sum_{i,\hat{j}} C^{i\hat{j}}(\mathbf{z})\mathcal{I}_i\,\hat{\mathcal{I}}_{\hat{j}}$$

$$C^{i\hat{j}}(\mathbf{z}) = C^{\alpha\hat{\beta}}\mathcal{B}^i_\alpha(\mathbf{z})\hat{\mathcal{B}}^{\hat{j}}_{Y^T\hat{\beta}}(\mathbf{z})$$

- In the special choice of basis blocks $\mathcal{I}_\alpha = \mathcal{G}_\alpha$, $C^{ij} = C^{\alpha\hat{\beta}}$. Since $C^{\alpha\beta}$ is constant, and since the KZ connection vanishes in this choice of basis, it follows that C^{ij} is covariantly constant.
- More mathematically, $C^{i\hat{j}}$, defined above, is a section in the product of original and level rank dual bundles. It is covariantly constant because it reduces to the (numerically constant matrix) $C^{\alpha\hat{\beta}}$ in irregular gauge.

Map between wave functions

- The formula
$$\phi^{\alpha^*} = C^{*\alpha^* \hat{\beta}^*} \hat{Q}_{\hat{\beta}^* \hat{\gamma}} \hat{\phi}^{\hat{\gamma}} \tag{9}$$
maps dual wave functions to (the conjugates of) usual wave functions. As Q and C are both covariantly constant, it follows immediately that if $\hat{\phi}^{\hat{\gamma}}$ obeys the dual Schrodinger equation, the complex conjugate of ϕ^{α^*} obeys the regular Schrodinger equation.
- The spin of the RHS of this map is $\hat{s}_i^{int} + \hat{h}_i$. The spin of the LHS equals $-\hat{s}_i^{int} - h_i$ (recall complex conjugation interchanges z and $\bar{z}$ and so flips spin). Consequently, spins match on both sides of (9) only if $s_i^{int} + \hat{s}_i^{int} = -(h_i + \hat{h}_i) = -H_i - n_i$ (we have used (8)).
- In other words, the formula (9) maps a wave function of $SU(N)_k$ charged particles with intrinsic spin $s_i^{\rm int}$, to wave functions for $U(|k|)_{N,N}$ particles with a different but related intrinsic spin. In the special case $H_i = \frac{1}{2}$, the two spins differ by $\pm\frac{1}{2}$.

Invertibility of the map

- In matrix notation, the map above is

$$\phi^* = C^* \hat{Q} \hat{\phi} \tag{10}$$

Upon complex conjugating

$$\phi = C \hat{Q}^* \hat{\phi}^* \tag{11}$$

- Since original and level rank dual sections are on equal footing, we have a similar map from regular sections to the complex conjugate of dual sections.

$$\hat{\phi}^* = C^\dagger Q \phi \tag{12}$$

It is natural to expect (12) to be the same equation as (11) (so that the map between sections is Z_2). This is the case provided

$$C \hat{Q}^* C^\dagger Q = I \tag{13}$$

It is natural to expect that (13) is true as a mathematical identity. Is this indeed the case?

Invertibility and the inner product

- While we suspect that a proof of (13) exists somewhere in the mathematics literature, we have not been able to find it. We have, however, found some independent evidence for the correctness of (13) proceeding as follows.
- By rearranging (13) we find an expression for $\hat{Q}$ in terms of C, $C^\dagger$ and Q. The RHS of this relation is covariantly constant. We independently know that $\hat{Q}$ is covariantly constant, i.e. is constant in the basis of blocks. If we could show that globally well defined bilinears of blocks and their complex conjugates are unique- i.e. that correlators in diagonal WZW theory are unique - we would have effectively proved (13).
- By explicit computation, we have verified this uniqueness for arbitrary representations in $SU(2)_k$ WZW theory, as well as for the special case of two fundamental and two antifundamental insertions in the general $SU(N)_k$ theory, giving some evidence for (13).

Conjecture and inner product

- For the general case, we do not yet have (or have not yet been able to locate) a proof of (13). We proceed conjecturing that this mathematical identity always holds.
- Using (13), it is a simple matter to show that our map between Hilbert Spaces preserves the inner product, and therefore matrix elements of the Hamiltonian.
- It follows, therefore, that (11) map is a map between the two quantum mechanical Hilbert spaces that preserves inner products and maps the two Hamiltonians to one another. I.e. that the two quantum problems are identical.

Interplay with exchange statistics

- Recall that any level rank pair of representations, R_i and $\hat{R}_i$, descends from the branching rule of some primary operator of $U(Nk)_1$ theory. Every such primary is a product of fermion fields. If the $U(N)_k$ primary is made up of an odd number of fermions, correlators of this primary are odd under interchange of two identical insertions. On the other hand, if the primary is built out of an even number of fermions, correlators of this primary are even under interchange of identical insertions.

Bose/Fermi or Bose/Bose - Fermi/Fermi dualities

- The observation of the previous slide can be used to show that our map between sections preserves statistics it the representations descend from an even number of fermions, but interchanges symmetry with antisymmetry if the representations descend from primaries with an odd number of Fermions. Restated, our map between Hilbert spaces is either of the Bose-Bose, Fermi-Fermi sort, or of the Bose-Fermi, Fermi-Bose sort, depending on reps.
- Let us now turn to the example we started the talk with. Let our $U(Nk)_1$ primary be simply the Fermion field itself. In this case the branching rules are really simple: this $U(Nk)_1$ representation simply equals the product of a $U(N)_{k,k}$ fundamental and the $SU(k)_N$ fundamental. Using the rule deduced above, we conclude that the duality is of the Bose Fermi variety. On the other hand, two box symmetric, two box antisymmetric or adjoint representations enjoy Bose -Bose or Fermi-Fermi dualities.

Parity

- One irritation in the construction above, is that our map takes sections to complex conjugate sections. Conjugate sections carry the opposite fluxes as compared to ordinary sections (can be seen from the KZ connection). This can be undone by performing a parity transformation, which flips the sign of the KZ connection. This parity tranfromation flips all spins. After performing this flip, the map between spins becomes

$$s_i^{int} = \hat{s}_i^{int} - H_i - n_i \qquad (14)$$

where the hatted side is assumed to have negative level.

- In summary, duality works when the participating particles have equal masses, transoform in level rank dual representations, and when
1) The intrinsic spins of the particles are related as above
2) The duality flips Bose/Fermi statistics if and only if H_i is a half (as opposed to full integer).

Comparison with UV dualities

- Consider $SU(N)$ theory coupled to fundamental Bosons/Fermions. These conjectured dualities have two massive phases. In the first, the level of the fermionic side has the same sign as its mass. In this phase the intrinsic spin of the bosons is zero, and the intrinsic spin of the fermions $-\frac{m_F}{2}$.
- The second phase occurs when the level of the fermions has the opposite sign as its mass. In this phase the Higgs mechansm turns the Bosons into vectors with spin $-\mathrm{sgn}(\mathrm{k_B}) = \mathrm{sgn}(\mathrm{k_F})$. The fermion continues to have spin equal to $-\frac{m_F}{2}$.
- In this exaple $H_i = \frac{1}{2}$ and $n_i = 0$. Easy to check that the relations beween spins does obey (14) in both phases. Also, of course, the relation between statistics. Similar agreement in the case of every conjectured matter Chern Simons duality. Our analysis can be thought of as first flow upwards to these UV dualities.

No UV dualities with large representations

- When going through the list of conjectured UV dualities involving matter CS theories, one is struck by the following fact.
- Each of these dualities have matter in small representations of the gauge group. One finds the fundamental, adjoint, two box symmetric .... . However there are no conjectured dualities invovling matter in, e.g. n boxes the first row of the Young Tableuax, with $n > 4$. One might wonder why this is the case
- (14) offers an explanation. If $H_i + n_i > 2$ then atleast one of $s_i^{\rm int}$ and $\hat{s}_i^{int}$ would have to have a spin of modulus greater than one. Field theories of this sort do not exist (or atleast have not yet been understood). This gives an explanation for this observation.

New Statistics?

- Recall that the quantum mechanics we constructed deviated from free particle quantum mechanics in two respects. First, the Hilbert space is a space of sections, whose fibres are the dimensionality of conformal blocks associated with the given number of insertions. Second, taking one particle around another yields a monodromy in the space of conformal blocks.
- One could ask the following question. Suppose one could find a situation in which the second effect was not important (parametrically suppressed), but the first effect - i.e. the counting factors associated with dimensionality - could not be ignored. There is some evidence from direct computation that such a situation arises in the study of fundamental particles in the t' Hooft limit, i.e in the limit $N \to \infty$, $k \to \infty$, on a sphere whose size is also taken to infinity (such that the volume is the largest parameter in the problem).

- In such a situation one would have an effectively free particle problem, but with a new effective statistics once we account for the number of conformal blocks. By manipulating well known formulae for the number of conformal blocks, and making some plausible assumptions, one then arrives at the following formula for the partition function of the theory, one arrives a new effective free statistics, one that effectively interpolates between free bosonic statistics at small N/k, and free (dual) Fermionic statistics at large N/k. Similar but dual statements apply to the fermionic theory.
- These new statistical formulae are obtained by replacing various numbers by q numbers in the usual formulae of Bose/Fermi statistics, and actually map to each other under duality.
- Consequently, it seems likely that the systems studied in this talk display completely new eqilibriation dynamics: this is my weak excuse for presenting this work in this conference.

Summary and Future directions

- Modulo establishing (13) in generality, we have presented a complete proof of duality between level rank dual pairs of Q Mech. Perhaps first step in flowing 'upwards' from the well understood level rank duality of topological theories to a proper understanding of dualities of UV QFTs
- To complete our work, it would be nice to find a clear proof of (13). Its possible that such a proof exists in the literature and just needs to be located.
- Would be interesting to try to solve our Schrodinger equations in simple contexts (two particles, four particles etc). Strong weak coupling duality described above may prove useful here, as it relates naively strongly coupled quantum mechanics to a nearly free theory.
- It would be interesting to rederive the duality of quantum mechanics using a path integral approach (w.i.p with T. Chakraborty and P. Ray). Generalize to field theory, for instance, in world line formalism?