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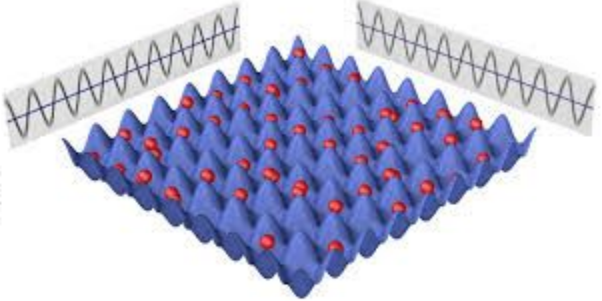

Resonance-Suppression Principle for Prethermalization Beyond Periodic Driving

[J. X. Sim, arXiv:2603.21540] update to come

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Heating under Driving

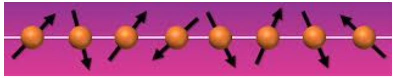
Questions:

For time-dependent many-body Hamiltonian $H(t) = H_0 + V(t)$

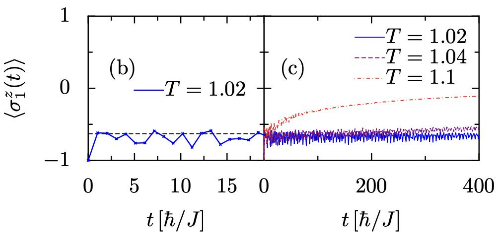
1) How to predict heating rate from Schrodinger Equation?

2) When does the system heat slowly (Prethermalization)?

$$i\hbar \frac{d}{dt}\psi(t) = H(t)\psi(t)$$

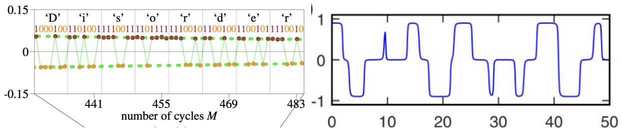
This talk: Lattice models, bounded Hilbert Space (Spin, Fermion) 

Talk Outline

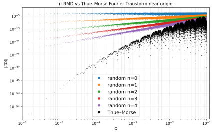


1) Recap on foundational results: Floquet Prethermalization

2) Examples of non-Floquet Prethermalization: Thue-Morse, Quasi-Floquet



3) My work: Resonance-Suppression Principle, frequency spectrum properties



4) Techniques: Linear Response (Kubo Formula) & Non-Perturbative Theory

$$\frac{dE}{dT} \sim \frac{g^2}{\sqrt{\phi''(\Omega_0)}} e^{-\phi(\Omega_0)} \qquad \|O\|_\kappa := \sup_{x\in\Lambda} \int_{\mathbb{R}} d\Omega \sum_{Z\ni x} e^{\kappa[|Z|+p(\frac{\Omega}{\lambda}))]} \|\tilde{O}_Z(\frac{\Omega}{\lambda})\|$$

5) Applications: Clarify subtleties in past examples and provide new examples

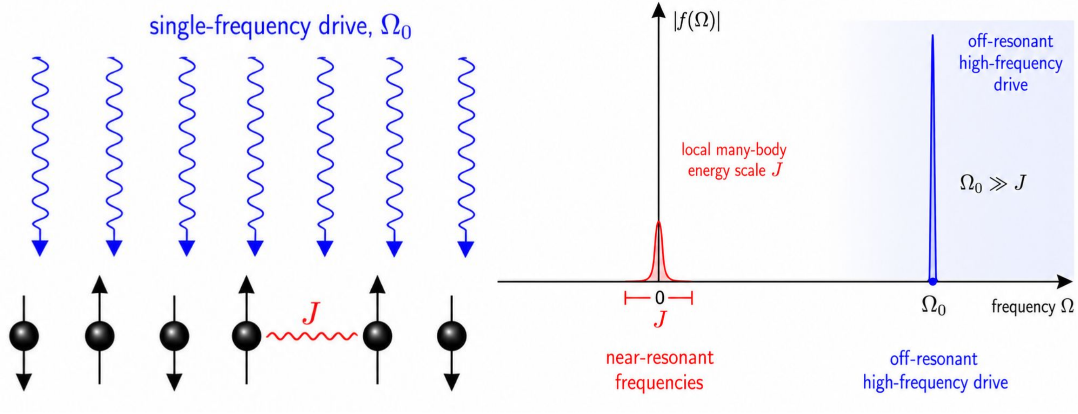
$$\tau_*^{LRT} \sim e^{\lambda} \;\Big|\; \tau_*^{NP} \sim e^{\lambda^{0.5}} \;\Big|\; V^F(t) := \sum_{k=1}^{\infty} V_k^F \sin\left(\frac{\lambda}{k!}t\right)$$

Floquet Prethermalization (Mori-Kuwahara-Saito, ADHH)

E.G. Spins on a lattice with finite connectivity

$$H_z = \sum_{i=1}^{N} \left[-J\sigma_i^z \sigma_{i+1}^z + B_z \sigma_i^z \right]$$

$$H_x = B_x \sum_{i=1}^{N} \sigma_i^x$$



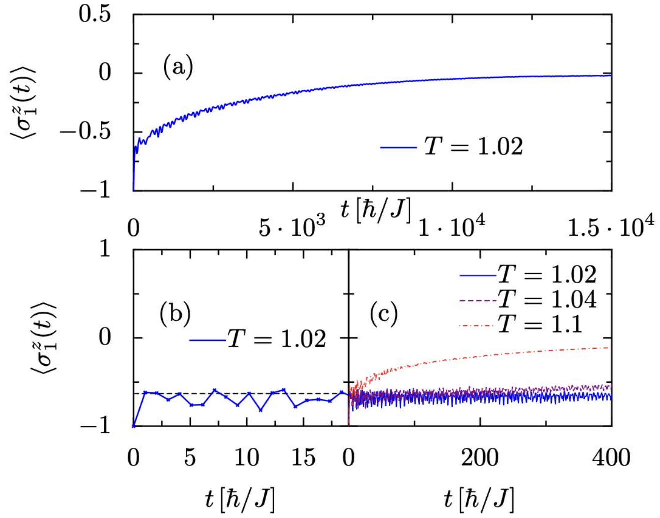


FIG. 1. (color online) Numerical demonstration of prethermalization-like phenomenon. (a): Relaxation in the long timescale, (b): initial relaxation, (c): transient time-evolution after the initial one. Parameters are $(J, B_x, B_z) = (1, 0.9045, 0.8090)$ and $N = 24$. The dotted line in (b) is the expectation value in the equilibrium state of $H_F^{(0)}$ at the inverse temperature $\beta = 0.85$, which is determined from the expectation value of $H_F^{(0)}$ at $t = 10$.

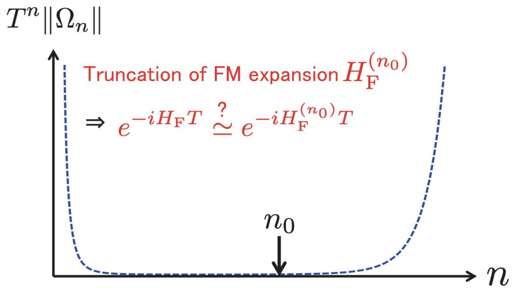
[MKS, PRL' 16: Initial state all down, look at 1st spin]

Result: Exponentially slow heating with drive frequency

Both in Linear Response & Non-Perturbative Theory.

(Linear Response: Abanin, De Roeck, Huveneers PRL'15)

Floquet Theorem: $U(t) = P(t)e^{-iH_F t}, \qquad P(t+T) = P(t).$



MKS: Rigorous analysis of Floquet-Magnus Expansion

$$e^{-iH_F^{(n)}T} \simeq e^{-iH_F T}, \quad \text{where} \quad H_F^{(n)} = \sum_{m=0}^{n} T^m \Omega_m$$

Floquet-Magnus Expansion: looks convergent up to $n \leq n_0 \sim \omega/(gk)$

$$\Omega_n = \sum_{\sigma} \frac{(-1)^{n-\theta[\sigma]}\theta[\sigma]!(n-\theta[\sigma])!}{i^n(n+1)^2 n! T^{n+1}} \int_0^T dt_{n+1} \cdots \int_0^{t_2} dt_1 [H(t_{\sigma(n+1)}), [H(t_{\sigma(n)}), \ldots, [H(t_{\sigma(2)}), H(t_{\sigma(1)})] \ldots]]$$

Termwise bound: $\|\Omega_n\| T^n \leq 2gN_V \dfrac{(2gkT)^n n!}{(n+1)^2}$

Mori-Kuwahara-Saito (PRL' 16): Slow Heating

$$\frac{1}{N}\|H_0(t) - H_0\| \leq \frac{N_V}{N}\left[16g^2 k 2^{-n_0} t + \mathcal{O}(T)\right]$$

Non-Floquet Driving

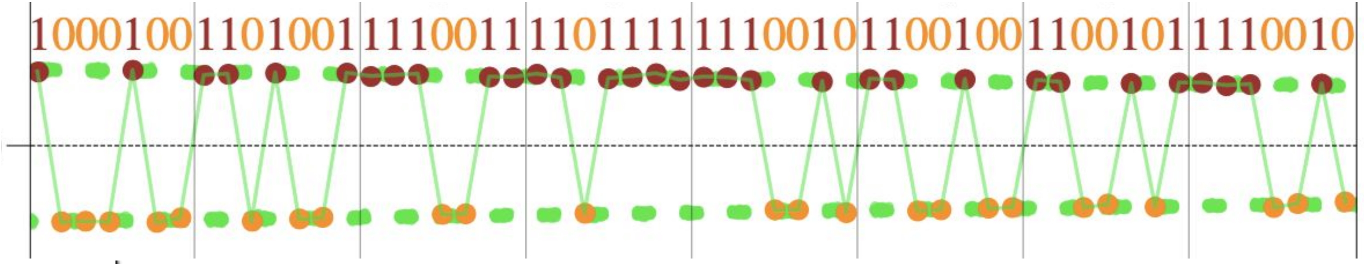
$\lambda^b \qquad e^{(\ln \lambda)^b} \qquad e^{\lambda^{b/(b+1)}}$

Non-exponential scaling with drive speed! Polynomial, Quasipolynomial, Stretch-expt.

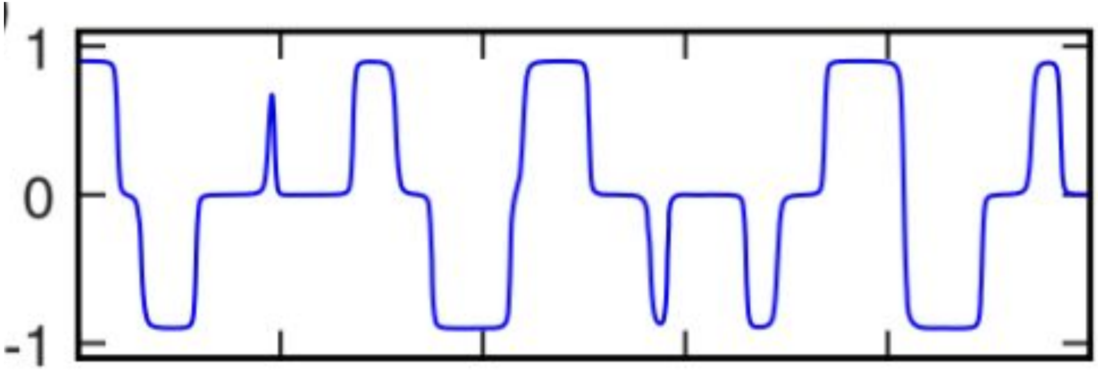
Previous results case-by-case analysis, physical picture not clear.

[Moon et al. NatPhys ‘25]

Thue-Morse Drive
(Mori et. al. PRL’21):
Generated by substitution rule 0 to 01, 1 to 10



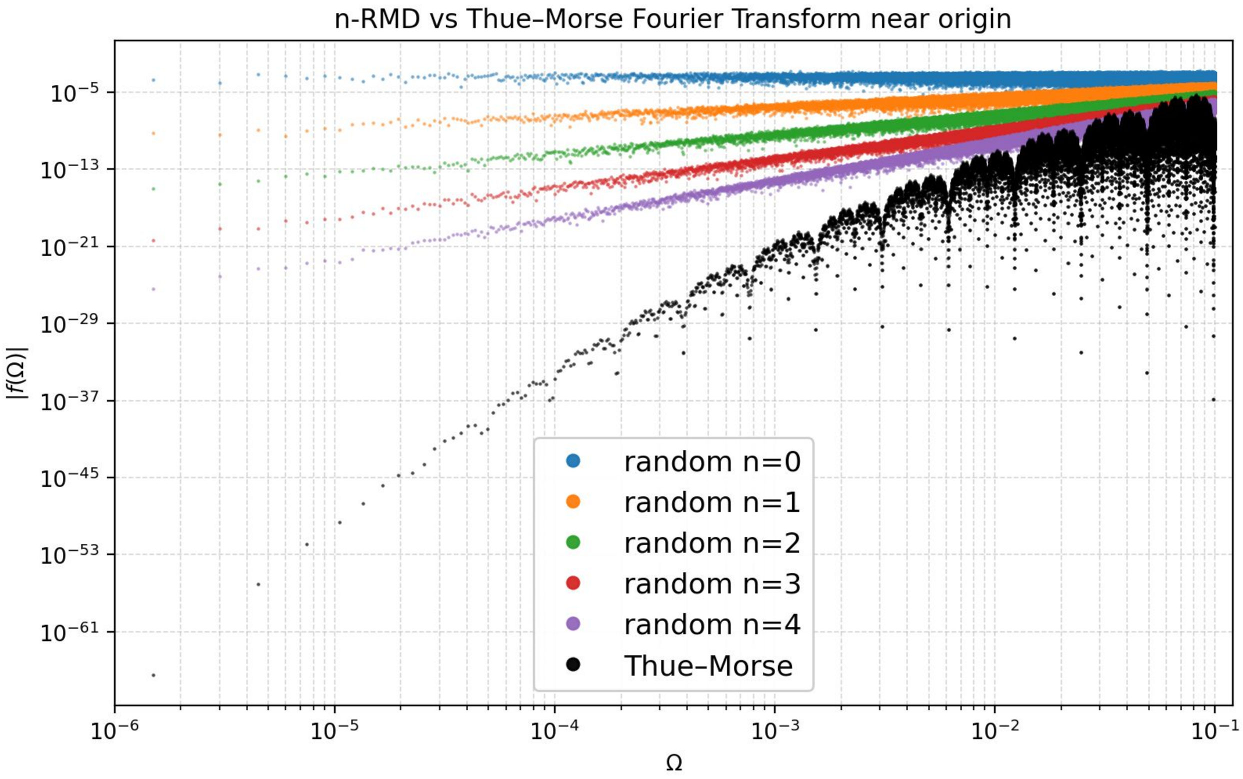
Quasi-Floquet Drive
(Else et. al. PRX’20) :
Multiple incommensurate frequencies



e.g. $V_{QF}(t) \sim \sin(\lambda t) + \sin(\varphi \lambda t), \qquad \varphi = \frac{\sqrt{5}-1}{2}$

Resonance-Suppression Principle: frequency domain

1) *Fourier Transform Suppression* suppresses **single-photon processes**
2) *Exceptional ‘subadditive’ spectrum arithmetic* suppresses **multi-photon processes**



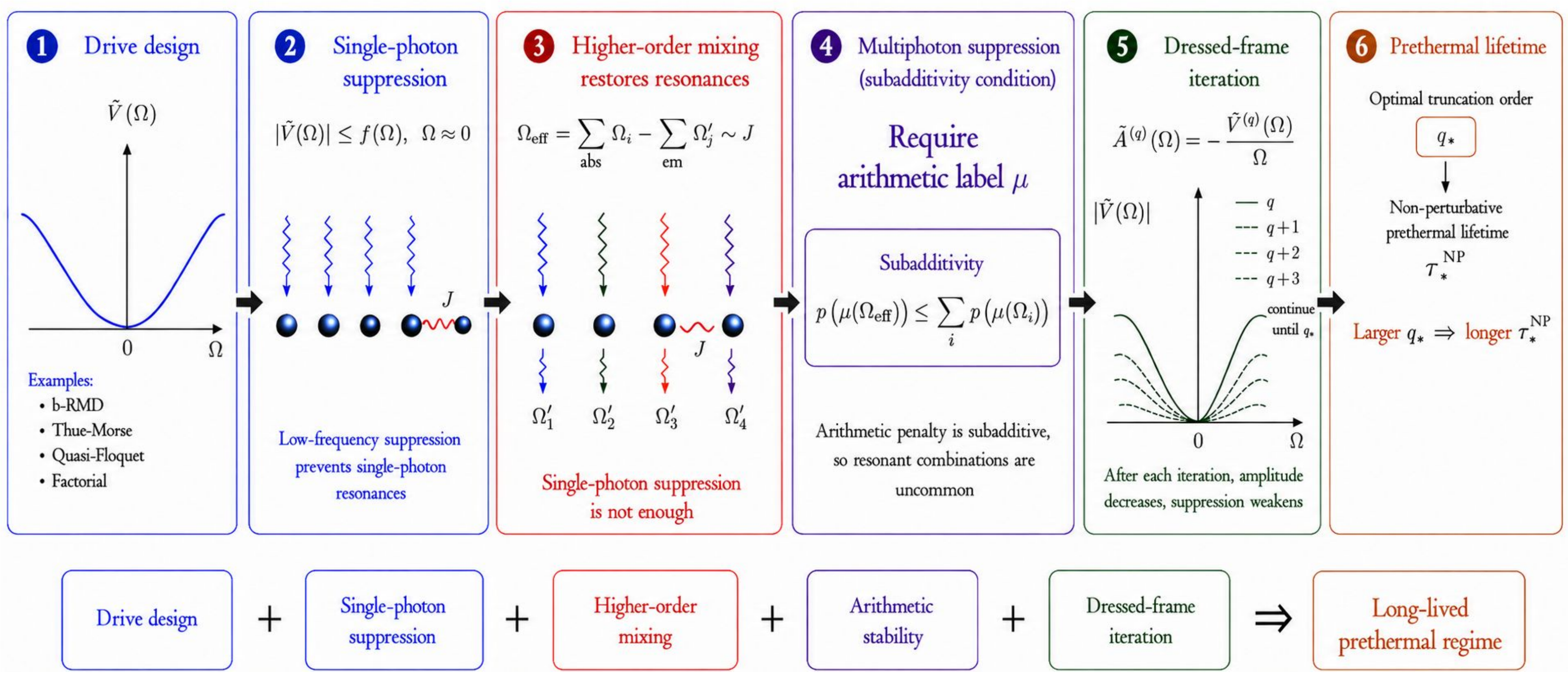
Main Claim:
If (2) holds, heating rate is fixed by (1).

Independent of microscopic spectral arithmetic details for why (2) holds.

Justified by Fer Expansion.

[https://arxiv.org/abs/2603.21540]

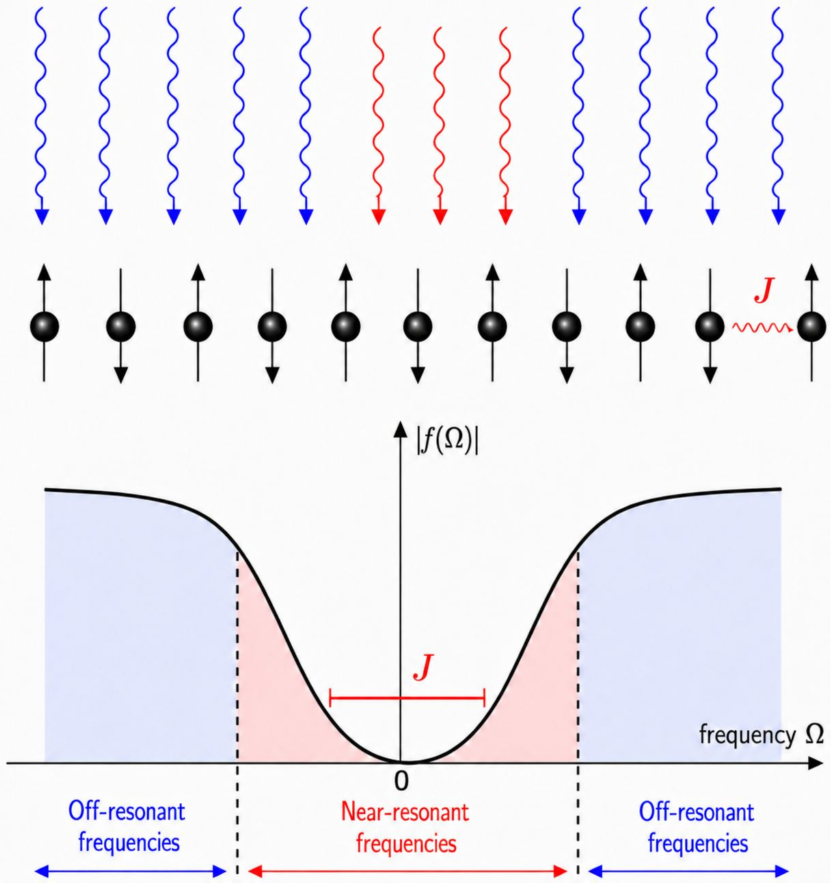
Roadmap of non-periodic theory [https://arxiv.org/abs/2603.21540]



Single-Photon Processes (non-Floquet)

TABLE I. Prethermal universality classes characterized by suppression law $f(\Omega)$ and corresponding lower bounds on heating-time scaling $\tau_*^{LRT}, \tau_*^{NP}$ in LRT and non-perturbatively.

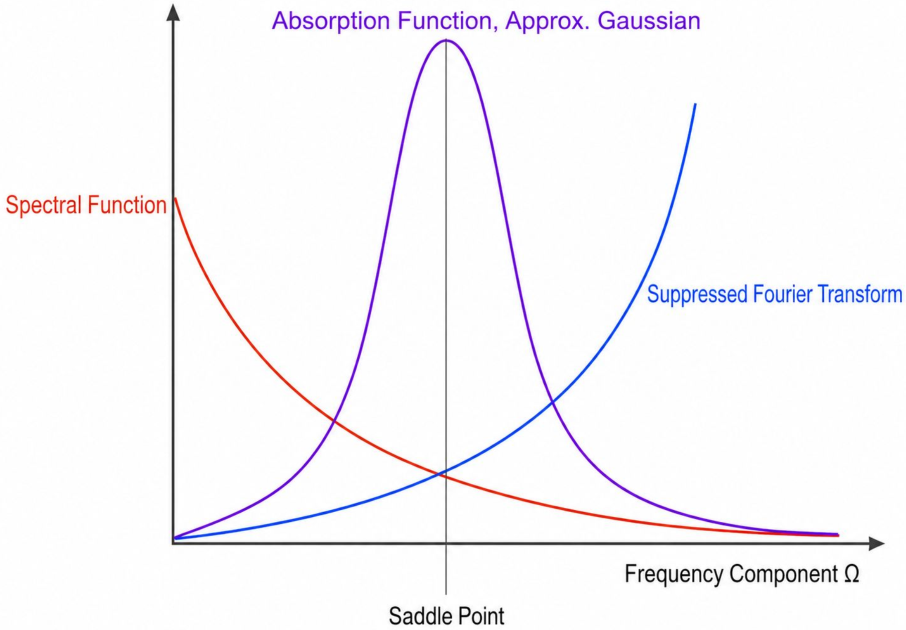
	Poly	Quasipoly	Stretch-Expt
$f(\Omega)$	$\|\Omega\|^b$	$e^{-\|\ln\Omega\|^b}$	$e^{-1/\|\Omega\|^b}$
τ_*^{LRT}	λ^{2b+1}	$e^{(\ln\lambda)^b}$	$e^{\lambda^{b/(b+1)}}$
τ_*^{NP}	λ^b	$e^{(\ln\lambda)^b}$	$e^{\lambda^{b/(b+1)}}$



Warm Up: Linear Response Theory, Kubo Formula

$$\frac{dE}{dt} \sim \int_{\mathbb{R}} d\Omega\, |f(\frac{\Omega}{\lambda})|^2 e^{-\frac{\Omega}{J}} = \frac{g^2}{\lambda} \int_{\mathbb{R}} d\Omega\, e^{2\ln[|f(\frac{\Omega}{\lambda})|]-\frac{\Omega}{J}}$$

$$\frac{dE}{dT} \sim \frac{g^2}{\sqrt{\phi''(\Omega_0)}} e^{-\phi(\Omega_0)} \Rightarrow \tau_* \sim \left(\frac{dE}{dt}\right)^{-1}$$



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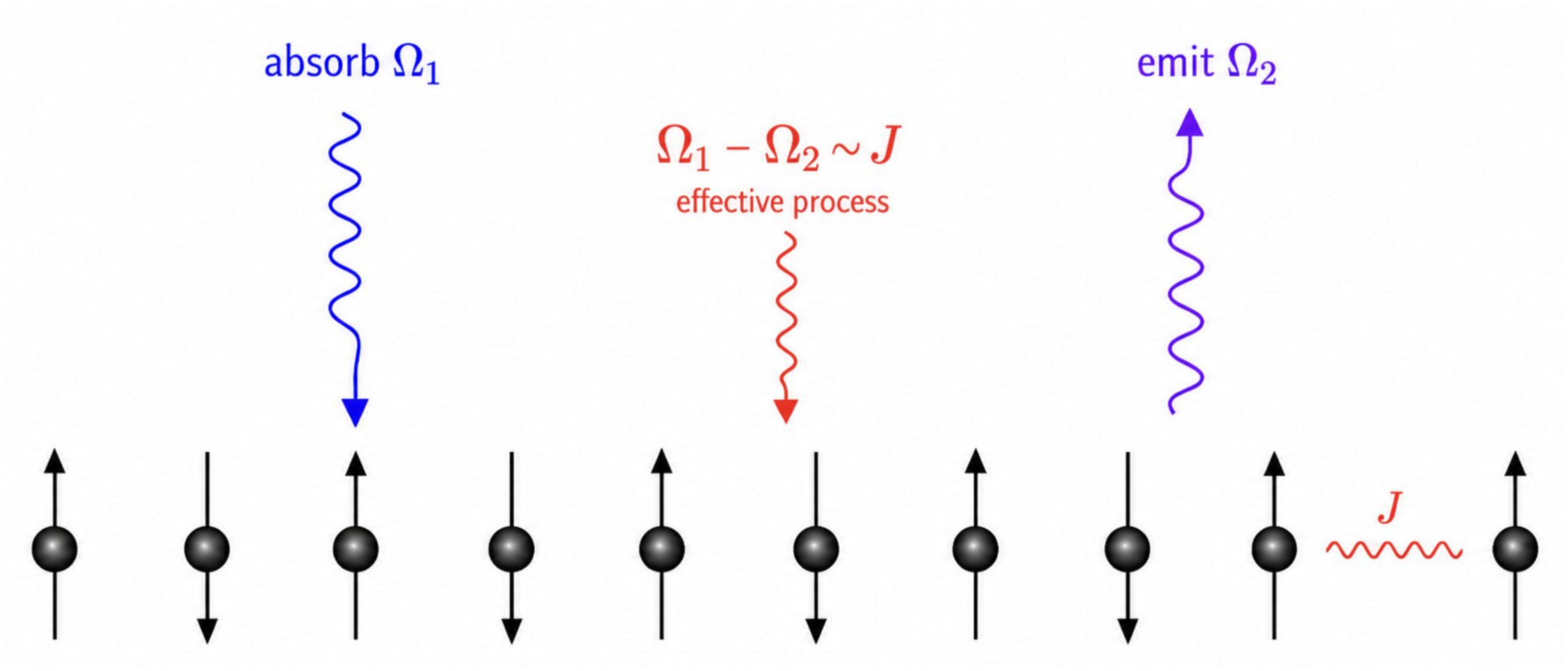
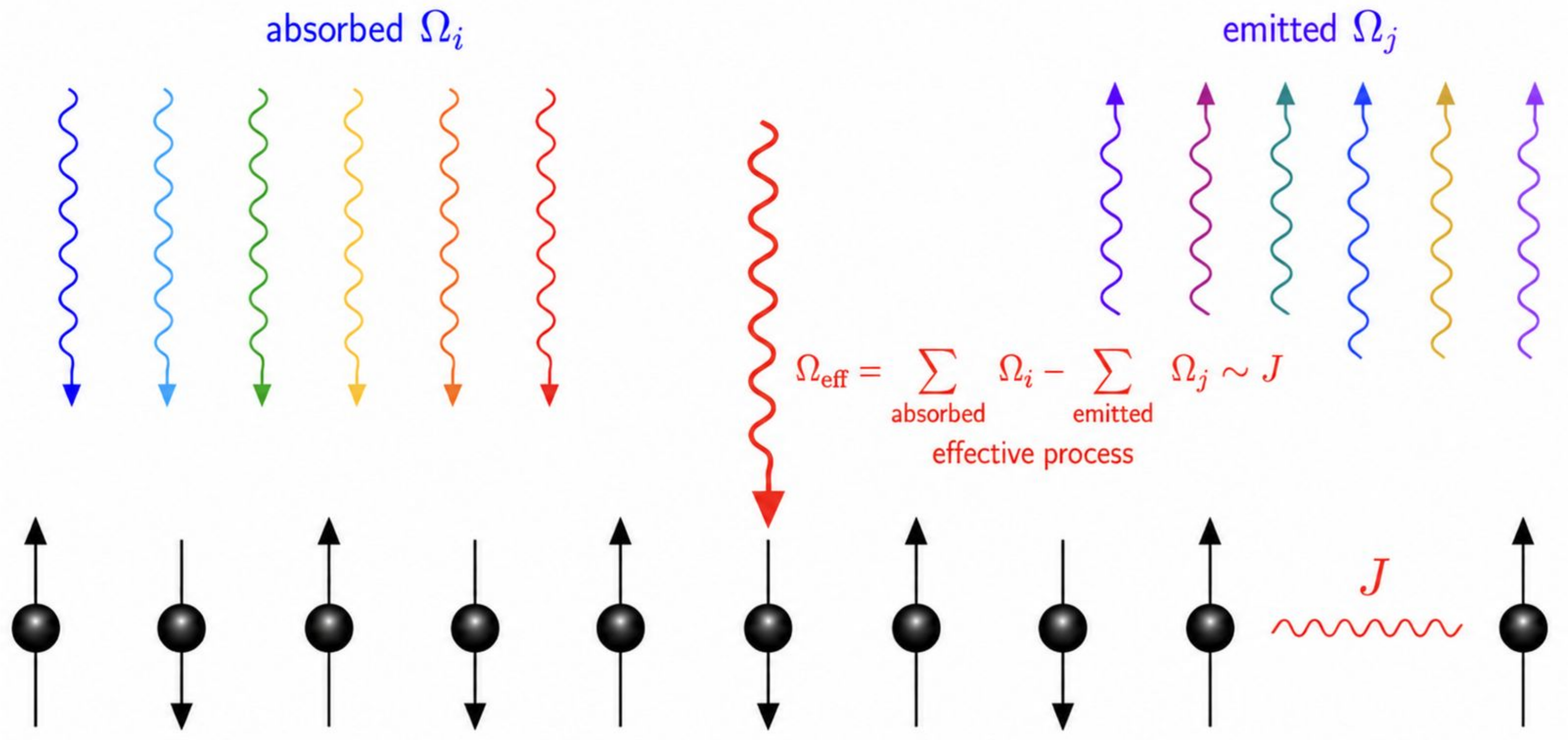
τ_*^{NP}	λ^b	$e^{(\ln\lambda)^b}$	$e^{\lambda^{b/(b+1)}}$

Problem: Multi-Photon Processes (non-Floquet)

Generally, resonances restored at second-order. Physics beyond linear response!

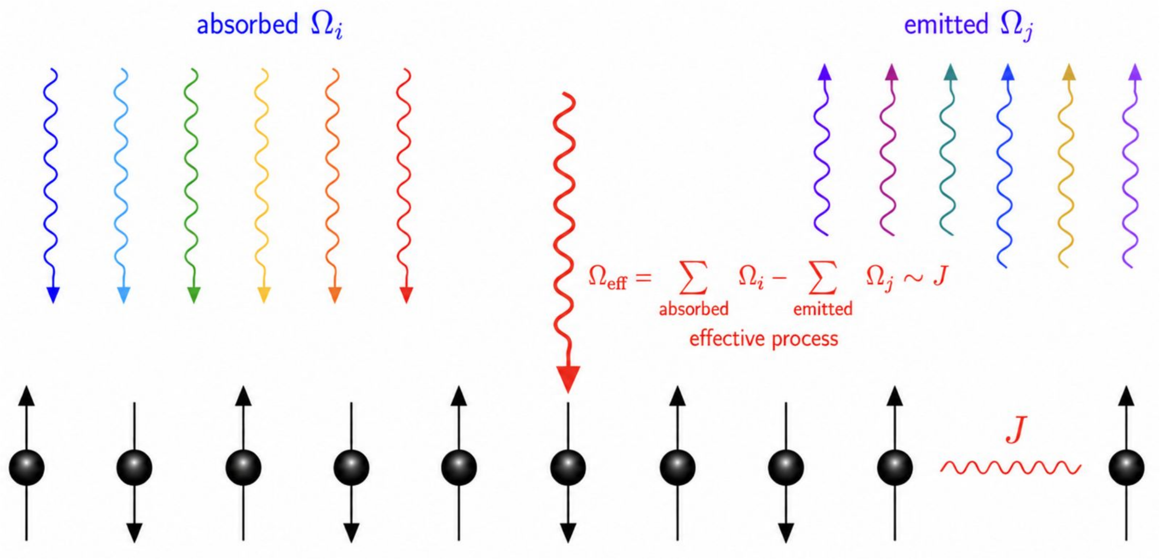


Resolution: Multi-Photon Processes (non-Floquet)



Subadditive Arithmetic Property controls resonances

$$p(\mu(\Omega_{\text{eff}})) \leq \sum_i p(\mu(\Omega_i))$$



Simple Explicit Example of Arithmetic Property: let k2 > k1.

$$|\frac{n_1}{k_1!} + \frac{n_2}{k_2!}| = |\frac{\frac{k_2!}{k_1!} n_1 + n_2}{k_2!}| \geq \frac{1}{k_2!}$$

more on this ‘arithmetic structure’ later.

Non-Perturbative Theory: Iterative Rotating-Frames

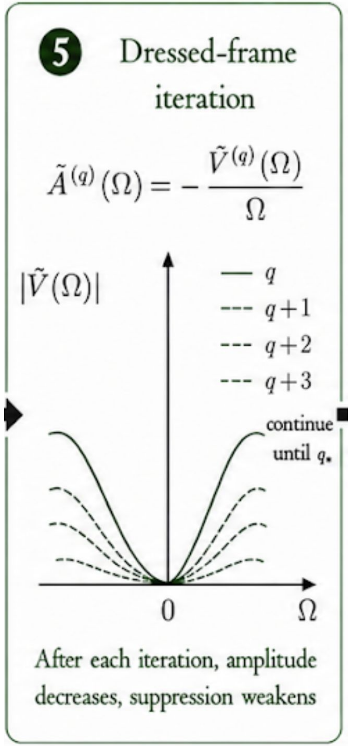
$$U(t) = P_*(t)\mathcal{T}(e^{-i\int_0^t dt'[D_*+V_*(t')]}) := P_*(t)U_*(t)$$

$$P_*(t) := e^{A^{(q_*-1)}(t)}...e^{A^{(0)}(t)}$$

Freq. Domain Recursion Equation (Small Divisor Problem!):

$$\tilde{A}^{(q)}(\Omega) = -\frac{\tilde{V}^{(q)}(\Omega)}{\Omega}$$

Dressed drive weak, rotating-frame ~1, thus heating slow.



Main Non-perturbative Result: Heating Theorem

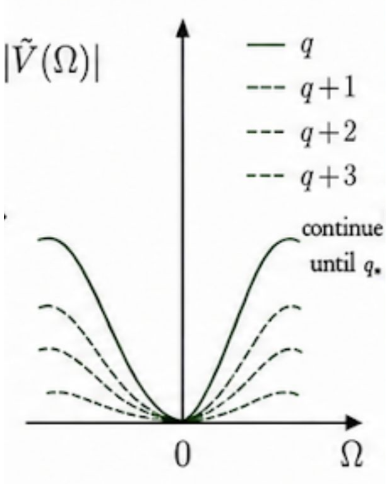
$$\frac{1}{J|\Lambda|}\|U^\dagger(t)\langle H\rangle U(t)-\langle H\rangle\| \leq \tilde{K}\frac{\|V_*\|_{\kappa_*}}{\|V\|_\kappa}t+O(J/\lambda)$$

⇩

$$\tau_* \sim \tilde{K}^{-1}\frac{\|V\|_{\kappa_0}}{\|V_*\|_{\kappa_*}}$$

⇩

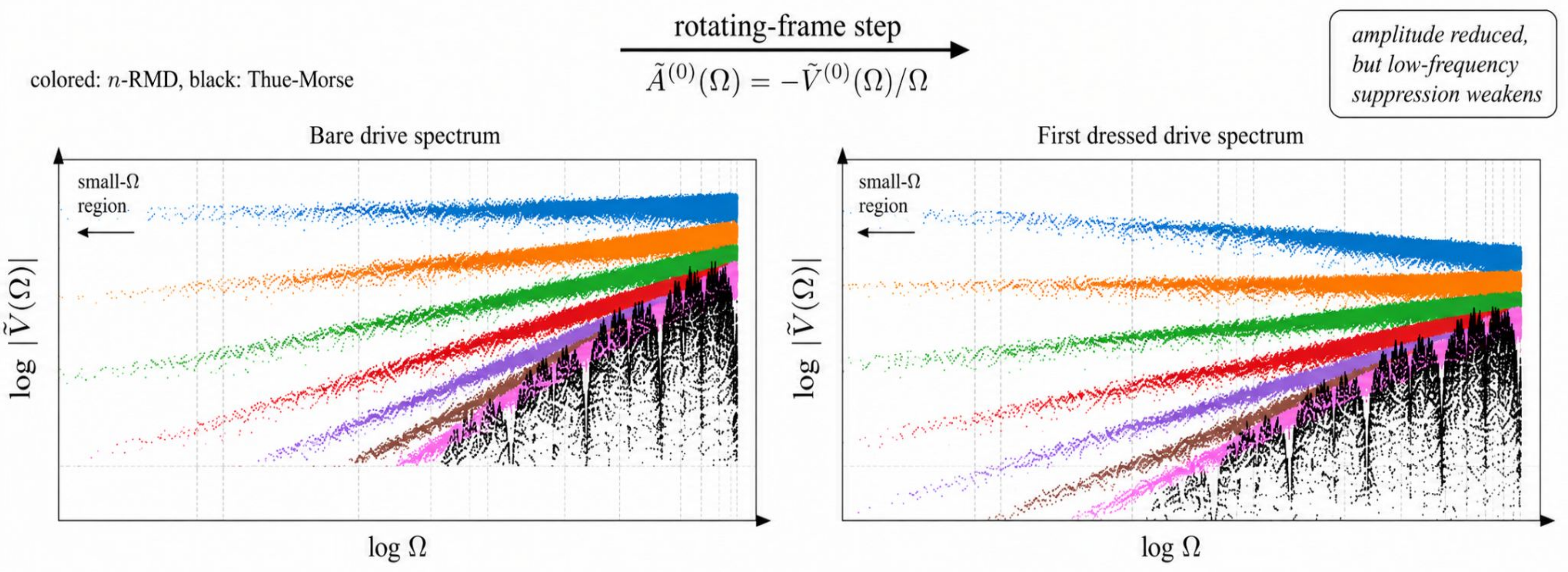
	Poly	Quasipoly	Stretch-Expt
$f(\Omega)$	$\|\Omega\|^b$	$e^{-\|\ln\Omega\|^b}$	$e^{-1/\|\Omega\|^b}$
τ_*^{NP}	λ^b	$e^{(\ln\lambda)^b}$	$e^{\lambda^{b/(b+1)}}$



Numerics: Loss of suppression and reduced amplitude

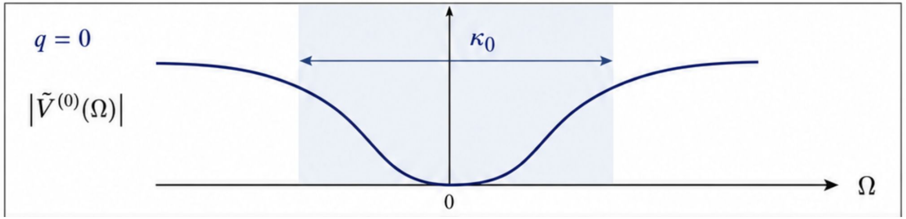
Numerics here are Fer Expansion for $H_n^{(0)} = J\,\sigma^z + g\,s_n\,\sigma^x, \; s_n \in \{\pm 1\}$

Magnus Expansion similar results



Quantify Locality, Resonance-Suppression by κ-norms

'Penalty' function $p(\frac{\Omega}{\lambda}) := -\ln f(\frac{\Omega}{\lambda})$



$$\|O\|_\kappa := \sup_{x\in\Lambda} \int_{\mathbb{R}} d\Omega \sum_{Z\ni x} e^{\kappa[|Z|+p(\frac{\Omega}{\lambda}))]} \|\tilde{O}_Z(\frac{\Omega}{\lambda})\|$$

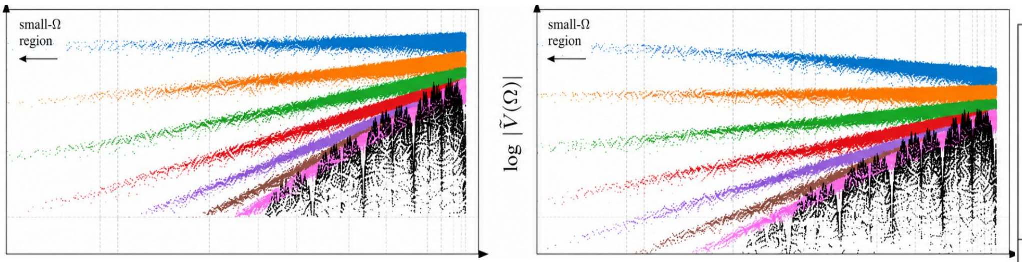
Intuition of scary formula: Penalize

1) Operators that are very spread out in space
2) Frequency components near the origin

e.g. Stretch-Expt $p(\frac{\Omega}{\lambda}) := (\frac{\lambda}{|\Omega|})^b \to \infty$ as $\Omega \to 0$

Small Divisor Problem: Relates drive & next rotating frame using single-photon suppression

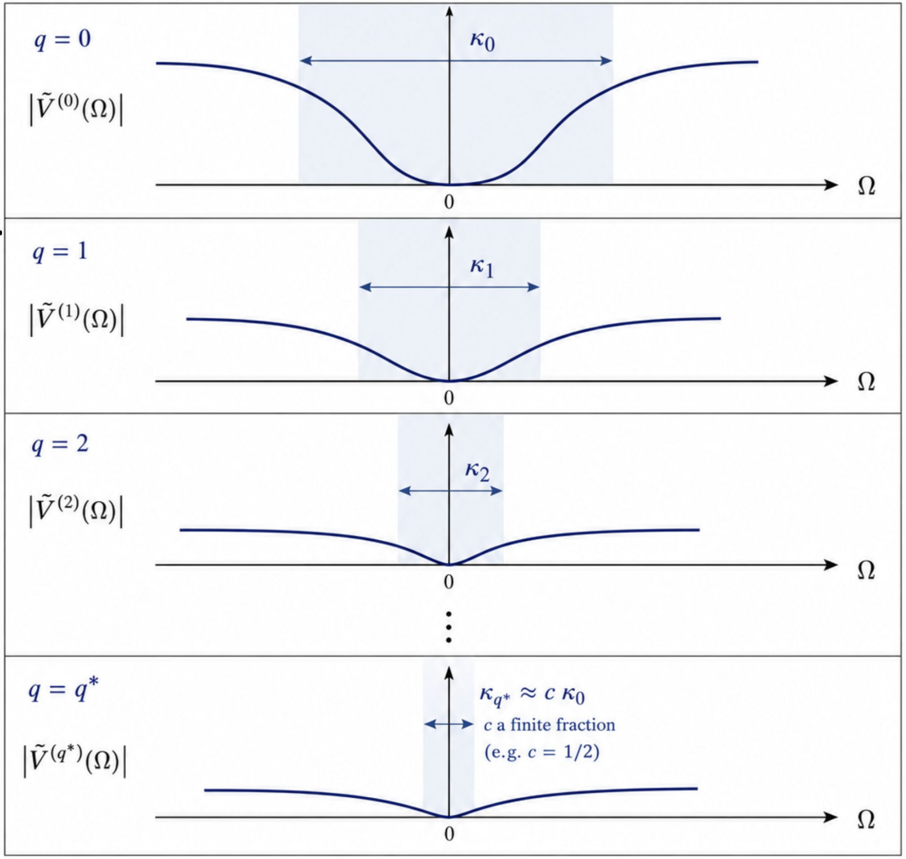
$$\|O\|_\kappa := \sup_{x\in\Lambda} \int_{\mathbb{R}} d\Omega \sum_{Z\ni x} e^{\kappa[|Z|+p(\frac{\Omega}{\lambda})]} \|\tilde{O}_Z(\frac{\Omega}{\lambda})\|$$



Lemma 1. *(Small Divisor Lemma) Let $\kappa - \kappa' > 0$. Compute a 'small divisor function', $h(\kappa-\kappa')$ below. I omit* $\sup_{x\in\Lambda}$ *in front for brevity.*

$$\|A^{(q)}\|_{\kappa'} \leq \frac{1}{\lambda} h(\kappa-\kappa') \|V^{(q)}\|_\kappa$$

Prethermal class	$h(\Delta\kappa)$	Condition
Polynomial	1	$\Delta\kappa \geq 1/b$
Quasipolynomial	$\exp[\Delta\kappa^{-1/(b-1)}]$	—
Stretched exponential	$\Delta\kappa^{-1/b}$	—



Multi-photon Processes: Do not take for granted!

Usually, subadditivity not satisfied! Prethermalization is uncommon.

$$\Omega_1 + \Omega_2 \sim J \Longrightarrow p(J) \gg p(\Omega_1) + p(\Omega_2)$$

With exceptional arithmetic frequency group, salvage subadditivity

$$\text{Quasi-Floquet:} \quad \Gamma = \mathbb{Z}^m := \{\mu = \vec{n} = (n_1, ..., n_m) |\ n_1, ..., n_m \in \mathbb{Z}\},\ \Omega(\vec{n}) = \vec{n} \cdot \vec{\omega}, ;$$

$$p\text{-adic/substitution:} \quad \Gamma = \mathbb{Z}[1/p] \ := \{\mu = \frac{k}{p^r} |\ k \in \mathbb{Z},\ r \in \mathbb{N}_0 := \{0, 1, 2, ...\}\},\ \Omega(r, k) = 2\pi\lambda\mu;$$

$$\text{Factorial (Rational):} \quad \Gamma = \mathbb{Q} = \bigcup_{k \geq 1} \frac{1}{k!}\mathbb{Z} := \{\mu = \frac{n}{k!} |\ n \in \mathbb{Z},\ k \in \mathbb{N}_0 := \{0, 1, 2, ...\}\}\ \Omega(k, n) = 2\pi\lambda\mu,$$

$$p(\mu(\Omega_1 + ... + \Omega_n)) \leq p(\mu(\Omega_1)) + ... + p(\mu(\Omega_n))$$

Put everything together: Bound the dressed frame

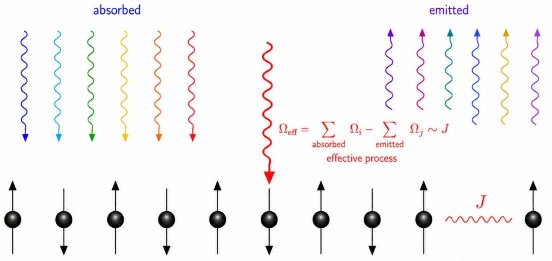
At every step of iteration $\tilde{A}^{(q)}(\Omega) = -\frac{\tilde{V}^{(q)}(\Omega)}{\Omega}$ we must consider dressed Hamiltonian

$$H^{(q+1)}(t) = {P^{(q)}}^{\dagger}(t) H^{(q)}(t) P^{(q)}(t) - i {P^{(q)}}^{\dagger}(t) \partial_t P^{(q)}(t)$$

To bound conjugated exponentials, under multi-photon suppression we have

Lemma 2. *(ADHH Iteration Lemma) For any $\kappa' < \kappa$ satisfying $3\|A\|_\kappa \le \kappa - \kappa'$, one has*

$$\|e^{\mathrm{ad}_A} O - O\|_{\kappa'} \le \frac{18}{\kappa'(\kappa - \kappa')} \|A\|_\kappa \|O\|_\kappa .$$



Finally, we apply Small Divisor Lemma $\|A^{(q)}\|_{\kappa'} \le \frac{1}{\lambda} h(\kappa - \kappa') \|V^{(q)}\|_\kappa$

Applications

1) **(i) Classify known examples in one conceptual picture**
Smooth or discontinuous Quasi-Floquet, random-multipolar, Thue-Morse, etc.

(ii) Clarify subtleties: LRT vs NP $V_{QF}(t) \sim \sin(\lambda t) + \sin(\varphi \lambda t), \qquad \varphi = \frac{\sqrt{5}-1}{2}$

e.g. Two-tone Quasi-Floquet drive, hard gap in bare spectrum

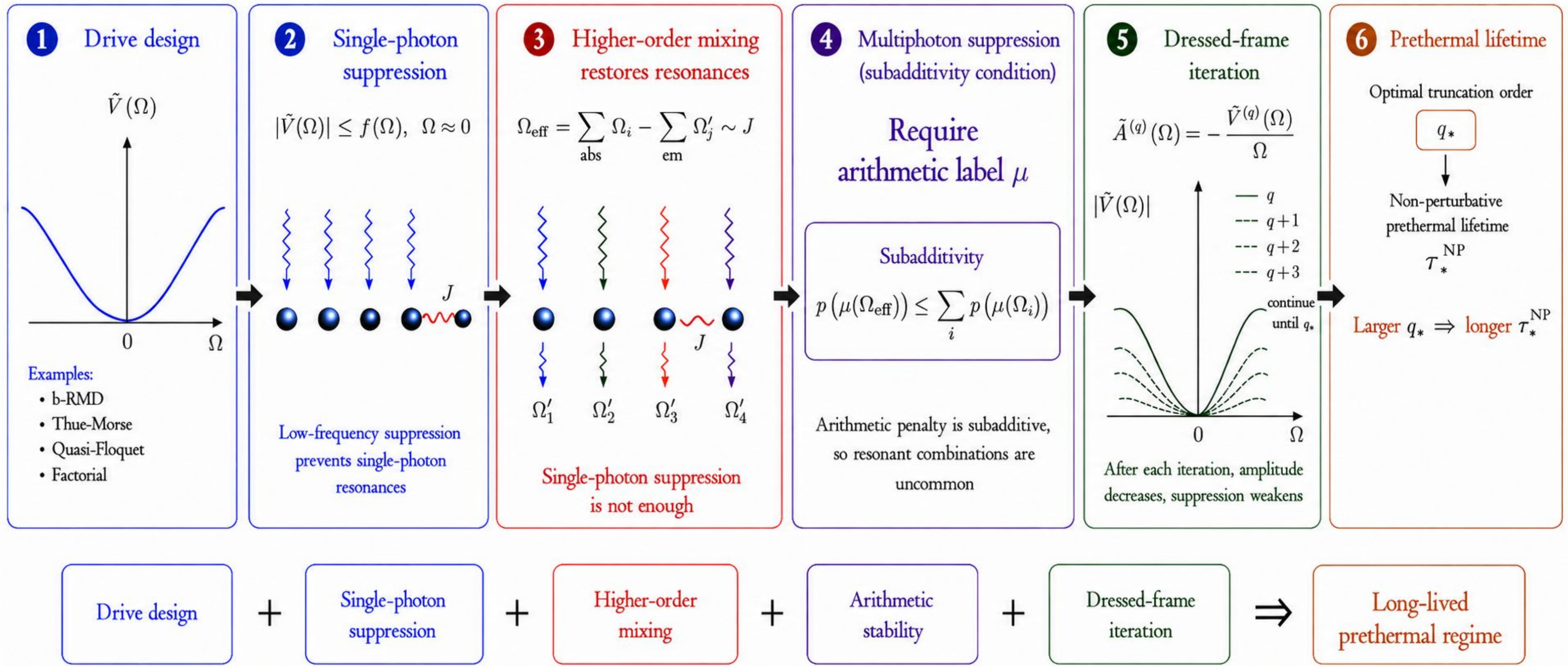
$$\tau_*^{LRT} \sim e^{\lambda} \,\Big|\, \tau_*^{NP} \sim e^{\lambda^{0.5}}$$

Reason: Multiphoton processes cause suppression breakdown $|\vec{n}_1 + \vec{n}_2|^\alpha \leq |\vec{n}_1|^\alpha + |\vec{n}_2|^\alpha$

2) **Provide new examples of non-periodic driving which have slow heating**

e.g. Factorial Drive: Well-behaved non-periodic function $V^F(t) := \sum_{k=1}^{\infty} V_k^F \sin\left(\frac{\lambda}{k!} t\right)$

Prethermal classes arise from $|V_k^F| \sim (k!)^{-b}, e^{-(\ln k!)^b}, e^{-(k!)^b}$



<u>Moving Forward:</u>
Arithmetic classifications,
Experimental protocols, applications in sensing, control, quantum gates etc
Computational complexity of dynamics?
Let's discuss!

Acknowledgements

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