An Observer-Based Approach to Thermalization

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June 4, 2026

https://www.observationalentropy.com

Reversible Dynamics, Irreversible Thermodynamics

An isolated quantum system evolves according to unitary dynamics, which are time-reversal symmetric.

$$\rho_t = U_t \rho_0 U_t^\dagger \qquad \Longrightarrow \qquad S_{\mathrm{vN}}(\rho_t) = S_{\mathrm{vN}}(\rho_0).$$

Therefore, the von Neumann entropy is conserved: it cannot increase alone.

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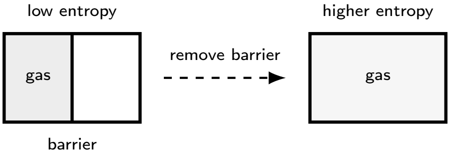

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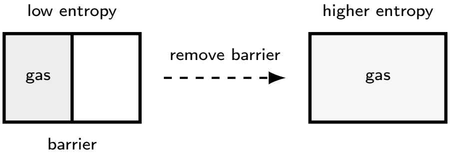

However, the second law states that entropy increases.

The question of this talk

If the microscopic dynamics conserves entropy, *where does the increase come from?*[1]

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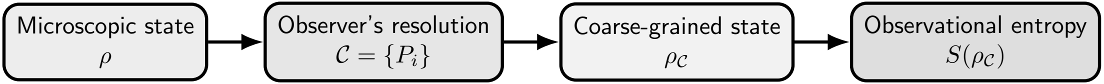
Observer's Description

Overview

A real observer has *finite resolution*, so it cannot read out all of ρ. We describe this resolution as a coarse graining. From it we build the *coarse-grained state* $\rho_{\mathcal{C}}$ and take its von Neumann entropy[2].



Microscopic state ρ

Observer's resolution $\mathcal{C} = \{P_i\}$

Coarse-grained state $\rho_{\mathcal{C}}$

Observational entropy $S(\rho_{\mathcal{C}})$

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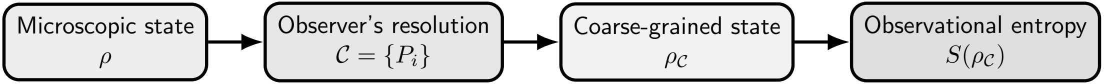

Microscopic state ρ → Observer's resolution $\mathcal{C} = \{P_i\}$ → Coarse-grained state $\rho_{\mathcal{C}}$ → Observational entropy $S(\rho_{\mathcal{C}})$

Main message

Entropy increases because *a finite-resolution observer redescribes the state*. Observational entropy is the von Neumann entropy of $\rho_{\mathcal{C}}$. Its growth under unitary evolution is the increase the observer sees[3].

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Microscopic State and Observer's Information

To make this precise, we start from the microscopic state, a density matrix

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Basic idea

We model this finite resolution as an orthogonal decomposition of the Hilbert space.

Projective Decomposition into Macrocells

We write the outcomes the observer can distinguish as

$$\mathcal{C} = \{P_i\}_{i=1}^{m}.$$

Here

$$P_i P_j = \delta_{ij} P_i, \qquad \sum_{i=1}^{m} P_i = I.$$

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Macrocell

P_i is the subspace of microscopic states the observer reads as outcome i.

Cell volume

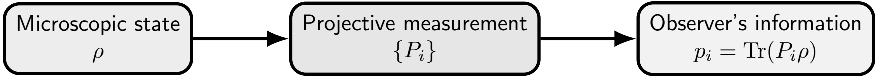
$$V_i := \operatorname{Tr} P_i = \dim(P_i).$$

V_i counts the microscopic states compatible with outcome i.

What the Observer Reads Out from the State

The observer reads out the distribution

$$p_i = \mathrm{Tr}(P_i \rho).$$



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Projective measurement
$\{P_i\}$

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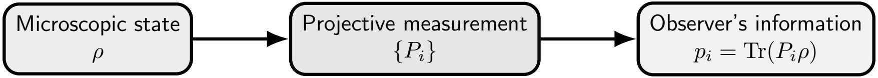

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Information lost here

p_i tells us which cell the system is in, but not the state inside it. So the observer's description is coarser than ρ.

Many States Give the Same Observed Outcomes

If two states ρ and σ satisfy

$$\mathrm{Tr}(P_i\rho) = \mathrm{Tr}(P_i\sigma) \qquad (i = 1, \ldots, m),$$

then the observer $\mathcal{C} = \{P_i\}_{i=1}^{m}$ cannot tell them the difference.

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Important consequence

So the observer should pick one *representative* of this class instead of ρ itself. *Which one?*

Maximum Ignorance

If the observer only knows the system is in cell i, the state inside is unknown. We then assign the maximally mixed state on P_i:

$$\tau_i := \frac{P_i}{V_i}.$$

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Meaning

τ_i says: we know the outcome is i, but nothing about the state inside the cell.

Definition of the Coarse-Grained (Macroscopic) State

We combine the known probabilities p_i with the maximally mixed states τ_i:

$$\rho_{\mathcal{C}} := \sum_i p_i \tau_i = \sum_i \mathrm{Tr}(P_i \rho) \frac{P_i}{V_i}.$$

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What does this state represent?

- The cell probabilities p_i match those of ρ.
- Information inside each cell is averaged out.
- To observer $\mathcal{C}$, ρ and $\rho_{\mathcal{C}}$ look the same, i.e., $\rho \sim_{\mathcal{C}} \rho_{\mathcal{C}}$.

Coarse-Graining Map

We can write the coarse graining as a linear map

$$\mathcal{G}_{\mathcal{C}}(\rho) := \sum_i \mathrm{Tr}(P_i \rho) \frac{P_i}{V_i} \, :$$

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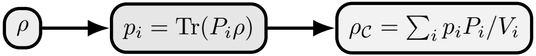
Measure

Read off the cell probabilities p_i with P_i.

Prepare

Replace the state inside each cell by P_i/V_i.



$$\rho \longrightarrow p_i = \mathrm{Tr}(P_i \rho) \longrightarrow \rho_{\mathcal{C}} = \textstyle\sum_i p_i P_i/V_i$$

The Coarse-Grained State Is the Maximum-Entropy State

This “maximum ignorance” rule is just the maximum-entropy principle. Among all states σ with the same cell probabilities

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Maximum entropy

$$\rho_{\mathcal{C}} = \operatorname*{argmax}_{\sigma\colon \mathrm{Tr}(P_i\sigma)=p_i} S_{\mathrm{vN}}(\sigma).$$

Derivation of Observational Entropy

Now we take the entropy of the coarse-grained state:

$$S(\rho_{\mathcal{C}}) = -\operatorname{Tr}\rho_{\mathcal{C}}\log\rho_{\mathcal{C}}.$$

In cell i, the eigenvalue p_i/V_i appears V_i times. Hence

$$\begin{aligned} S(\rho_{\mathcal{C}}) &= -\sum_i V_i \frac{p_i}{V_i}\log\frac{p_i}{V_i} \\ &= -\sum_i p_i \log\frac{p_i}{V_i}. \end{aligned}$$

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Observational entropy

$$S_{\mathrm{obs}}(\rho;\mathcal{C}) := S(\rho_{\mathcal{C}}) = -\sum_i p_i \log\frac{p_i}{V_i}.$$

Decomposition into Two Terms

Observational entropy splits into two terms:

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Uncertainty about which cell the system is in.

Internal uncertainty of each cell

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Meaning

The more probability moves into larger cells, the more entropy the observer sees.

Expression in Terms of Relative Entropy

Let the dimension of the full Hilbert space be

$$d = \mathrm{Tr}\, I = \sum_i V_i,$$

and define the volume-weighted distribution

$$\pi_i := \frac{V_i}{d}.$$

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Interpretation

Entropy increases as the cell distribution p approaches the volume distribution π.

Relation to Microscopic Entropy

The map $\mathcal{G}_{\mathcal{C}}$ throws away information the observer ignores. So

$$S_{\mathrm{vN}}(\rho) \leq S_{\mathrm{obs}}(\rho; \mathcal{C}) \leq \log d.$$

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Left inequality

Coarse graining reduces information, so entropy does not decrease.

Right inequality

The maximum is reached by the fully mixed state on the whole space.

Two Extreme Cases

One formula gives the classical entropies as limiting cases of the observer's resolution.

Finest graining: Gibbs/Shannon

If each cell is one-dimensional, then $V_i = 1$.
In this case

$$S_{\text{obs}} = -\sum_i p_i \log p_i.$$

This is the Shannon (Gibbs) entropy of the outcomes.

Coarsest graining: Boltzmann

If there is only one cell, then $P_1 = I$, $V_1 = d$, and $p_1 = 1$.
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Summary

Observational entropy interpolates between Shannon/Gibbs and Boltzmann entropy, depending on the observer's resolution.

Unitary Evolution and the Coarse-Grained State

The microscopic state evolves unitarily,

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Hence

$$S_{\mathrm{vN}}(\rho_t) = S_{\mathrm{vN}}(\rho_0).$$

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The key difference

The entropy of ρ_t is conserved, but the entropy of $\rho_{\mathcal{C}}(t)$ can change.

Entropy Increase

Entropy increases when time evolution spreads probability into larger cells.

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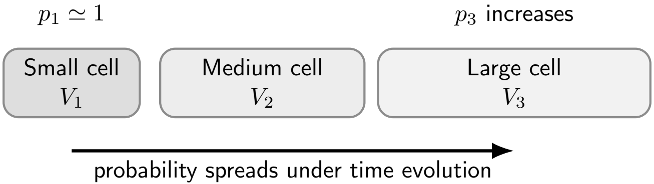
Irreversibility does not come from breaking unitarity. It appears when the observer redescribes the state by coarse graining.

From Smaller Cells to Larger Cells

A low-entropy state has its probability concentrated in a few small cells.



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Medium cell V_2

Large cell V_3

probability spreads under time evolution

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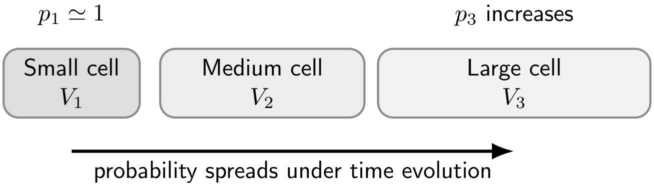

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Condition favoring increase

If the probabilities $p_i(t)$ spread into cells with larger V_i, then

$$\sum_i p_i(t) \log V_i$$

tends to increase.

What Looks Irreversible?

Microscopically, U_t is reversible and loses no information. But coarse graining maps many microscopic states to the same $\rho_{\mathcal{C}}$:

$$\rho \longmapsto \rho_{\mathcal{C}}.$$

Microscopic states
many degrees of freedom

Coarse graining
$\mathcal{G}_{\mathcal{C}}$

Coarse-grained state
few probabilities p_i

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Irreversibility

Irreversibility comes not from the dynamics, but from the observer's description, which lumps many microscopic states together.

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❺ Observational entropy is

$$S_{\mathrm{obs}}(\rho; \mathcal{C}) = S(\rho_{\mathcal{C}}) = -\sum_i p_i \log \frac{p_i}{V_i} \ .$$

Entropy Increase: Summary

Microscopic description

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Physical interpretation

Entropy increases as the observer's distribution $p_i(t)$ approaches the equilibrium one, proportional to cell volume,

$$\pi_i = V_i/d.$$

Observer-Dependence in Gravity

The same principle applies in (quantum) gravity: entropy is defined in relation to an observer.

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[4] De Vuyst et al., arXiv:2405.00114; de la Hamette, arXiv:2603.23598.

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This talk

- Observer = a coarse graining $\mathcal{C} = \{P_i\}$ (a measurement)
- Well-defined entropy $S(\rho_{\mathcal{C}})$
- Depends on the resolution

Gravity

- Observer = a quantum reference frame (a clock, i.e., quantum measurement)
- Including it turns a Type III algebra into Type II, giving a density matrix and a finite entropy

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Different observers see different entropies. Fixing the observer makes entropy well-defined (whether the observer is a coarse graining or a quantum reference frame).[4]

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Take-Home Message

Observational (von Neumann macroscopic) entropy is naturally derived as **the von Neumann entropy of the coarse-grained state**[5].

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[5] Von Neumann, Mathematical Foundations of Quantum Mechanics (1932) and Teruaki Nagasawa et al., Phys. Rev. Research **6**, 043327 (2024), Rep. Prog. Phys. **88** 117601 (2025)

Take-Home Message

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Formulas

$$\rho \xrightarrow{\mathcal{G}_{\mathcal{C}}} \rho_{\mathcal{C}} = \sum_i \mathrm{Tr}(P_i \rho) \frac{P_i}{V_i}$$

$$S_{\mathrm{obs}}(\rho; \mathcal{C}) = S(\rho_{\mathcal{C}}) = -\sum_i p_i \log \frac{p_i}{V_i} = -\sum_i p_i \log p_i + \sum_i p_i \log V_i.$$

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