
Thermalization and its breakdown in closed quantum systems

Ryusuke Hamazaki (RIKEN, Japan)

Quantum thermalization, Hydrodynamics and Gravity
@ YITP, Kyoto (June 2026)



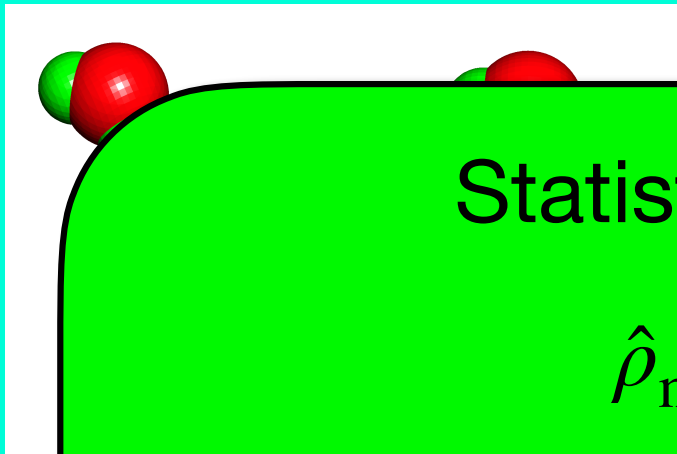
ERATO Sagawa Information-to-Energy
Interconversion Project



Motivation of this tutorial talk

(Quantum) mechanics
[microscale]

Dynamics of atoms/molecules



Thermodynamics
[macroscale]

Temperature, pressure, ...



Statistical mechanics

$$\hat{\rho}_{\text{mic}} \propto \delta(E - \hat{H})$$

Why is equilibrium stat. mech. valid?
How can we go beyond equilibrium?

Motivation of this tutorial talk

(Quantum) mechanics
[microscale]

Dynamics of atoms/molecules



**Universality of nonequilibrium
many-body dynamics
from quantum mechanics?**

Statistical mechanics

$$\hat{\rho}_{\text{mic}} \propto \delta(E - \hat{H})$$

Why is equilibrium stat. mech. valid?
How can we go beyond equilibrium?

From quantum mechanics to statistical mechanics

von
Neumann



Zeitschrift für Physik 57, 30 (1929)
“Beweis des Ergodensatzes und
des H-Theorems in der neuen Mechanik”

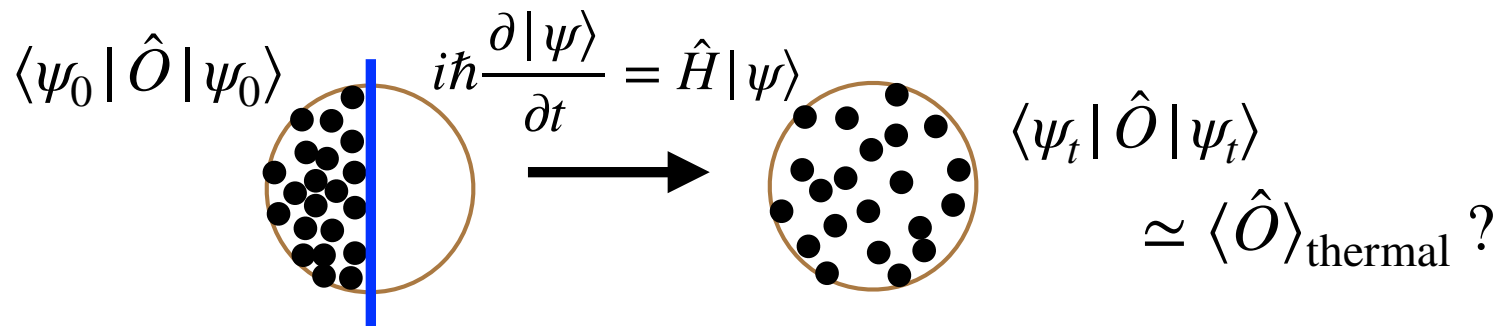


Boltzmann
(H theorem/stat.mech, 1872)

Schrödinger
(quantum mech., 1926)



Thermalization in isolated systems



He essentially provided modern ideas
(eigenstate thermalization, random matrices, ...)

Experiments of nonequilibrium quantum many-body dynamics

Artificial quantum systems: quantum simulators/computers

cold atom, superconducting qubits, trapped ions, ...

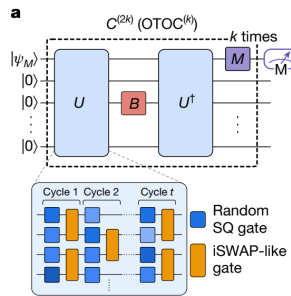
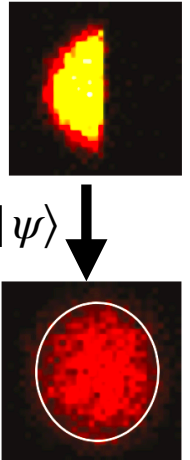
Noiseless systems with extreme controllability

→ Platform to investigate non-equilibrium physics from quantum mechanics

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = \hat{H} |\psi\rangle$$

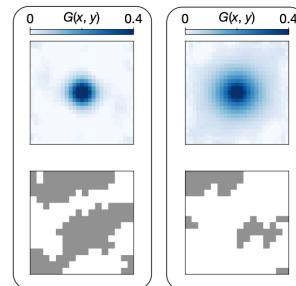
Thermalization

J. Choi et al.,
Science (2016)



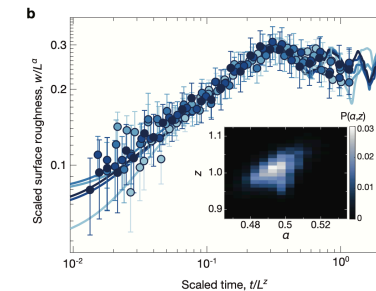
OTOC
(Google)

Google Quantum AI
and Collaborators, Nature (2025)



Coarsening
(Harvard)

T. Manovitz et al.,
Nature (2025)



“Surface” growth
(KAIST)

K. Kwon, ..., **RH**, ...
arXiv (2026)

Frontier of modern quantum science!

Outline

Part I. Closed quantum systems

Basics of thermalization and eigenstate thermalization

Thermalization, non-integrability, and symmetry

Breakdown of thermalization

Beyond thermalization: second law of thermodynamics

Part II. Open quantum systems

Measurement theory and GKSL equations

Stationary states and relaxation in GKSL dynamics

Measurement-induced transitions and Lyapunov spectrum

Thermalization and eigenstate thermalization hypothesis

Review articles:

(Rigorous discussions)

*T. Mori et al., *Journal of Physics B* 51 (11), 112001 (2018)

*C. Gogolin and J. Eisert, *Reports on Progress in Physics* 79.5 (2016)

(ETH and quantum chaos)

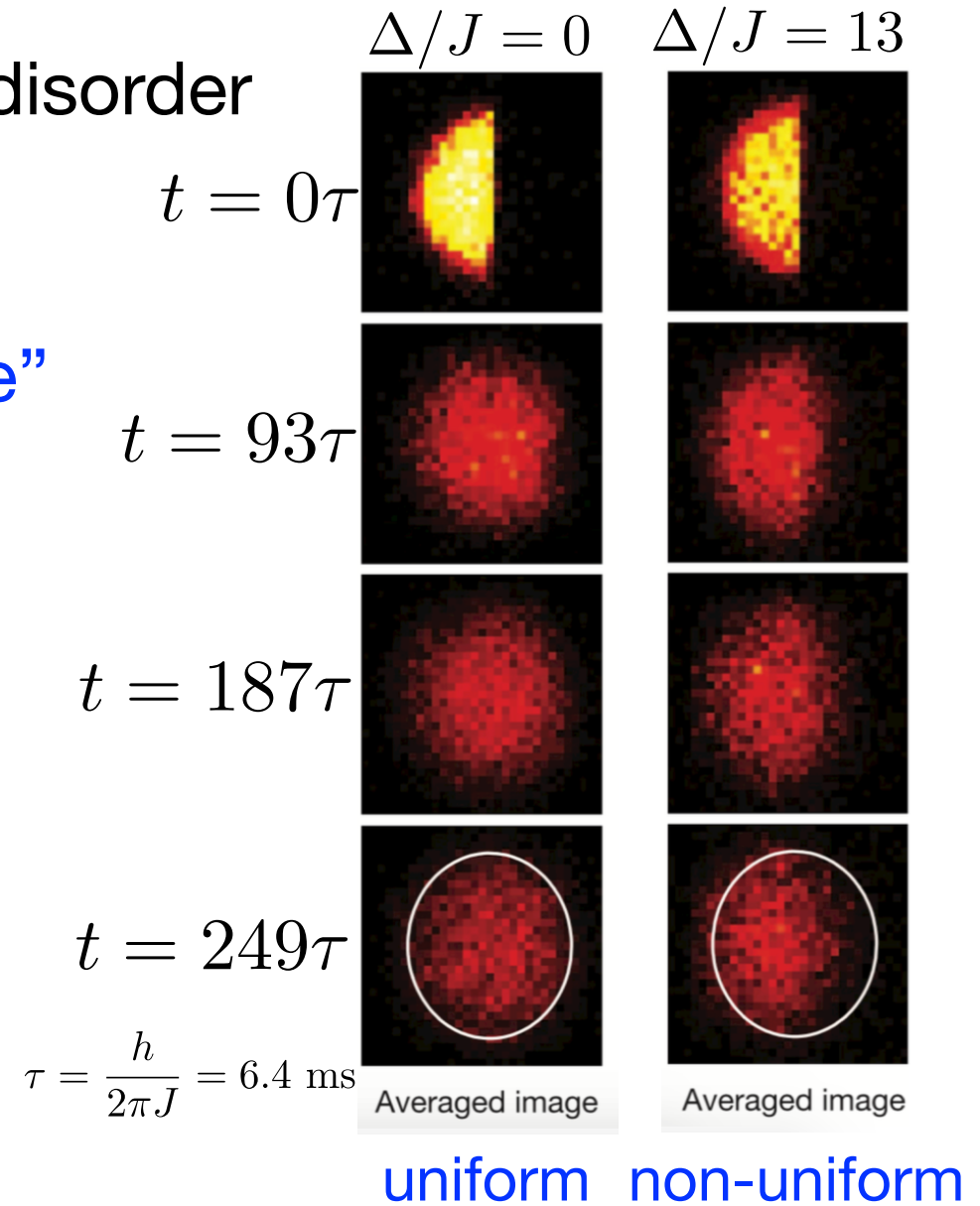
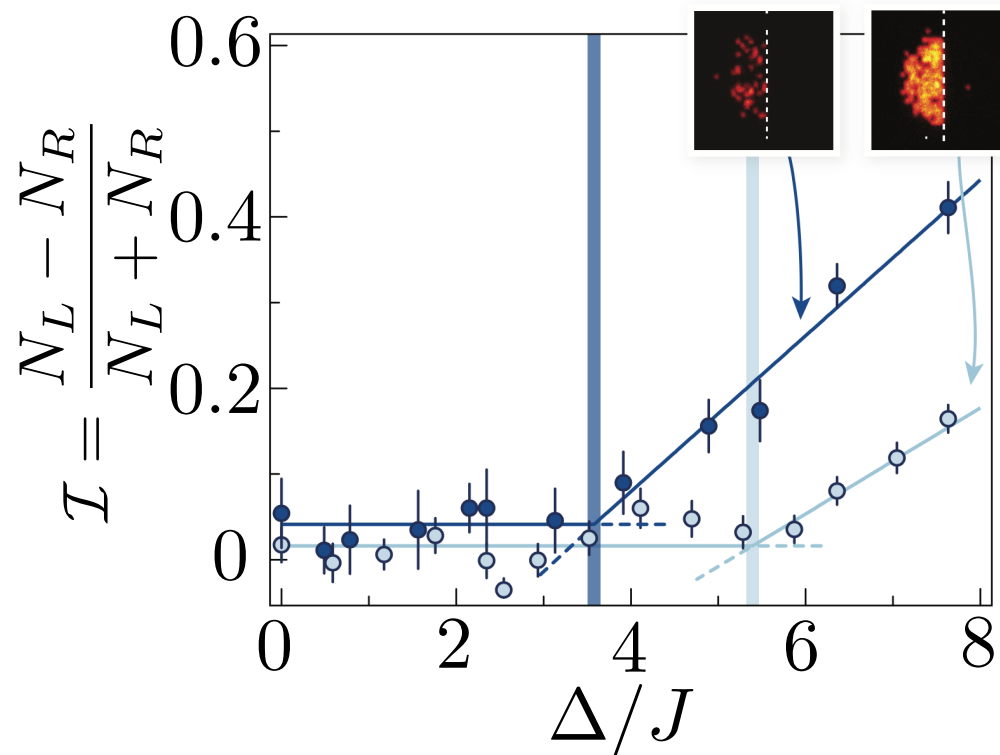
*L. D'Alessio et al., *Advances in Physics* 65.3 (2016)

Thermalization and its breakdown

2D Bose-Hubbard model with disorder
(disorder strength Δ)

J. Choi et al., Science 352.6293 (2016)

Observation of “MBL regime”



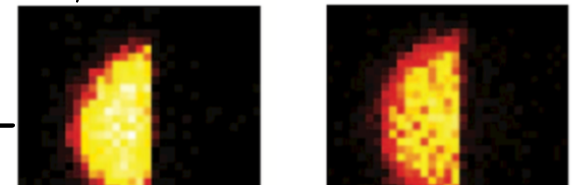
Thermalization and its breakdown

J. Choi et al., Science 352.6293 (2016)

2D Bose-Hubbard model with disorder
(disorder strength Δ)

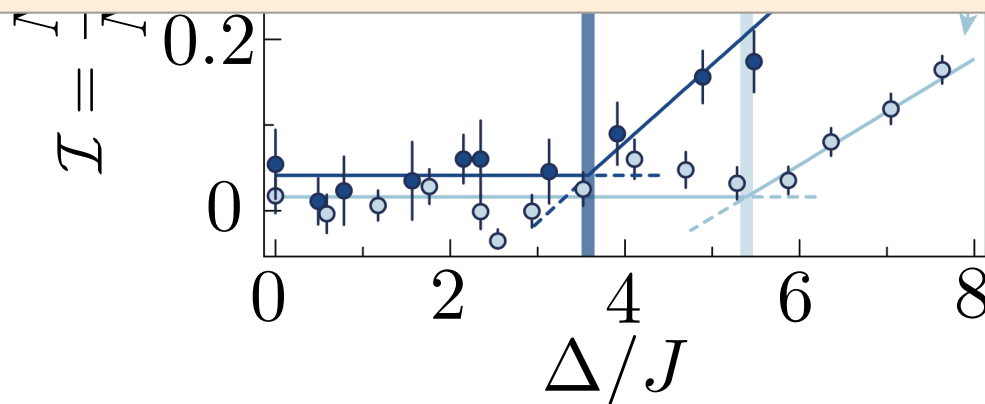
$\Delta/J = 0$ $\Delta/J = 13$

$t = 0\tau$



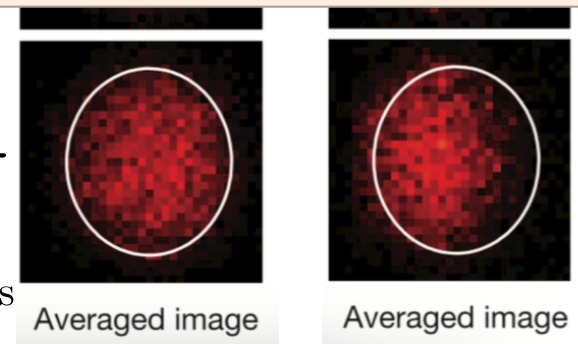
Problem of quantum thermalization is much more nontrivial than we naively think!

Eigenstate thermalization hypothesis (ETH) is the key for understanding thermalization



$t = 249\tau$

$$\tau = \frac{h}{2\pi J} = 6.4 \text{ ms}$$



uniform non-uniform

Thermalization of quantum pure states

Initial pure quantum states $|\psi\rangle$

→ Remain pure after time evolutions $|\psi_t\rangle = e^{-i\hat{H}t} |\psi\rangle$

Different from microcanonical ensemble at the level of density matrices

$$\hat{\rho}_t = |\psi_t\rangle\langle\psi_t| \neq \hat{\rho}_{\text{mic}} \quad \hat{\rho}_{\text{mic}}(E) = \frac{1}{d_{\text{mic}}} \sum_{\alpha: |E_\alpha - E| \leq \delta E} |E_\alpha\rangle\langle E_\alpha|$$

If one focuses on specific observables \hat{A}

(e.g., macroscopic ones), $\hat{\rho}_t$ can give thermal values

$$A(t) = \text{Tr}[\hat{\rho}_t \hat{A}] \simeq \text{Tr}[\hat{\rho}_{\text{mic}} \hat{A}]$$

↑
subextensive correction is neglected in the thermodynamic limit

→ Discuss thermalization at the level of observable(s)!

How to characterize thermalization?

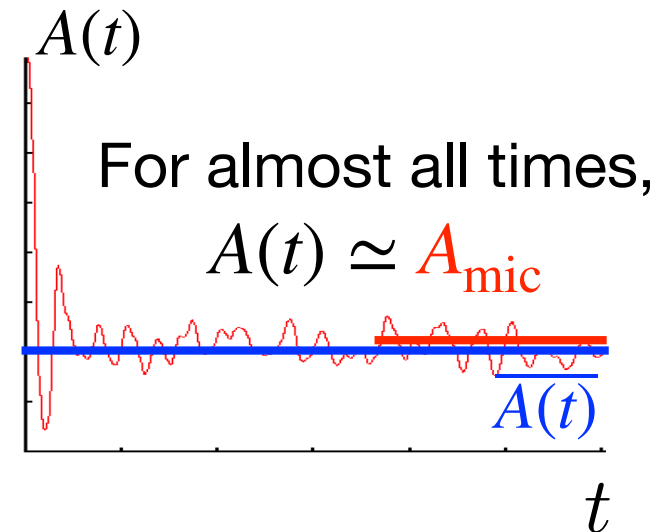
In this talk, we focus on long-time thermalization of observables in large but finite isolated quantum systems

Definition of thermalization

(Note: the definition differs across the literature)

When $A(t) \simeq A_{\text{mic}}$ for almost all times, we say that \hat{A} thermalizes

Avoid the issue of quantum recurrence



Assume that $A(t) \simeq \overline{A(t)}$ for almost all times, i.e., the temporal fluctuation is small

$$\overline{A} := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T A(t) dt$$

Long-time average

→ We focus on thermalization on temporal average $\overline{A(t)} \simeq A_{\text{mic}}$

Spectral decomposition

Thermalization on temporal average $\overline{A(t)} \simeq A_{\text{mic}}$?

Spectral decomposition $\hat{H} = \sum_{\alpha} E_{\alpha} |E_{\alpha}\rangle\langle E_{\alpha}|$

$$\rightarrow A(t) = \sum_{\alpha\beta} c_{\alpha}^* c_{\beta} e^{i(E_{\alpha}-E_{\beta})t} A_{\alpha\beta} \quad \begin{array}{l} A_{\alpha\beta} = \langle E_{\alpha} | \hat{A} | E_{\beta} \rangle \\ c_{\alpha} = \langle E_{\alpha} | \psi_0 \rangle \end{array}$$

If there is no degeneracy, the condition is rewritten as

$$\overline{A(t)} = \sum_{\alpha} |c_{\alpha}|^2 A_{\alpha\alpha} \simeq A_{\text{mic}}$$

If ETH is assumed for the matrix elements $A_{\alpha\alpha}$,
this is satisfied regardless of the details of the initial state!

Eigenstate thermalization hypothesis (ETH)

ETH for diagonal matrix elements of \hat{A}

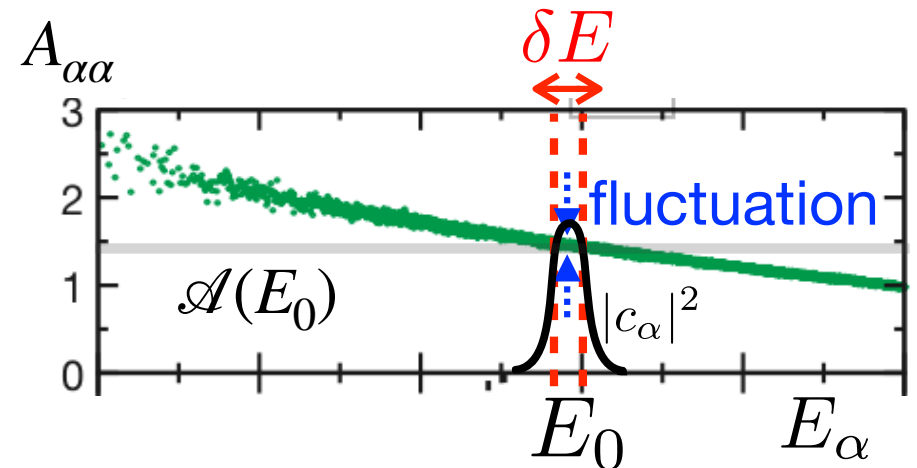
For all energy eigenstates $|E_\alpha\rangle, |E_\beta\rangle$ in the energy shell $[E - \delta E/2, E + \delta E/2]$ subextensive
 $|A_{\alpha\alpha} - A_{\beta\beta}|$ becomes negligible in the thermodynamic limit
 $[\forall \alpha \ A_{\alpha\alpha} \rightarrow \mathcal{A}(E_\alpha)]$

→ For all initial states with small energy fluctuations,
 $\overline{A(t)} \simeq A_{\text{mic}}$ is justified

$$\overline{A(t)} = \frac{\sum_{\alpha} |c_{\alpha}|^2 \langle E_{\alpha} | \hat{A} | E_{\alpha} \rangle}{\sum_{\alpha} |c_{\alpha}|^2} \simeq A_{\text{mic}}$$

$\underbrace{\qquad\qquad\qquad}_{= 1}$ $\underbrace{\qquad\qquad\qquad}_{\simeq \mathcal{A}(E_0)}$ $\underbrace{\qquad\qquad\qquad}_{\text{ETH}}$

normalization



Thermalization, nonintegrability and symmetry

Review articles:

(ETH and quantum chaos)

*L. D'Alessio et al., *Advances in Physics* 65.3 (2016)

(Dynamics of integrable systems)

*L. Vidmar and M. Rigol, *J. Stat. Mech.* (2016) 064007

*F. H. L. Essler and M. Fagotti, *J. Stat. Mech.* (2016) 064002

Classification of quantum many-body systems

Nonintegrable systems

$$\hat{H} = \sum_i J \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z + g \hat{\sigma}_i^x + h \hat{\sigma}_i^z$$

→ Many models are found to satisfy ETH/thermalization

ETH holds in
many systems



Integrable systems

- Noninteracting systems
- Systems mappable to quadratic forms of quasiparticles
- Systems solvable via Bethe ansatz

Breakdown of ETH

conserved quantities given by
(an extensive sum of) local operators



Many local conserved quantities

→ ETH and thermalization fail

The stationary state is given by the generalized Gibbs ensemble

Nonintegrability and ETH

XXZ model on a ladder: tunable integrability

System-size dependence of the ETH?

W. Beugeling et al., Phys. Rev. E 89 (2014).

\hat{A} : spin-z component

$(L, N_{\uparrow}) = (11, 5)$

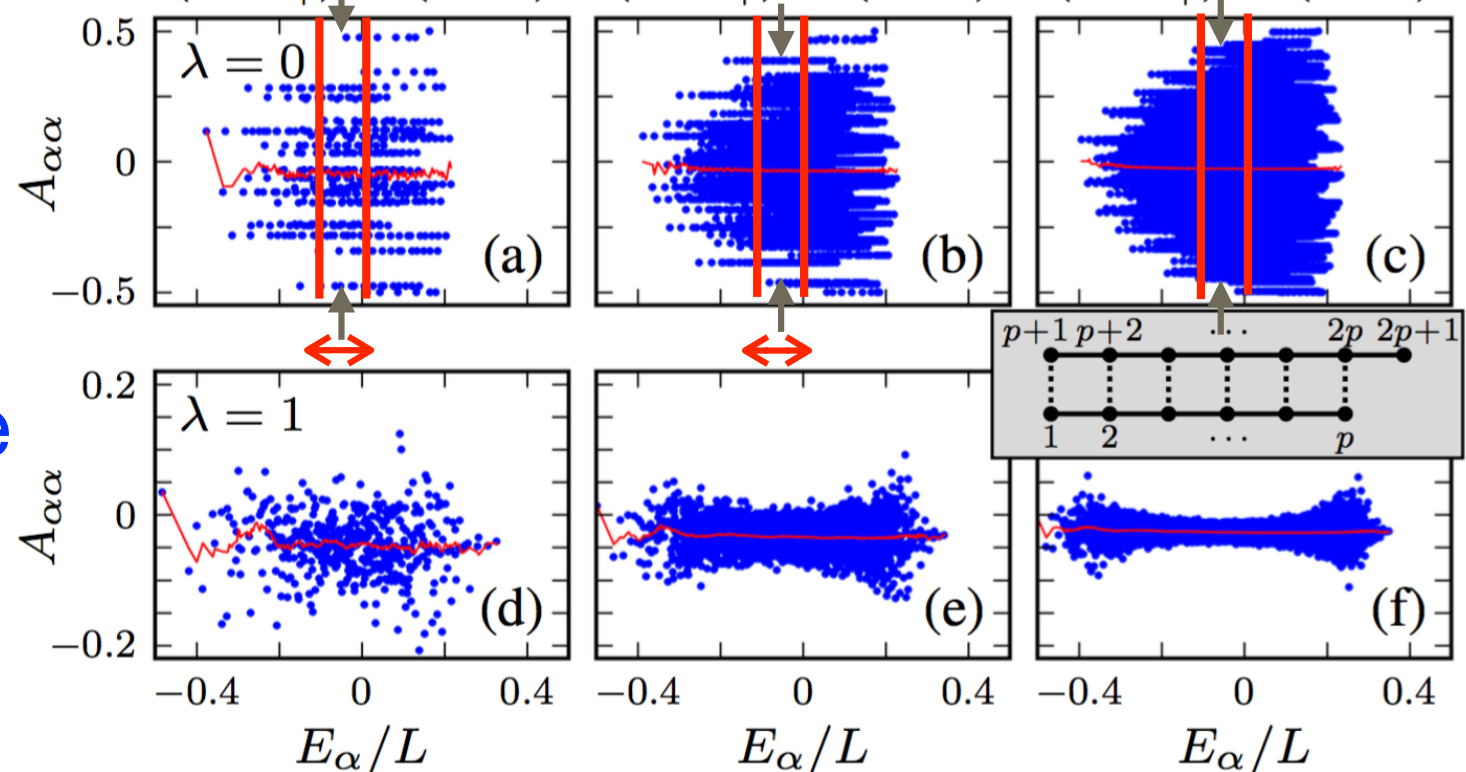
$(L, N_{\uparrow}) = (15, 7)$

$(L, N_{\uparrow}) = (19, 9)$

Integrable

No ETH

Nonintegrable
(chaos)



Nonintegrability and ETH

XXZ model on a ladder: tunable integrability

System-size dependence of the ETH?

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\hat{A} : spin-z component

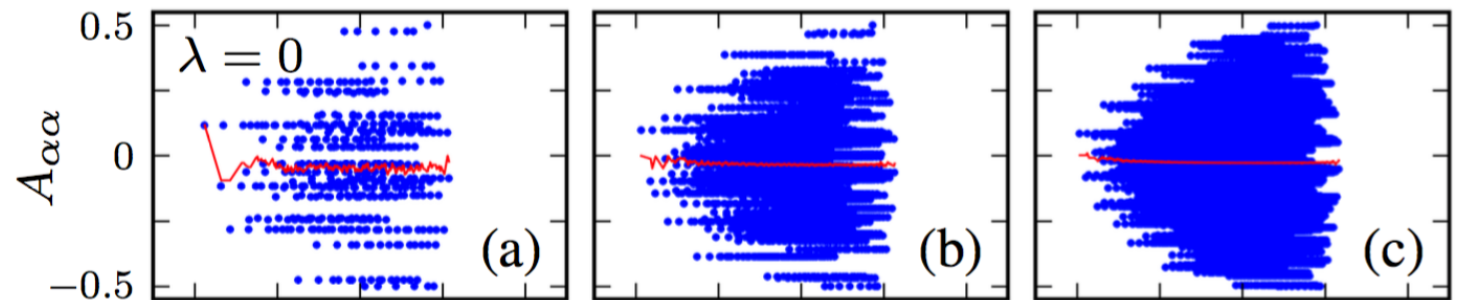
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Integrable

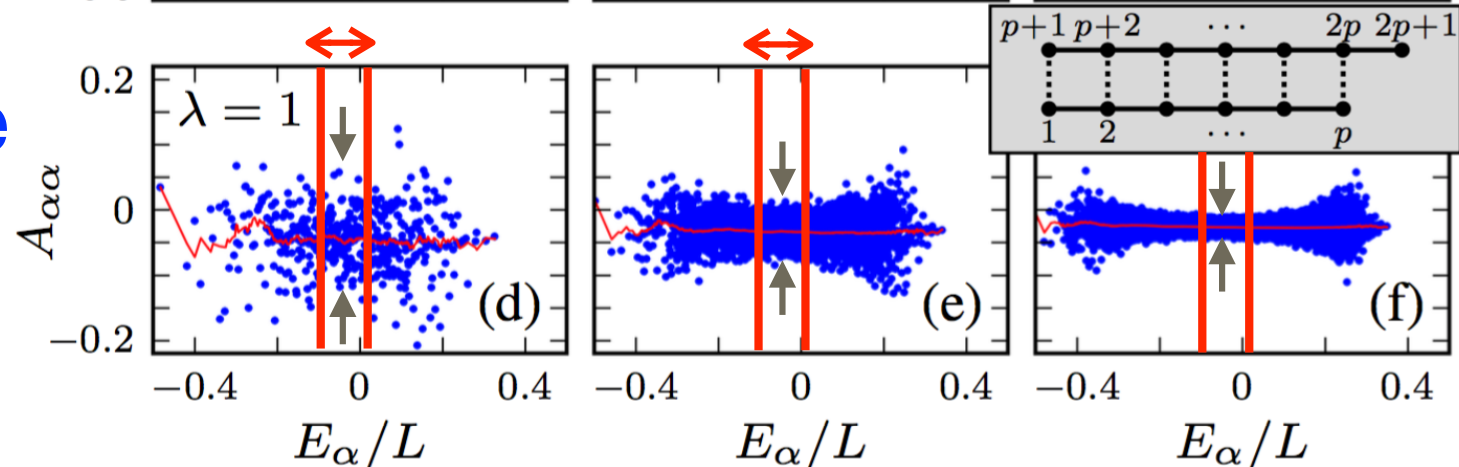
No ETH



Nonintegrable
(chaos)

ETH:

fluctuations decay exponentially fast!



Srednicki's conjecture

In chaotic quantum systems,
matrix elements at finite sizes are conjectured to follow:

M. Srednicki, J. Phys. A: Math. Gen. 32 1163 (1999)

$$A_{\alpha\beta} = \mathcal{A}(E_\alpha)\delta_{\alpha\beta} + e^{-S(\bar{E})/2}f(\bar{E}, \omega)R_{\alpha\beta} \quad \begin{array}{l} \bar{E} = (E_\alpha + E_\beta)/2 \\ \omega = E_\beta - E_\alpha \end{array}$$

$S(E) = \mathcal{O}(N)$: thermodynamic entropy

$f(E, \omega)$: smooth function (related with two-point correlator $\langle \hat{A}(t)\hat{A} \rangle$)

$R_{\alpha\beta}$: quasi-random variables with mean 0, variance 1

At finite sizes, the second term decays exponentially $e^{-\mathcal{O}(N)}$
← recovering the ETH in the thermodynamic limit

Within sufficiently small energy shell $\omega \ll 1$,
matrix elements obey statistics predicted by random-matrix theory

See also A. Dymarsky, PRL (2022) for the scale for ω

Srednicki's conjecture

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Remark: several directions beyond this conjecture

*Full ETH: higher-order correlations of matrix elements $A_{\alpha_1\alpha_2}A_{\alpha_2\alpha_3}A_{\alpha_3\alpha_4}\dots$,
(necessary to account for multi-point time correlators, e.g., OTOC)

L. Foini and J. Kurchan, PRE (2019); A. Chan et al., PRL (2019);
C. Murthy and M. Srednicki, PRL (2019); S. Pappalardi et al., PRL (2022)

*Hydrodynamics can give information on matrix elements (e.g., $f(E, \omega)$)

L. Capizzi et al., PRX (2025); J. Wang et al., PRL (2026)

See also A. Dymarsky, PRL (2022) for the scale for ω

Spectral statistics and nonintegrability

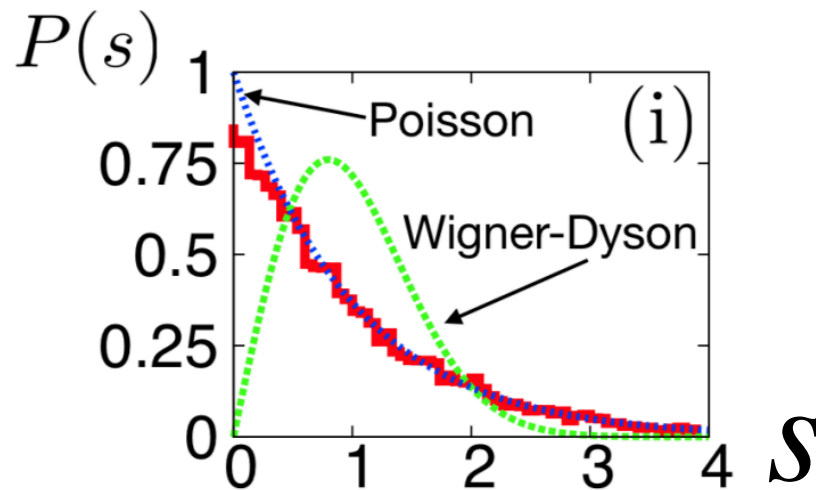
ETH and Srednicki's conjecture

← Motivation from random-matrix theory

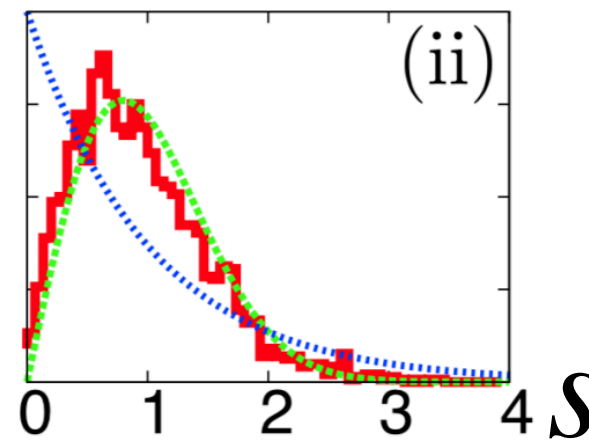
$$\hat{H} |E_\alpha\rangle = E_\alpha |E_\alpha\rangle$$

Relevant for eigenstates, but how about eigenvalues?

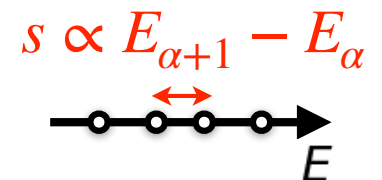
Level-spacing distributions $P(s)$



Integrable
→ Poisson



Non-integrable (no symmetry)
→ Random-matrix theory



cf) Analytical approaches for spectral form factor:

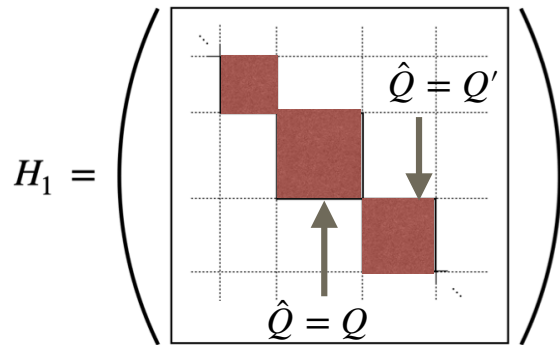
S. Müller, S. Heusler, P. Braun, F. Haake, A. Altland, PRL (2004); PRE (2005) [semiclassical];

B. Bertini, P. Kos, T Prosen, PRL (2018) [self-dual circuit; cf. B. Bertini's talk]

Symmetry and ETH in nonintegrable systems

Continuous symmetry: $U(1)$ $\hat{H} = \sum_{i=1}^{L-1} J \vec{\hat{\sigma}}_i \cdot \vec{\hat{\sigma}}_{i+1} + \sum_{i=1}^L h_i \hat{\sigma}_i^z$ h_i : small disorder

Nonintegrable, but with a *local* conserved quantity $\hat{Q} = \sum_{i=1}^L \hat{\sigma}_i^z$
 $[\hat{H}, \hat{Q}] = 0$



Block diagonalization
by the total magnetization

$$\mathcal{H} = \bigoplus_Q \mathcal{H}^Q$$

In the mixed symmetry sectors, ETH breaks down

$$\langle E_\alpha^{(Q)} | \hat{A} | E_\alpha^{(Q)} \rangle \neq \langle E_\beta^{(Q')} | \hat{A} | E_\beta^{(Q')} \rangle \quad Q = \langle E_\alpha^{(Q)} | \hat{Q} | E_\alpha^{(Q)} \rangle \neq \langle E_\beta^{(Q')} | \hat{Q} | E_\beta^{(Q')} \rangle = Q'$$

Within sectors specified by \hat{Q} , ETH may be recovered

$$\langle E_\alpha^{(Q)} | \hat{A} | E_\alpha^{(Q)} \rangle = \langle E_\beta^{(Q)} | \hat{A} | E_\beta^{(Q)} \rangle ?$$

Symmetry and ETH in nonintegrable systems

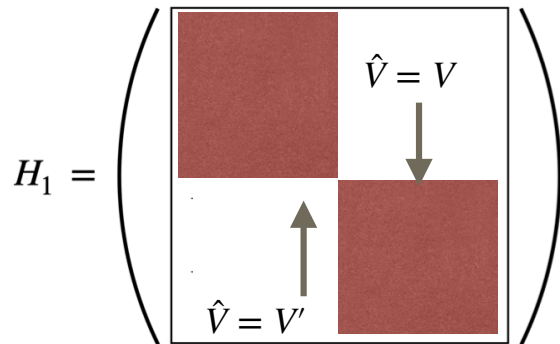
How about discrete symmetries?
 e.g., transverse-field Ising models (TFIM)

$$\hat{H} = \sum_{\langle i,j \rangle} J \hat{\sigma}_i^z \hat{\sigma}_j^z + \sum_i h_i \hat{\sigma}_i^x$$

Non-local conserved quantity $\hat{V} = \prod_i \hat{\sigma}_i^x$

$$[\hat{H}, \hat{V}] = 0$$

\mathbb{Z}_2 symmetry



Block diagonalization
 by the symmetry

$$\mathcal{H} = \mathcal{H}^{V=+1} \oplus \mathcal{H}^{V=-1}$$

ETH trivially breaks for \hat{V} for mixed sectors

$$V = \langle E_\alpha^{(V)} | \hat{V} | E_\alpha^{(V)} \rangle \neq \langle E_\beta^{(V')} | \hat{V} | E_\beta^{(V')} \rangle = V'$$

Typically, ETH can still hold for *local* observables even for the mixed sectors

$$\langle E_\alpha^{(V)} | \hat{A}_{\text{loc}} | E_\alpha^{(V)} \rangle \simeq \langle E_\beta^{(V')} | \hat{A}_{\text{loc}} | E_\beta^{(V')} \rangle$$

e.g.) $\hat{A}_{\text{loc}} = \hat{\sigma}_1^x \hat{\sigma}_2^x$

Locality of observables and conserved quantities

Local conserved quantities break ETH for local observables

Non-local conserved quantities (from discrete symmetry) typically do not break ETH for local observables

$$\langle E_\alpha^{(V)} | \hat{A}_{\text{loc}} | E_\alpha^{(V)} \rangle \simeq \langle E_\beta^{(V)} | \hat{A}_{\text{loc}} | E_\beta^{(V)} \rangle$$

*Breakdown of ETH for local observables without local conserved quantities?

← Quantum scars, Hilbert-space fragmentation

*Do non-local conserved quantities lead to breakdown of ETH for nontrivial non-local observables different from \hat{V} ?

$$\langle E_\alpha^{(V)} | \hat{A}_{\text{nloc}} | E_\alpha^{(V)} \rangle \neq \langle E_\beta^{(V)} | \hat{A}_{\text{nloc}} | E_\beta^{(V)} \rangle ?$$

Locality of observables and conserved quantities

Local conserved quantities break ETH for local observables

Non-local conserved quantities (from discrete symmetry) typically do not break ETH for local observables

A class of non-local observables are proved to break the ETH for mixed sectors due to discrete symmetry

[O. Fukushima and **RH**, Physical Review Letters 131 (13), 131602 (2023).]

← Quantum scars, Hilbert-space fragmentation

*Do non-local conserved quantities lead to breakdown of ETH for nontrivial non-local observables different from \hat{V} ?

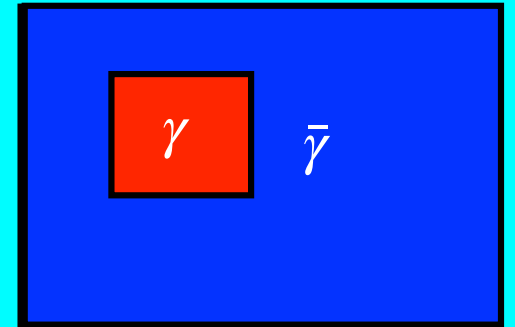
$$\langle E_{\alpha}^{(V)} | \hat{A}_{\text{nloc}} | E_{\alpha}^{(V)} \rangle \neq \langle E_{\beta}^{(V)} | \hat{A}_{\text{nloc}} | E_{\beta}^{(V)} \rangle ?$$

Discrete symmetries lead to breakdown of thermalization for non-local observables

General result: for symmetry $\hat{V}_{\tilde{\Omega}}$ ($\tilde{\Omega} \subseteq \Omega$), assume that symmetry operator is decomposed as $\hat{V}_{\tilde{\Omega}} = \hat{V}_{\gamma} \hat{V}_{\bar{\gamma}}$ $\bar{\gamma} = \tilde{\Omega} \setminus \gamma$

→ After additional (reasonable) assumptions, we prove that either \hat{V}_{γ} or $\hat{V}_{\bar{\gamma}}$ breaks the ETH for mixed symmetry sectors

$$\Omega = \tilde{\Omega}$$



Take γ local: then the ETH for \hat{V}_{γ} is typically expected
→ Non-local observable $\hat{V}_{\bar{\gamma}}$ breaks the ETH

The case for the higher-form symmetries

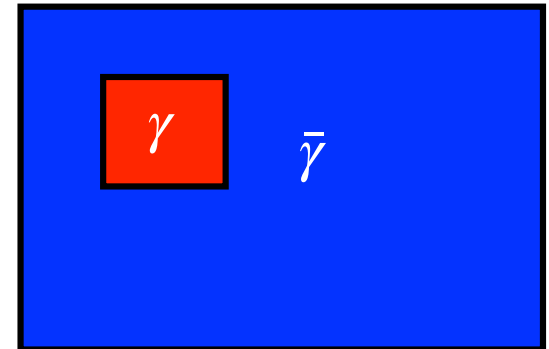
How large is $\hat{V}_{\bar{\gamma}}$, compared with the total system size?

Conventional global symmetries: $\dim[\tilde{\Omega}] = \dim[\Omega]$

(Operator size of $\hat{V}_{\bar{\gamma}}$) $\simeq \text{Vol}[\Omega]$

Breakdown of the ETH may be attributed to the lack of the large “bath” in this case

$$\Omega = \tilde{\Omega}$$

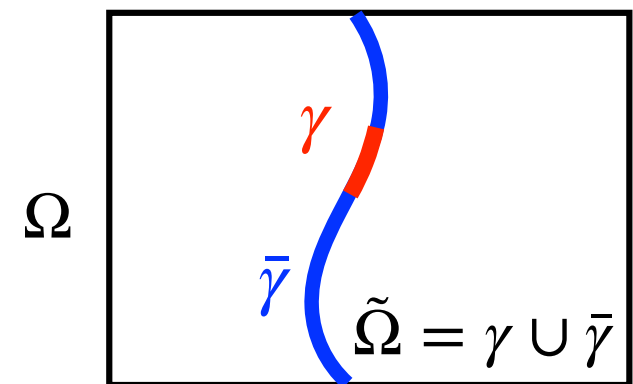


Higher-form symmetries:

$\dim[\tilde{\Omega}] < \dim[\Omega]$

(Operator size of $\hat{V}_{\bar{\gamma}}$) $\ll \text{Vol}[\Omega]$

$\hat{V}_{\bar{\gamma}}$ breaks the ETH due to symmetry, despite the large bath



Demonstration: 2D Lattice \mathbb{Z}_2 gauge theory

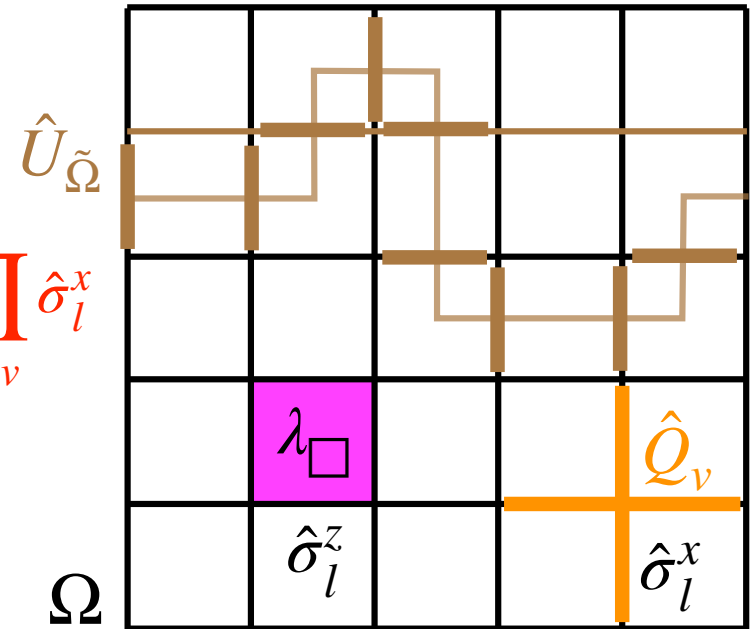
$$\hat{H} = - \sum_{\square} \lambda_{\square} \prod_{l \in \square} \hat{\sigma}_l^z - g \sum_{l \in \Omega} \hat{\sigma}_l^x$$

Gauge symmetry (local): $[\hat{H}, \hat{Q}_v] = 0$, $\hat{Q}_v = \prod_{l \in v} \hat{\sigma}_l^x$

→ Take the sector with $Q_v = +1$ (for all v)

Higher-form symmetry:

$[\hat{H}, \hat{U}_{\tilde{\Omega}}] = 0$, $\hat{U}_{\tilde{\Omega}} = \prod_{l \in \tilde{\Omega}} \hat{\sigma}_l^x$, $\tilde{\Omega}$ is obtained by a closed loop



Demonstration: 2D Lattice \mathbb{Z}_2 gauge theory

$$\hat{H} = - \sum_{\square} \lambda_{\square} \prod_{l \in \square} \hat{\sigma}_l^z - g \sum_{l \in \Omega} \hat{\sigma}_l^x$$

Gauge symmetry (local): $[\hat{H}, \hat{Q}_v] = 0$, $\hat{Q}_v = \prod_{l \in v} \hat{\sigma}_l^x$

→ Take the sector with $Q_v = +1$ (for all v)

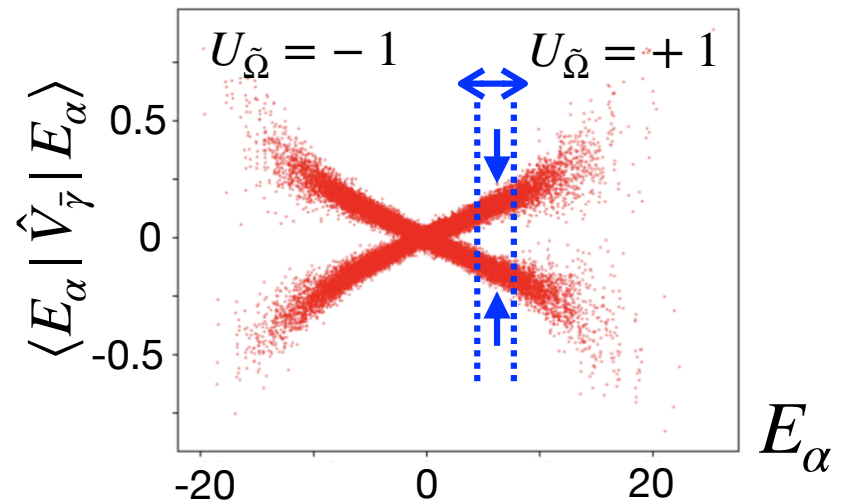
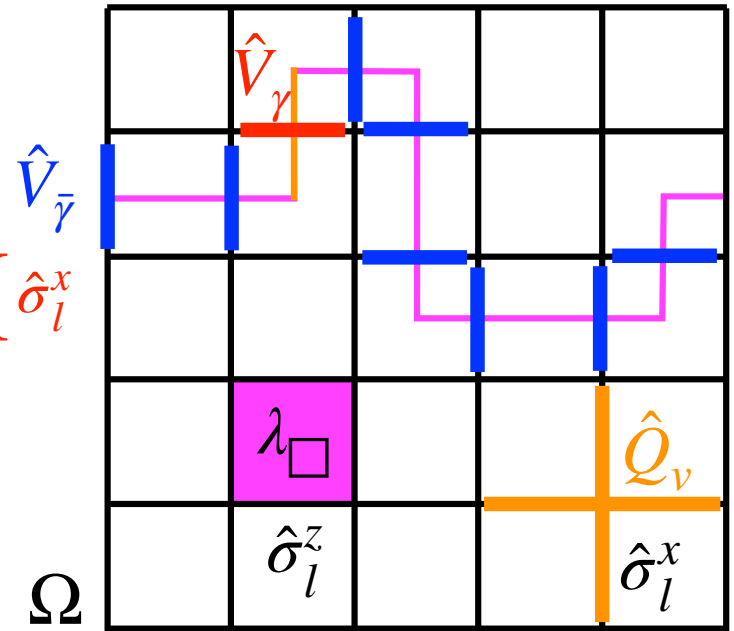
Higher-form symmetry:

$[\hat{H}, \hat{U}_{\tilde{\Omega}}] = 0$, $\hat{U}_{\tilde{\Omega}} = \prod_{l \in \tilde{\Omega}} \hat{\sigma}_l^x$, $\tilde{\Omega}$ is obtained by a closed loop

$\hat{U}_{\tilde{\Omega}} = \hat{V}_{\gamma} \hat{V}_{\bar{\gamma}} \rightarrow \hat{V}_{\bar{\gamma}}$ breaks the ETH,
while \hat{V}_{γ} shows the ETH

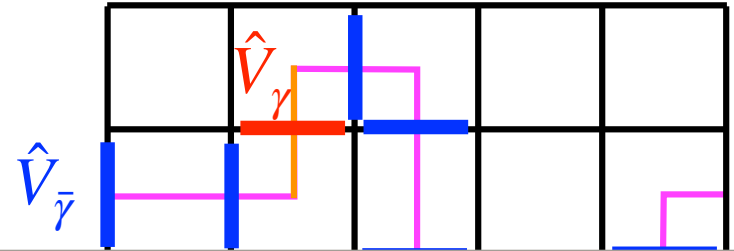
$$\text{Vol}[\bar{\gamma}] \ll \text{Vol}[\Omega]$$

[O. Fukushima and **RH**, PRL 131 (13), 131602 (2023).]



Demonstration: 2D Lattice \mathbb{Z}_2 gauge theory

$$\hat{H} = - \sum_{\square} \lambda_{\square} \prod_{l \in \square} \hat{\sigma}_l^z - g \sum_{l \in \Omega} \hat{\sigma}_l^x$$



Remark: other aspects of symmetries

*Non-commutative symmetry operators
 → ETH for degenerated eigenstates

Non-Abelian symmetry, Abelian symmetry with projective representation

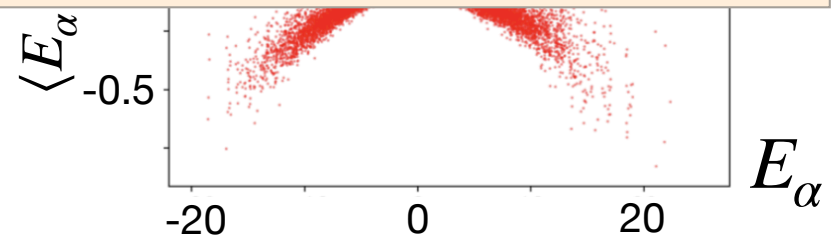
C. Murthy et al., PRL (2023), S Onoda, O Fukushima, **RH**, O Morikawa, JHEP, in press (2026)

*Symmetry breaking at finite energy density

KR. Fratus and M. Srednicki, PRE (2015); arXiv (2016)

WHILE $\hat{V}_{\bar{\gamma}}$ SHOWS THE ETH

$$\text{Vol}[\bar{\gamma}] \ll \text{Vol}[\Omega]$$



[O. Fukushima and **RH**, PRL 131 (13), 131602 (2023).]

Other mechanisms of breakdown of thermalization

Review articles:

(Many-body localization)

*R. Nandkishore, D. A. Huse,

Annu. Rev. Condens. Matter Phys. 6 (1), 15-38 (2018)

*P. Sierant et al., Rep. Prog. Phys. 88 026502 (2025)

(Quantum many-body scars, Hilbert-space fragmentation)

*M. Serbyn et al., Nature Physics 17, 675–685 (2021)

*S. Moudgalya et al., Rep. Prog. Phys. 85 086501 (2022)

Classification of quantum many-body systems

Nonintegrable systems

ETH holds in

Nontrivial mechanisms are found for the breakdown of ETH
beyond the classification via nonintegrability

Integrable systems

Breakdown of ETH

Many-body localization (MBL):

localization via disorder

Quantum many-body scars:

embedded nonthermal eigenstates / subspaces

Hilbert-space fragmentation:

exponentially large splitting of the Hilbert space

Many-body localization (MBL)

Due to strong disorder, thermalization is hindered

e.g.)
$$\hat{H} = \sum_{i=1}^{L-1} J \vec{\hat{\sigma}}_i \cdot \vec{\hat{\sigma}}_{i+1} + \sum_{i=1}^L h_i \hat{\sigma}_i^z \quad h_i \in [-h, h]$$

Delocalized regime

MBL regime

disorder
strength h

ETH

ETH seems absent

The existence of a stable MBL *phase*
in the thermodynamic limit is still controversial

MBL *regime* certainly exists at experimentally relevant
finite system sizes and offers interesting physics

P. Sierant et al., Rep. Prog. Phys. 88 026502 (2025)

Nonthermalization in nonintegrable systems: quantum many-body scars

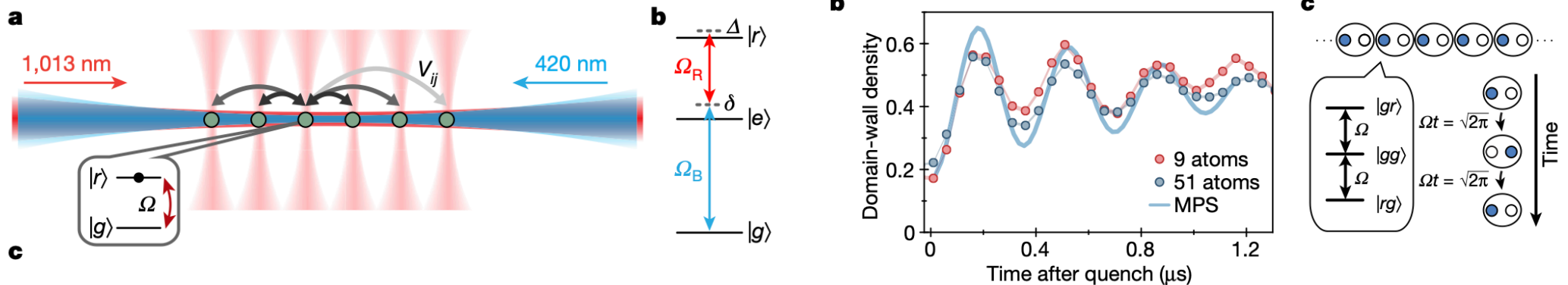
Array of Rydberg atoms in one dimension

$$\hat{H} = \sum_i \frac{\Omega}{2} \hat{\sigma}_i^x - \sum_i \Delta \hat{n}_i + \sum_i V \hat{n}_i \hat{n}_{i+1}$$

$\Omega, \Delta \ll V$: Adjacent excitations are suppressed (Rydberg blockade)

$|\uparrow\uparrow\rangle$

H. Bernien et al., Nature 551, 579–584 (2017)



Starting from the \mathbb{Z}_2 state $|\uparrow\downarrow\cdots\uparrow\downarrow\rangle$,
oscillations persist for a long time

Nonthermalization in nonintegrable systems: quantum many-body scars

$$\hat{H} = \sum_i \frac{\Omega}{2} \hat{\sigma}_i^x - \sum_i \Delta \hat{n}_i + \sum_i V \hat{n}_i \hat{n}_{i+1} \quad \Delta = 0, \Omega = 2 \ll V \quad \hat{P}_i = \frac{1 - \hat{\sigma}_i^z}{2}$$

Rydberg blockade
 $\rightarrow \hat{H}_{\text{eff}} = \sum_i \hat{P}_i \hat{\sigma}_{i+1}^x \hat{P}_{i+2}$
PXP model
(non-integrable)

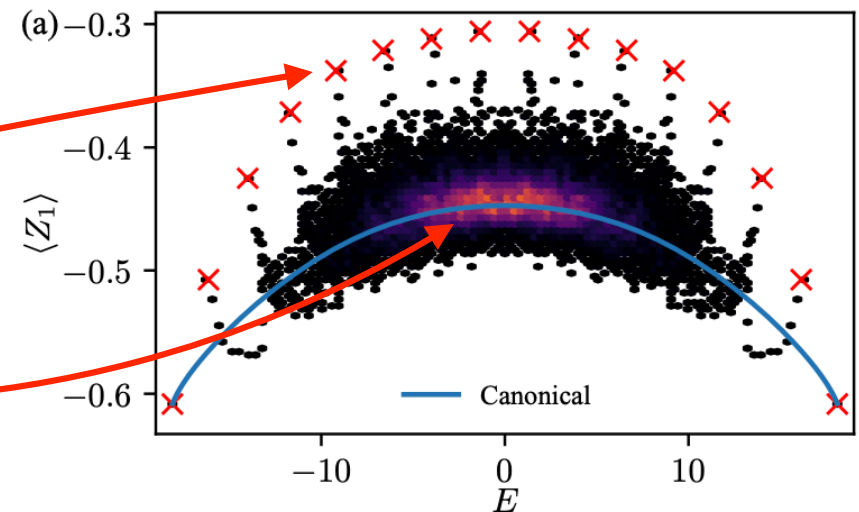
CJ. Turner et al., Nat. Phys. (2018)

"Quantum many-body scars"

*There exist eigenstates that do not satisfy ETH
 \rightarrow they affect dynamics of observables

*Most eigenstates satisfy ETH
 \rightarrow Typical initial states thermalize,
 but initial states with large overlap with the scars do not thermalize

CJ. Turner et al., Phys. Rev. B 98, 155134 (2018)



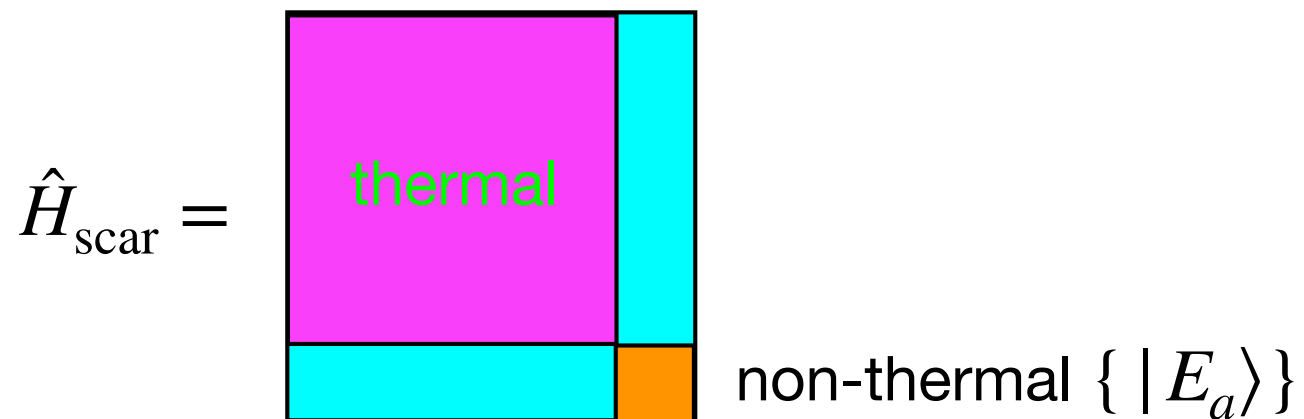
cf. many scar states at nonzero energy in the PXP model are approximate ones

Nonthermalization in nonintegrable systems: quantum many-body scars

Various mechanisms for the emergence of quantum many-body scars have been proposed

- *Restricted spectrum-generating algebra S. Moudgalya et al., PRB (2020)
- *Shiraishi-Mori embedding N. Shiraishi and T. Mori, PRL (2017)
- *Commutant algebra S. Moudgalya and OI. Motrunich, PRX (2022, 2024)
- etc... See also C. Matsui's talk

In general, the Hamiltonian is block-diagonalized:
non-thermal eigenstates are embedded into thermal spectrum
without local conserved quantities

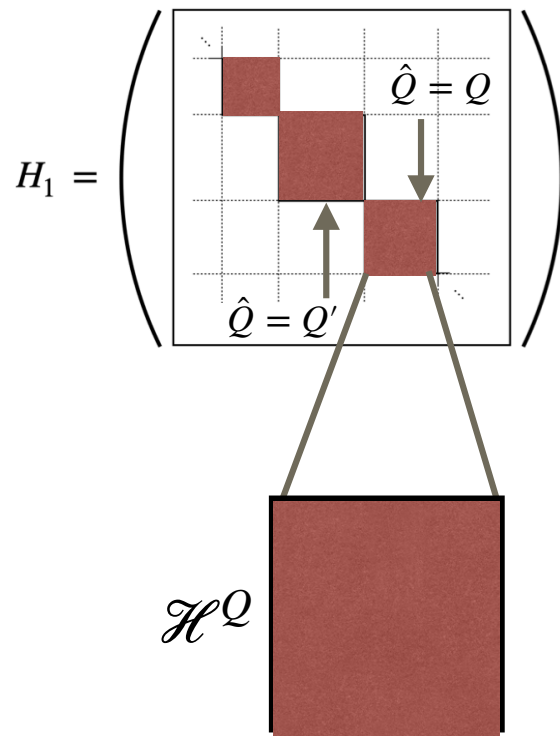


Hilbert-space fragmentation (HSF)

Continuous symmetries: $U(1)$

Nonintegrable, but with *local* conserved quantities \hat{Q}

$$[\hat{H}, \hat{Q}] = 0$$



Block diagonalization by
local conserved quantities

$$\mathcal{H} = \bigoplus_Q \mathcal{H}^Q$$

Within a sector specified by \hat{Q} , ETH often holds true

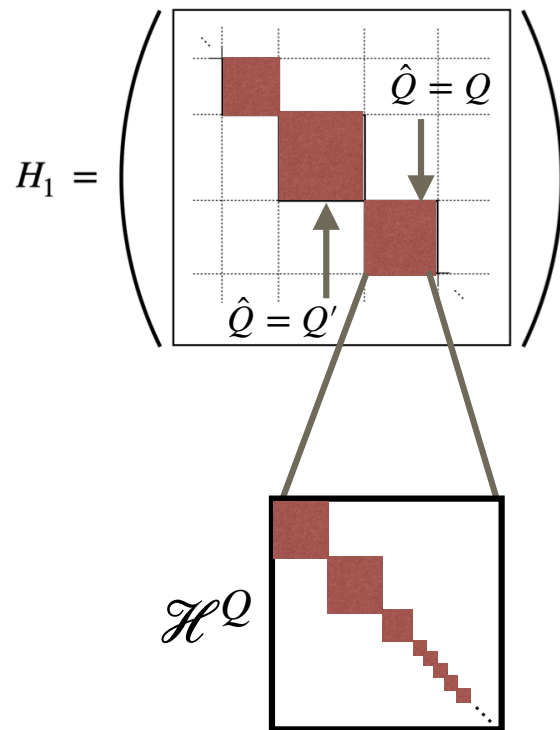
$$\langle E_\alpha^{(Q)} | \hat{A} | E_\alpha^{(Q)} \rangle = \langle E_\beta^{(Q)} | \hat{A} | E_\beta^{(Q)} \rangle$$

Hilbert-space fragmentation (HSF)

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$$\langle E_\alpha^{(Q)} | \hat{A} | E_\alpha^{(Q)} \rangle = \langle E_\beta^{(Q)} | \hat{A} | E_\beta^{(Q)} \rangle$$

Even after specifying all local conserved quantities,
 \mathcal{H}^Q can be further block diagonalized exponentially

→ Hilbert space fragmentation (HSF)
ETH breaks down for local observables

$$\langle E_\alpha^{(Q)} | \hat{A} | E_\alpha^{(Q)} \rangle \neq \langle E_\beta^{(Q)} | \hat{A} | E_\beta^{(Q)} \rangle$$

Hilbert-space fragmentation (HSF)

Continuous symmetries: $U(1)$

Nonintegrable, but with *local* conserved quantities \hat{Q}

HSF was first discussed for models with charge and dipole-moment conservation laws

S. Pai et al., PRX (2019); P. Sala et al., PRX (2020)

Here, I will discuss HSF obtained from high-dimensional quantum Ising models, another standard model in quantum many-body systems

A. Yoshinaga, H. Hakoshima, I. Imoto, Y. Matsuzaki, and **RH**, PRL 129, 090602 (2022)

→ Hilbert space fragmentation (HSF)
ETH breaks down for local observables

$$\langle E_{\alpha}^{(Q)} | \hat{A} | E_{\alpha}^{(Q)} \rangle = \langle E_{\beta}^{(Q)} | \hat{A} | E_{\beta}^{(Q)} \rangle$$

HSF in the quantum Ising models

High-dimensional TFIM

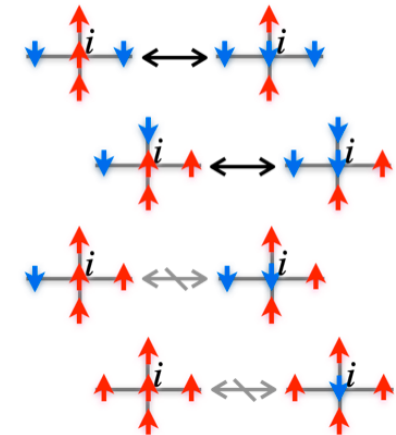
$$\hat{H} = \hat{H}_{\text{DW}} + g \sum_i \hat{\sigma}_i^x \quad \hat{H}_{\text{DW}} = - \sum_{\langle i,j \rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z$$

Effective model at $g \rightarrow 0$ (non-integrable)

$$\hat{H} = \hat{H}_{\text{DW}} + g \sum_i \hat{\sigma}_i^x \hat{Q}_i$$

\hat{Q}_i : projection operator

- 1 if the sum of the surrounding spins is zero;
- 0 otherwise



Nontrivial constraints in dynamics

HSF in the quantum Ising models

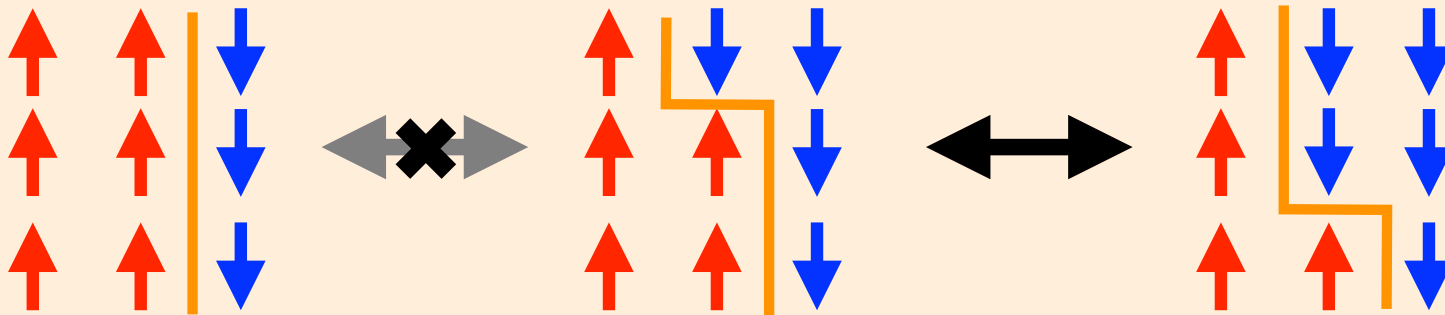
High-dimensional TFIM

$$\hat{H} = \hat{H}_{\text{DW}} + g \sum_i \hat{\sigma}_i^x \quad \hat{H}_{\text{DW}} = - \sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z$$

\hat{H}_{DW} is dominant over the transverse field

→ Only local conserved quantity (except for energy)

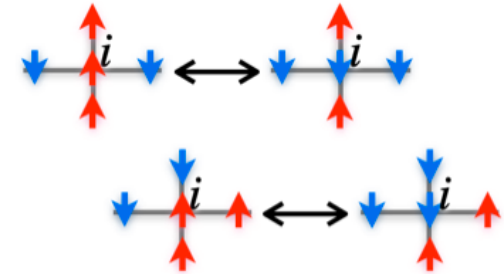
Transitions that do not change the domain-wall number are allowed



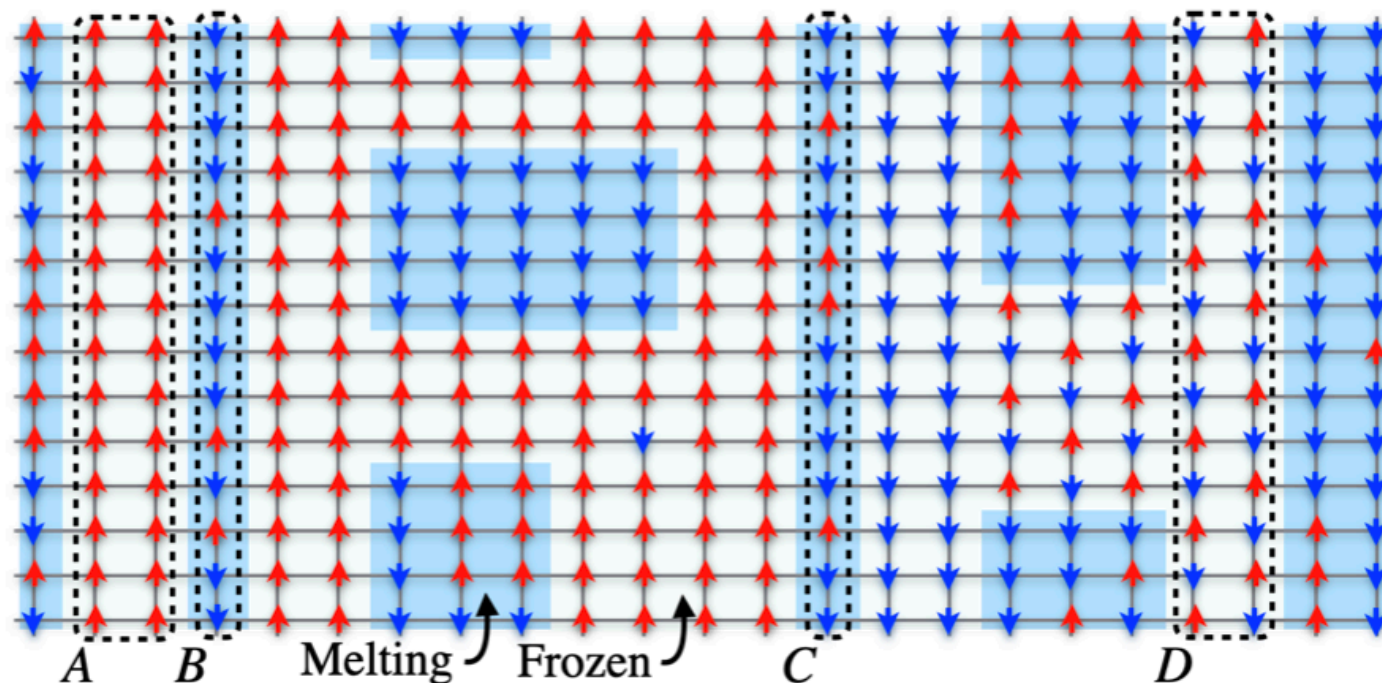
Nontrivial constraints in dynamics

Frozen regions

$$\hat{H} = \hat{H}_{\text{DW}} + h_x \sum_i \hat{\sigma}_i^x \hat{Q}_i$$

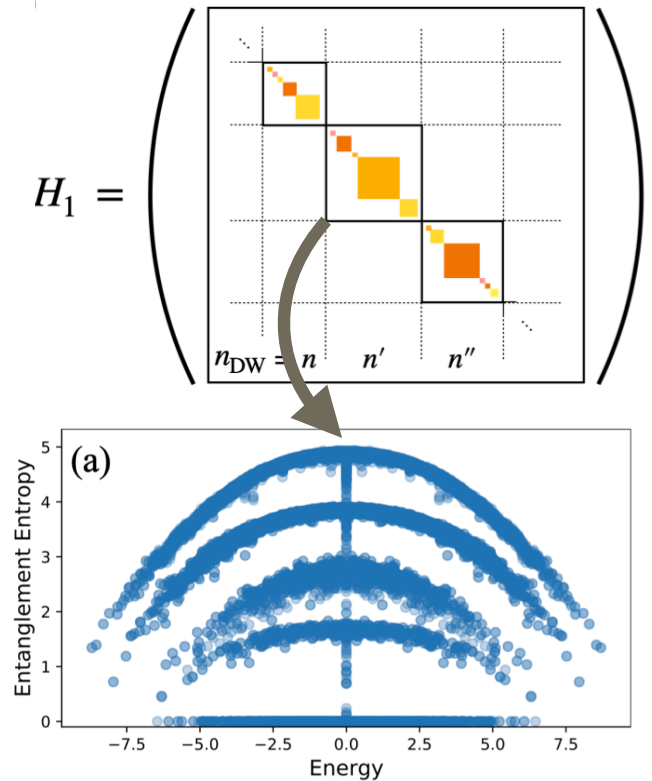
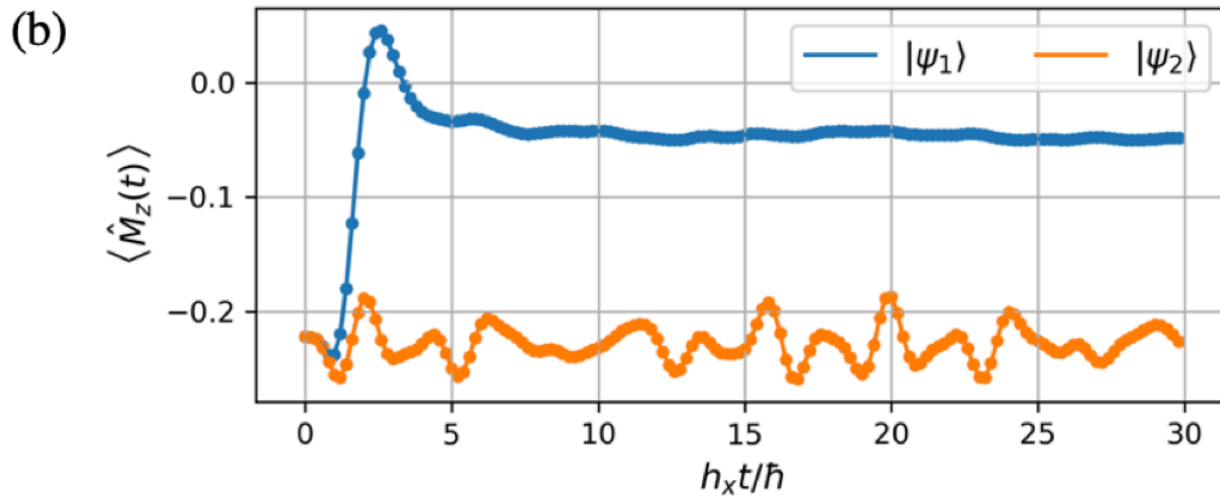
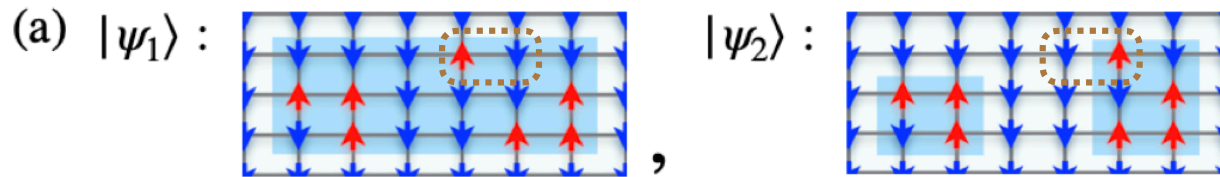


Due to dynamical constraints,
invariant (= frozen) regions exist, spatially dividing the system



Non-thermal dynamics

Spatial separation by frozen regions lead to HSF
(ETH breaks down even after fixing domain-wall numbers)



→ Non-thermal dynamics depending on initial states

Classification of quantum many-body systems

Nonintegrable systems

ETH holds in
many systems

Nontrivial mechanisms are found for the breakdown of ETH
beyond the classification via nonintegrability

Integrable systems

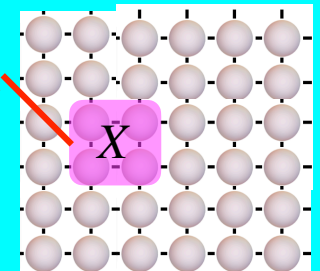
Still, generic many-body systems with local interactions
are expected to satisfy the ETH

Numerical verification using Hamiltonians
described by local random-matrix ensembles
(ETH holds with probability 1 in the thermodynamic limit)

S. Sugimoto, [RH](#), M. Ueda, PRL (2021)

S. Sugimoto, [RH](#), M. Ueda, PRL (2022) [long-range interactions]

random \hat{h}_X



$$\hat{H}_{\text{loc}} = \sum_X \hat{h}_X$$

Summary: thermalization

ETH provides a sufficient condition for thermalization

ETH holds for many non-integrable systems, while it generally fails for integrable systems

Locality of observables and conserved quantities (e.g., due to symmetries) plays a crucial role in thermalization

Beyond integrability, some mechanisms for the breakdown of ETH are known, e.g., MBL, many-body scars, and HSF

ETH and hydrodynamics

Towards thermalization:

- (I) After fast local relaxation, hydrodynamics sets in
- (II) Hydrodynamic modes slowly relax, leading to global thermalization

(II) affects the structure of matrix elements
in the low-frequency limit

e.g., L. Capizzi et al., PRX (2025)

$$A_{\alpha\beta} = \mathcal{A}(E_\alpha)\delta_{\alpha\beta} + e^{-S(\bar{E})/2}f(\bar{E}, \omega)R_{\alpha\beta}$$

(I) is not derived from standard ETH
(related approach: operator spreading/hydrodynamics)

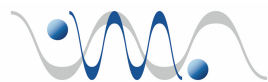
V. Khemani et al., PRX (2018); T. Rakovszky et al., PRX (2018)

Outlook: Can ETH be extended to account for
local thermalization/hydrodynamization?

Relaxation and its breakdown in open quantum systems

Ryusuke Hamazaki (RIKEN, Japan)

Quantum thermalization, Hydrodynamics and Gravity
@ YITP, Kyoto (June 2026)



ERATO Sagawa Information-to-Energy
Interconversion Project



Outline

Part I. Closed quantum systems

Beyond thermalization: second law of thermodynamics

Thermalization, non-integrability, and symmetry

Breakdown of thermalization

Second law of thermodynamics

Part II. Open quantum systems

Measurement theory and GKSL equations

Stationary states and relaxation in GKSL dynamics

Measurement-induced transitions and Lyapunov spectrum

Beyond thermalization: second law of thermodynamics in closed quantum systems

Review articles:

(General quantum thermodynamics)

*J. Goold et al 2016 J. Phys. A: Math. Theor. 49 143001 (2016).

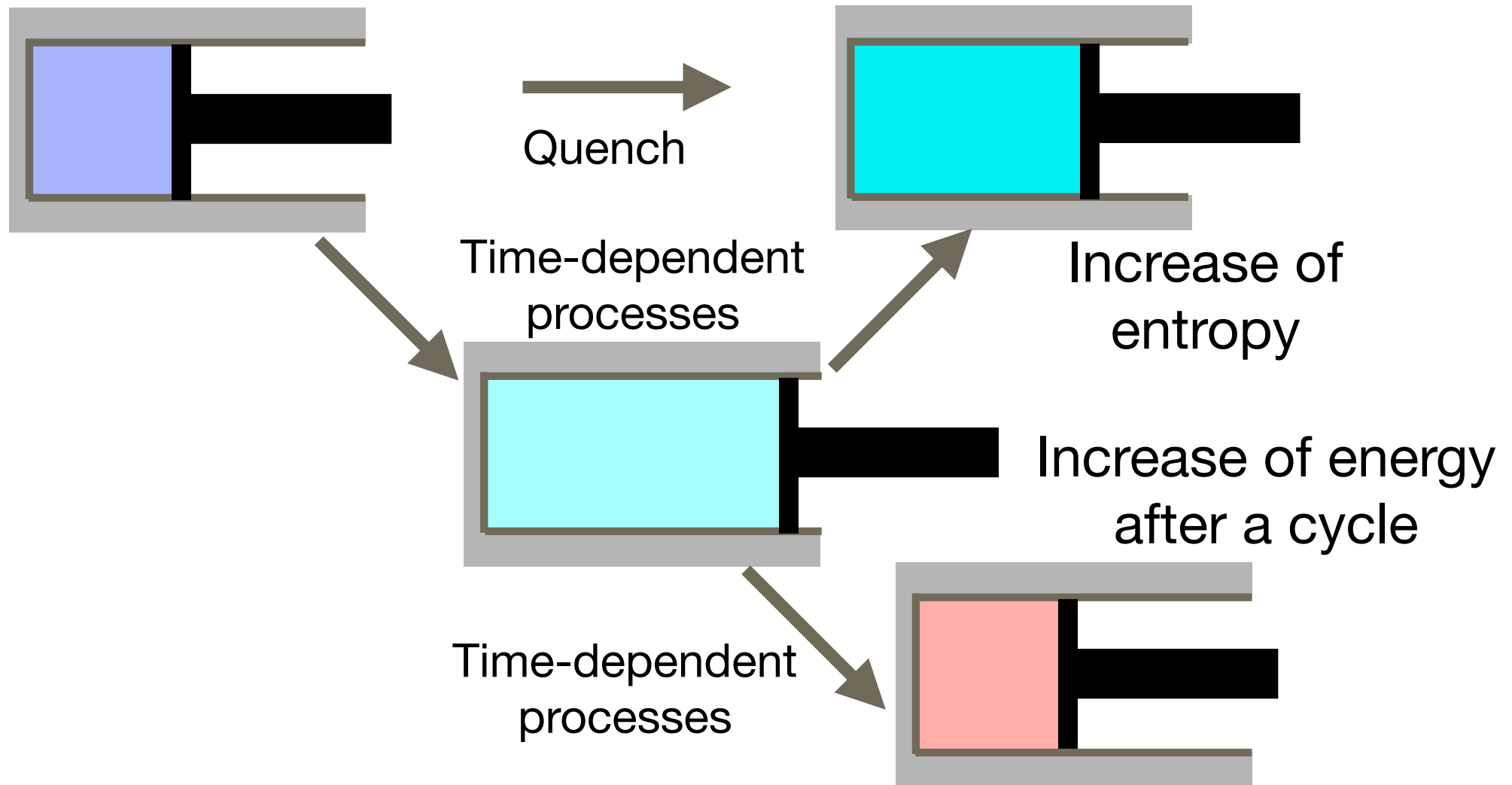
(Quantum batteries)

*F. Campaioli et al., Rev. Mod. Phys. 96, 031001 (2024).

Second law of thermodynamics in closed dynamics

Other laws of thermodynamics beyond thermalization?

Second law of thermodynamics in closed systems



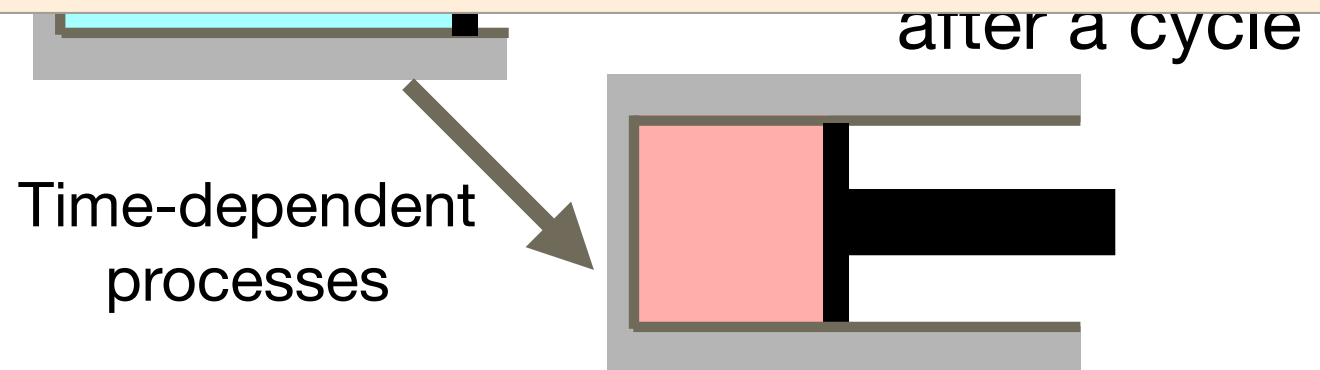
Second law of thermodynamics in closed dynamics

Other laws of thermodynamics beyond thermalization?

Second law of thermodynamics in closed systems

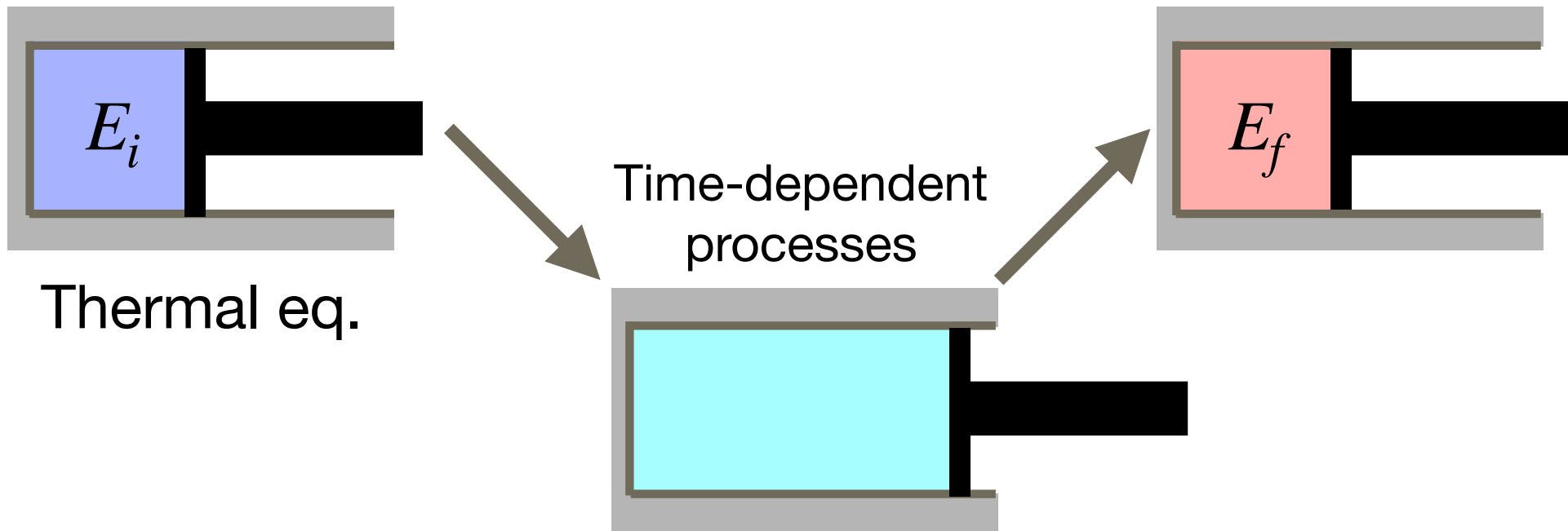
Second law of thermodynamics can include general time-dependent operations, making distinction from the conventional study of thermalization

Many aspects of the second law in quantum many-body systems have still been open!



Planck's principle from quantum mechanics?

Planck's principle: In an adiabatic thermodynamic cycle, no work can be extracted from thermal equilibrium

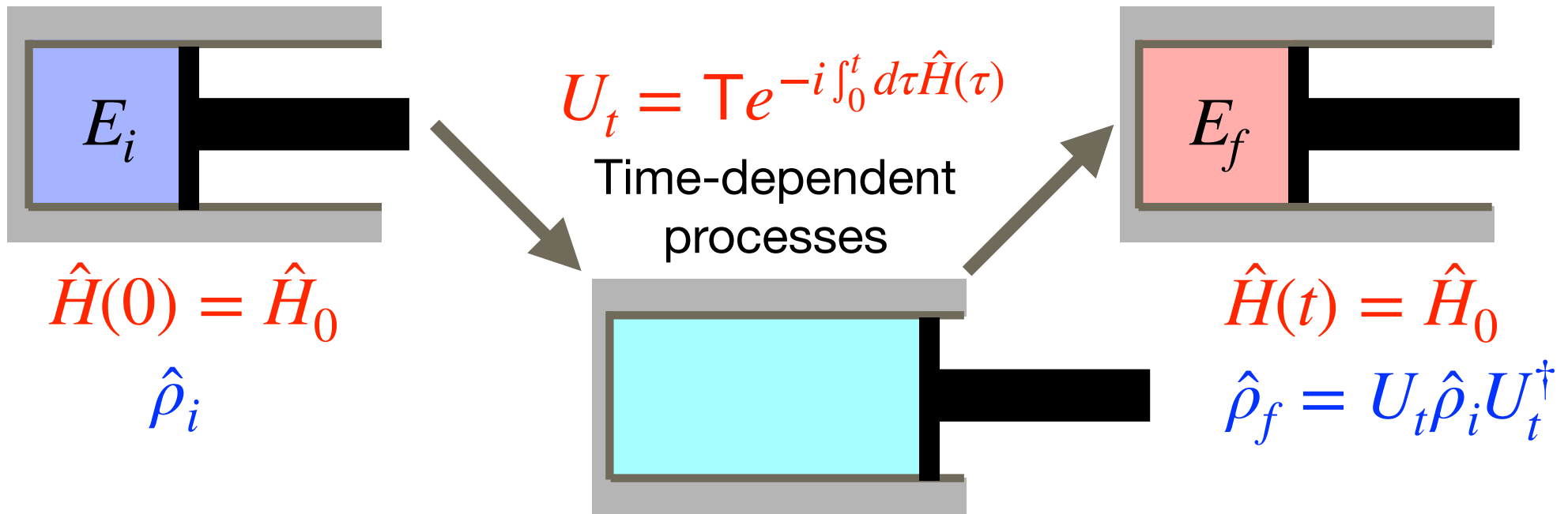


$$W_{\text{ext}} = E_i - E_f \leq 0$$

Planck's principle from quantum mechanics?

Planck's principle: In adiabatic thermodynamic processes, no work can be extracted from thermal equilibrium

Understanding from closed (unitary) quantum dynamics?



$$W_{\text{ext}} = \text{Tr}[\hat{H}_0 \hat{\rho}_i] - \text{Tr}[\hat{H}_0 \hat{\rho}_f] \leq 0 ?$$

Passive states

Taking $\hat{\rho}_i = \frac{e^{-\beta\hat{H}_0}}{Z}$ ($\beta > 0$), we indeed obtain

$$\begin{aligned} W_{\text{ext}} &= \text{Tr}[\hat{H}_0\hat{\rho}_i] - \text{Tr}[\hat{H}_0\hat{\rho}_f] \\ &= -\beta^{-1}\text{Tr}[\hat{\rho}_i \ln \hat{\rho}_i] - \beta^{-1} \ln Z + \beta^{-1}\text{Tr}[\hat{\rho}_f \ln \hat{\rho}_i] + \beta^{-1} \ln Z \\ &= -\beta^{-1}D(\hat{\rho}_f\|\hat{\rho}_i) \leq 0 \quad D(\hat{\rho}\|\hat{\sigma}) = \text{Tr}[\hat{\rho} \ln \hat{\rho}] - \text{Tr}[\hat{\rho} \ln \hat{\sigma}] \geq 0 \end{aligned}$$

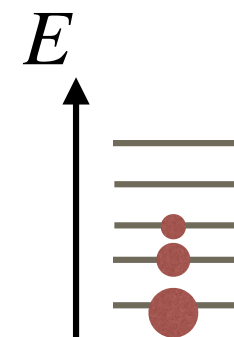
States satisfying $W_{\text{ext}} \leq 0$ is called *passive states*:

A. Lenard, Journal of Statistical Physics 19, 575 (1978).

If $\hat{\rho}_i = \sum_{\alpha=1}^d p_{\alpha} |E_{\alpha}\rangle\langle E_{\alpha}|$ with $p_1 \geq p_2 \geq \dots$

and $E_1 \leq E_2 \leq \dots$, we have $W_{\text{ext}} \leq 0$

← unitary operations can only permute the population $\{p_{\alpha}\}$ without changing its set



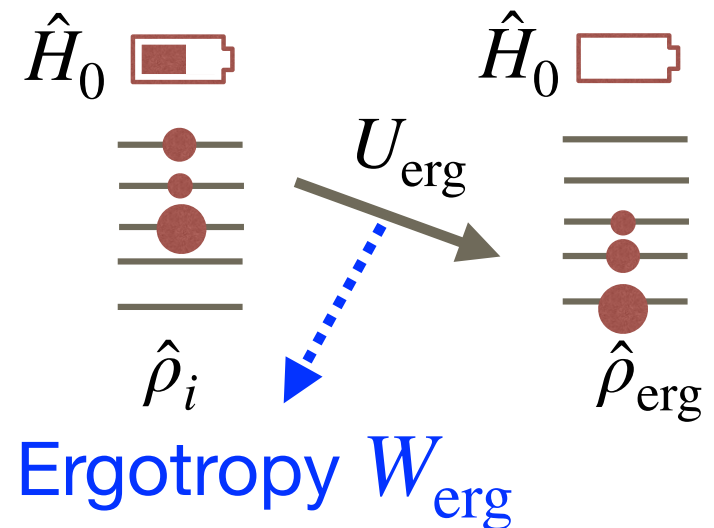
Ergotropy

From non-passive states, we can extract energy $W_{\text{ext}} > 0$ if arbitrary unitary operations are possible

→ Maximal extractable work is called ergotropy

$$W_{\text{erg}} = \max_U W_{\text{ext}} = \text{Tr}[\hat{H}_0 \hat{\rho}_i] - \text{Tr}[\hat{H}_0 \hat{\rho}_{\text{erg}}]$$

A. E. Allahverdyan et al., Europhys. Lett., 67(4):565 (2004)



$\hat{\rho}_{\text{erg}}$ is a passive state obtained via optimal unitary operation

$$\hat{\rho}_i = \sum_{\alpha} q_{\alpha} |q_{\alpha}\rangle \langle q_{\alpha}| \quad q_1 \geq q_2 \geq \dots \quad \hat{\rho}_{\text{erg}} = \sum_{\alpha} q_{\alpha} |E_{\alpha}\rangle \langle E_{\alpha}|$$

$$\hat{H}_0 = \sum_{\alpha} E_{\alpha} |E_{\alpha}\rangle \langle E_{\alpha}| \quad E_1 \leq E_2 \leq \dots \quad U_{\text{erg}} = \sum_{\alpha} e^{i\phi_{\alpha}} |E_{\alpha}\rangle \langle q_{\alpha}|$$

Paradox in quantum many-body systems

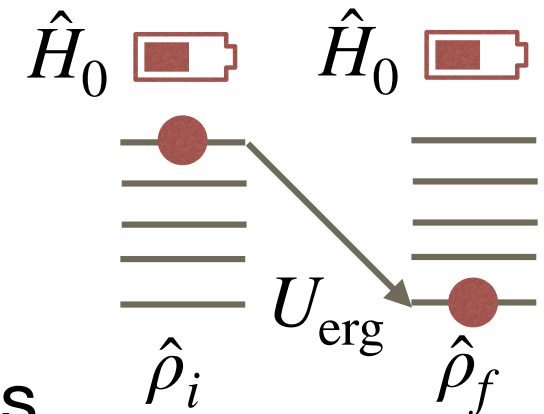
Let's consider pure states in quantum many-body systems, which are thermal at the level of observables (in the spirit of thermalization we discussed previously)

$$\hat{\rho}_i = |\psi_0\rangle\langle\psi_0| \quad \langle\psi_0|\hat{A}|\psi_0\rangle \simeq \langle\hat{A}\rangle_{\text{mic}}$$

We then find an extensive ergotropy, inconsistent with the second law of thermodynamics

$$W_{\text{erg}} = \langle\psi_0|\hat{H}|\psi_0\rangle - \langle E_1|\hat{H}|E_1\rangle = O(N)$$

$$U_{\text{erg}}|\psi_0\rangle = |E_1\rangle$$



Key point: in quantum many-body systems, it is usually difficult to realize the optimal unitary U_{erg}

Passivity of pure quantum states at finite times

Idea to resolve the paradox:

Focus on realistic quantum many-body dynamics,
whose Hamiltonians are local / macroscopic

(Local) A. Hokkyo and M. Ueda, Phys. Rev. Lett. 134, 010406 (2025)

(Macroscopic) Y. Chiba, Y. Yoneta, **RH**, A. Shimizu, arXiv:2602.06657 (2026) [Yuuya's talk]

$$\begin{array}{l} |\psi_0\rangle \\ \quad \curvearrowright \\ |\psi_t\rangle \end{array} \quad \langle \psi_0 | \hat{A} | \psi_0 \rangle \simeq \langle \hat{A} \rangle_{\text{mic}} \quad \hat{A}: \text{macroscopic observables}$$
$$U_t = \mathcal{T} e^{-i \int_0^t d\tau \hat{H}(\tau)} \quad \hat{H}(\tau) = \hat{H}_0 + \sum_{\mu} f_{\mu}(\tau) \hat{B}_{\mu}$$

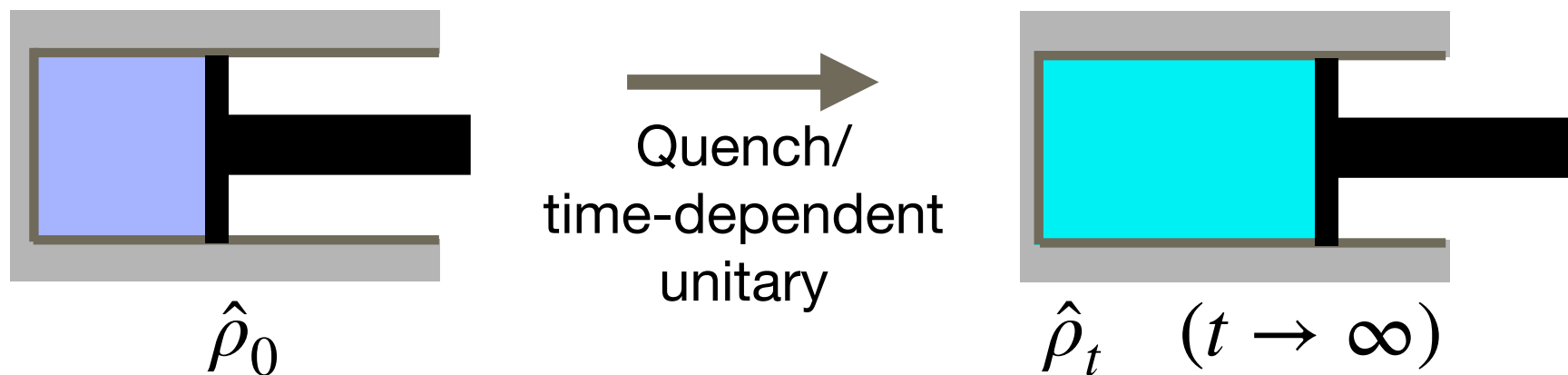
macroscopic operations

If t is independent of N , an extensive energy cannot be extractable:

$$\lim_{N \rightarrow \infty} \frac{W_{\text{ext}}}{N} = \lim_{N \rightarrow \infty} \frac{\langle \psi_0 | \hat{H} | \psi_0 \rangle - \langle \psi_t | \hat{H} | \psi_t \rangle}{N} \leq 0$$

Increase of entropy from quantum pure states?

Increase of entropy from unitary quantum dynamics?



How to define entropy in closed quantum systems?

Von Neumann entropy cannot increase: $-\text{Tr}[\hat{\rho}_0 \ln \hat{\rho}_0] = -\text{Tr}[\hat{\rho}_t \ln \hat{\rho}_t]$

Second law has been discussed for alternative candidates:

(Quench) Diagonal entropy, entanglement entropy, observational entropy

TN. Ikeda et al., *Annals of Physics* (2015), K. Kaneko et al., *PRE* (2017),
F. Meier et al., *PRX Quantum* (2026), ...

(Time-dependence) Macroscopic entropy density

Y. Chiba, Y. Yoneta, [RH](#), A. Shimizu, arXiv:2602.06657 (2026) [Yuuya's talk]

Summary: the second law in closed quantum dynamics

Second law of thermodynamics is another essential issue for foundations of thermodynamics

In quantum thermodynamics, Planck's principle is often discussed via passivity and ergotropy, but it seems inconsistent with pure-state thermal equilibrium at first

There are recent developments of justifying the second law even for pure states in quantum many-body systems

Outline

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Second law of thermodynamics

Part II. Open quantum systems

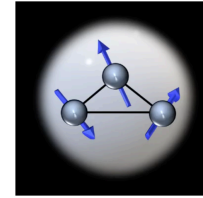
Measurement theory and GKSL equations

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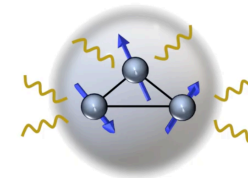
Nonequilibrium phenomena in open systems

We have discussed closed systems



isolated from environment

Open systems → Emergence of richer nonequilibrium states

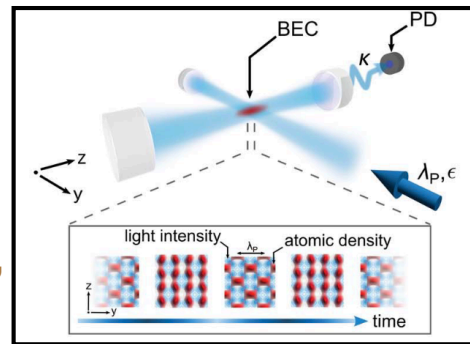


attached with external systems

Even dissipation and measurement can now be experimentally controlled

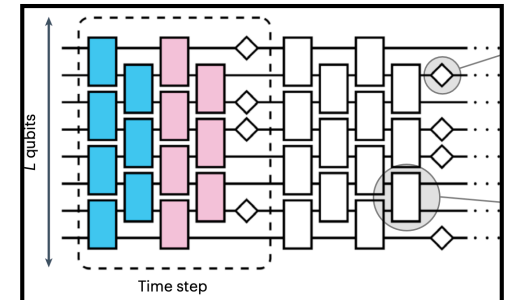
Dissipative time crystal

P. Kongkhambut et al.,
Science (2021)



Measurement-induced phase transitions

J.M. Koh et al.,
Nat. Phys. (2023)



What happens to long-time dynamics?

Quantum phases unique to nonequilibrium open systems?

Lessons from part I: spectrum and symmetry

For thermalization in isolated systems,
spectrum and symmetry are important

Spectrum:

$$i \frac{d|\psi_t\rangle}{dt} = \hat{H} |\psi_t\rangle \longrightarrow \hat{H} |E_\alpha\rangle = E_\alpha |E_\alpha\rangle$$

E_α : energy eigenvalue
($E_1 \leq E_2 \leq \dots$)

$|E_\alpha\rangle$: energy eigenstate

Symmetry: $[\hat{H}, \hat{O}] = 0 \rightarrow$ conserved quantities $\langle \hat{O}(t) \rangle = \langle \hat{O}(0) \rangle$

Relaxation dynamics in open systems?

Spectrum: Liouvillian spectrum, Lyapunov spectrum

Symmetry: strong symmetry, which leads to conserved quantities

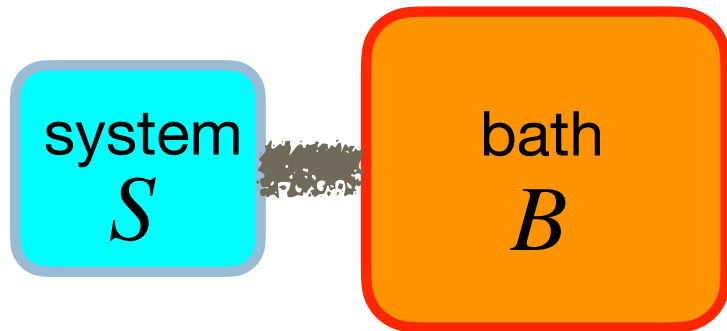
Measurement theory and GKSL equations

Book/Review articles:

- *HM. Wiseman and GJ. Milburn, Quantum Measurement and Control, Cambridge University Press (2011).
 - *GT. Landi et al., PRX Quantum 5, 020201 (2024).
 - *RH, K Mochizuki, H Oshima, Y Fuji, PTEP, ptag055 (2026).
-

Dissipation and quantum measurement

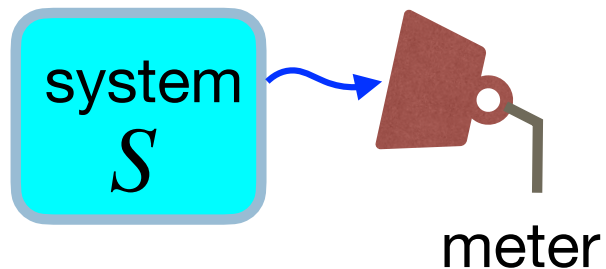
Dissipation due to, e.g., environment



System loses its information
irreversibly in time

Initial pure states tend to be mixed in time

Quantum measurement



Information is stored as
measurement outcomes

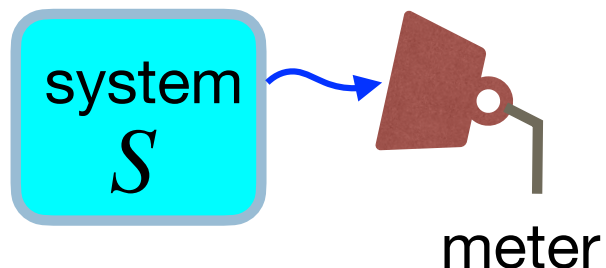
Unless measurement outcomes are
discarded, initial pure states
remain to be pure

Dissipation and quantum measurement

Dissipation due to, e.g., environment

Quantum trajectory for individual measurement outcomes,
and GKSL equation after ensemble average

Quantum measurement



Information is stored as
measurement outcomes

Unless measurement outcomes are
discarded, initial pure states
remain to be pure

Quantum systems under measurement

Review: projective measurement

The system state $\hat{\rho}_S$ is projectively measured by a meter



Probability of finding $k = 0, 1, \dots, : p_k = \text{Tr}[\hat{\rho}_S \hat{P}_k]$

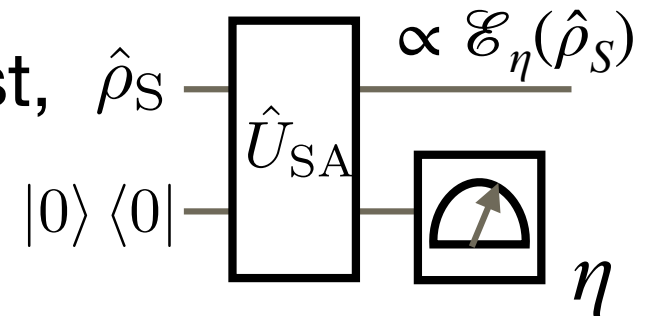
Post-measurement state (Born's rule): $\hat{\rho}'_{Sk} = \frac{\hat{P}_k \hat{\rho}_S \hat{P}_k}{p_k}$

→ Ensemble average leads to $\rho'_S = \sum_k p_k \hat{\rho}'_{Sk} = \sum_k \hat{P}_k \hat{\rho}_S \hat{P}_k$

Quantum systems under measurement

Indirect measurement:

Entangle system and auxiliary system first,
and measure the auxiliary system



Probability of finding $\eta = 0, 1, \dots, : p_\eta = \text{Tr}[\hat{\rho}_S \hat{M}_\eta^\dagger \hat{M}_\eta] \quad \hat{P}_\eta = |\eta\rangle\langle\eta|$

Post-measurement state: $\frac{\mathcal{E}_\eta(\hat{\rho}_S)}{p_\eta} = \frac{\hat{M}_\eta \hat{\rho}_S \hat{M}_\eta^\dagger}{p_\eta}$

Measurement operator

$$\hat{M}_\eta = \langle\eta| \hat{U}_{SA} |0\rangle$$

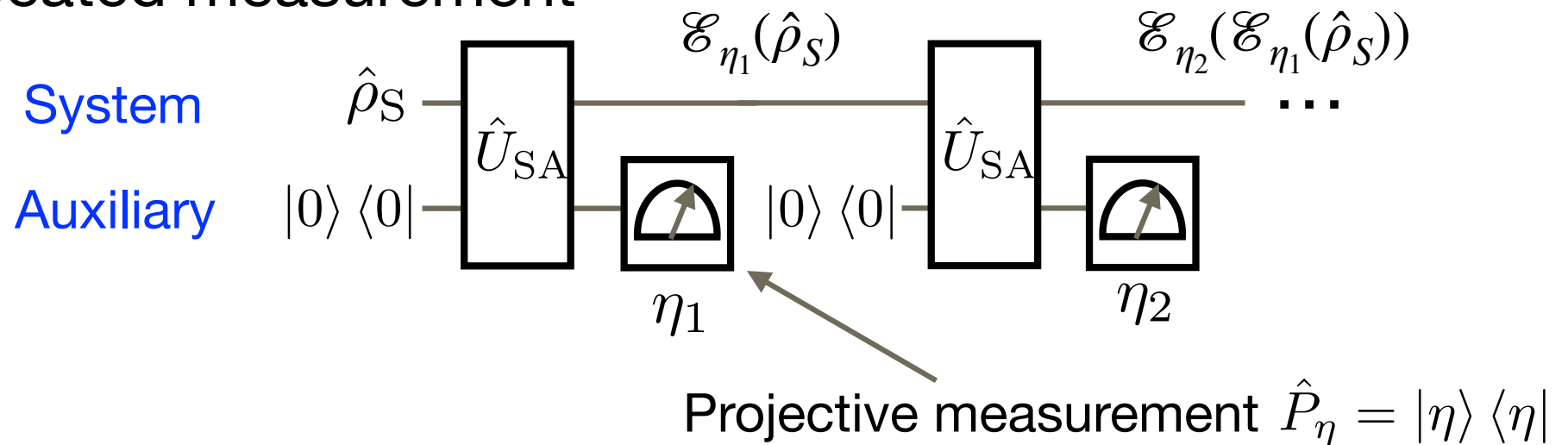
$$\sum_\eta \hat{M}_\eta^\dagger \hat{M}_\eta = \hat{I}_S$$

→ Ensemble average leads to $\rho' = \sum_\eta \mathcal{E}_\eta(\hat{\rho}) = \sum_\eta \hat{M}_\eta \hat{\rho} \hat{M}_\eta^\dagger$

Kraus representation

Repeated measurement and quantum trajectories

Repeated measurement



If the measurement outcomes are $\eta_1, \eta_2, \dots, \eta_n$
we obtain a state conditioned on measurement outcomes

$$\hat{\rho}_S(n; \eta_1, \dots, \eta_n) \propto \mathcal{E}_{\eta_n} \circ \dots \circ \mathcal{E}_{\eta_2} \circ \mathcal{E}_{\eta_1}(\hat{\rho}_S)$$

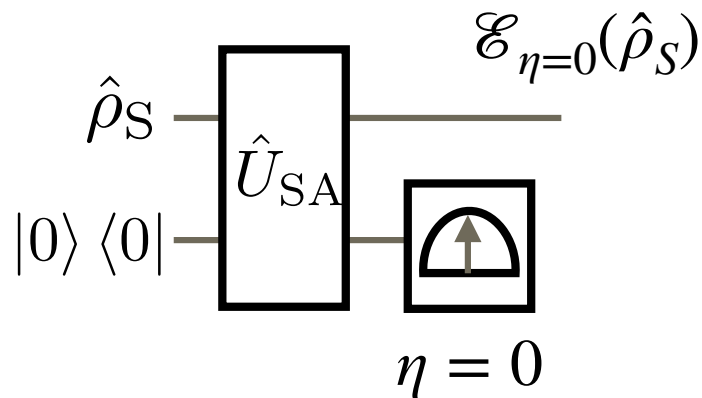
Quantum trajectories (initially pure states remain pure)

Continuous-time limit and GKSL equation

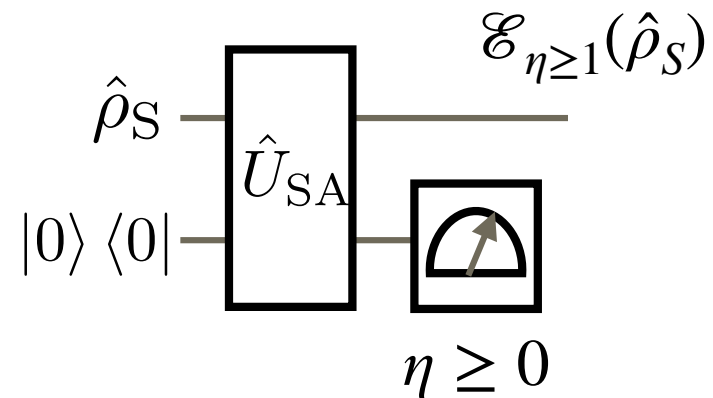
For weak measurement (small interaction time δt), we expect the probability of finding $\eta \neq 0$ is $p_{\eta \geq 1} \propto \delta t$

→ We set $\hat{M}_{\eta \geq 1} = \hat{L}_\eta \sqrt{\delta t}$, $\hat{M}_{\eta=0} = 1 - \frac{1}{2} \sum_{\eta \geq 1} \hat{L}_\eta^\dagger \hat{L}_\eta \delta t$

$$\hat{U}_{SA} = \hat{\mathbb{I}} + \mathcal{O}(\delta t)$$



Most cases



Probability $\propto \delta t$
(quantum jumps)

Continuous-time limit and GKSL equation

For weak measurement (small interaction time δt),
we expect the probability of finding $\eta \neq 0$ is $p_{\eta \geq 1} \propto \delta t$

→ We set $\hat{M}_{\eta \geq 1} = \hat{L}_\eta \sqrt{\delta t}$, $\hat{M}_{\eta=0} = 1 - \frac{1}{2} \sum_{\eta \geq 1} \hat{L}_\eta^\dagger \hat{L}_\eta \delta t$

Measurement outcomes	$\eta \geq 1$	$\eta = 0$
Measurement operator	$\hat{M}_{\eta \geq 1} = \hat{L}_\eta \sqrt{\delta t}$	$\hat{M}_{\eta=0} = 1 - \frac{1}{2} \sum_{\eta \geq 1} \hat{L}_\eta^\dagger \hat{L}_\eta \delta t$
Measurement probability	$p_{\eta \geq 1} = \langle \hat{L}_\eta^\dagger \hat{L}_\eta \rangle \delta t + \mathcal{O}(\delta t^2)$	$p_{\eta=0} = 1 - \sum_{\eta \geq 1} \langle \hat{L}_\eta^\dagger \hat{L}_\eta \rangle \delta t + \mathcal{O}(\delta t^2)$
State after measurement	$\hat{\rho}_S^{\eta \geq 1}(t + \delta t) = \frac{\hat{M}_\eta \hat{\rho}_S \hat{M}_\eta^\dagger}{p_\eta}$	$\hat{\rho}_S^{\eta=0}(t + \delta t) = \frac{\hat{M}_0 \hat{\rho}_S \hat{M}_0^\dagger}{p_0}$

Repeated measurement and GKSL equation

Average $\hat{\rho}_S = \sum_{\eta} p_{\eta} \hat{\rho}_S^{\eta}$ satisfies the following time evolution

$$\frac{d\hat{\rho}_S}{dt} = \sum_{\eta \geq 1} \hat{L}_{\eta} \hat{\rho}_S \hat{L}_{\eta}^{\dagger} - \frac{1}{2} \{ \hat{L}_{\eta}^{\dagger} \hat{L}_{\eta}, \hat{\rho}_S \}$$

By adding Hamiltonian dynamics for $\hat{\rho}_S$ itself, we have

$$\begin{aligned} \frac{d\hat{\rho}_S}{dt} &= -i[\hat{H}, \hat{\rho}_S] + \sum_{\eta \geq 1} \hat{L}_{\eta} \hat{\rho}_S \hat{L}_{\eta}^{\dagger} - \frac{1}{2} \{ \hat{L}_{\eta}^{\dagger} \hat{L}_{\eta}, \hat{\rho}_S \} \\ &= -i(\hat{H}_{\text{NH}} \hat{\rho}_S - \hat{\rho}_S \hat{H}_{\text{NH}}^{\dagger}) + \sum_{\eta \geq 1} \hat{L}_{\eta} \hat{\rho}_S \hat{L}_{\eta}^{\dagger} \quad \hat{H}_{\text{NH}} = \hat{H} - \frac{i}{2} \sum_{\eta \geq 1} \hat{L}_{\eta}^{\dagger} \hat{L}_{\eta} \end{aligned}$$

Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) equation

Repeated measurement and GKSL equation

Average $\hat{\rho}_S = \sum_n p_n \hat{\rho}_S^\eta$ satisfies the following time evolution

Remark:

If we postselect trajectories with $\eta = 0$, we obtain time evolution due to non-Hermitian Hamiltonians

$$\frac{d\hat{\rho}_S}{dt} \propto -i(\hat{H}_{\text{NH}}\hat{\rho}_S - \hat{\rho}_S\hat{H}_{\text{NH}}^\dagger) \quad \text{cf) K. Takasan's talk}$$

$$= -i(\hat{H}_{\text{NH}}\hat{\rho}_S - \hat{\rho}_S\hat{H}_{\text{NH}}^\dagger) + \sum_{\eta \geq 1} \hat{L}_\eta \hat{\rho}_S \hat{L}_\eta^\dagger \quad \hat{H}_{\text{NH}} = \hat{H} - \frac{i}{2} \sum_{\eta \geq 1} \hat{L}_\eta^\dagger \hat{L}_\eta$$

Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) equation

Spectral decomposition of GKSL equation

$$\frac{d\hat{\rho}}{dt} = \mathcal{L}[\hat{\rho}] = -i[\hat{H}, \hat{\rho}] + \sum_{\eta} \hat{L}_{\eta} \hat{\rho} \hat{L}_{\eta}^{\dagger} - \frac{1}{2} \{ \hat{L}_{\eta}^{\dagger} \hat{L}_{\eta}, \hat{\rho} \}$$

\mathcal{L} is linear: spectral decomposition is often useful

For simplicity, we assume the diagonalizability of \mathcal{L}

Eigenequation ($1 \leq a \leq \dim[\mathcal{H}]^2$)

$$\mathcal{L}[\hat{\phi}_a] = \lambda_a \hat{\phi}_a \quad \hat{\phi}_a: \text{right eigenmode}$$

$$\mathcal{L}^{\dagger}[\hat{\chi}_a] = \lambda_a^* \hat{\chi}_a \quad \hat{\chi}_a: \text{left eigenmode}$$

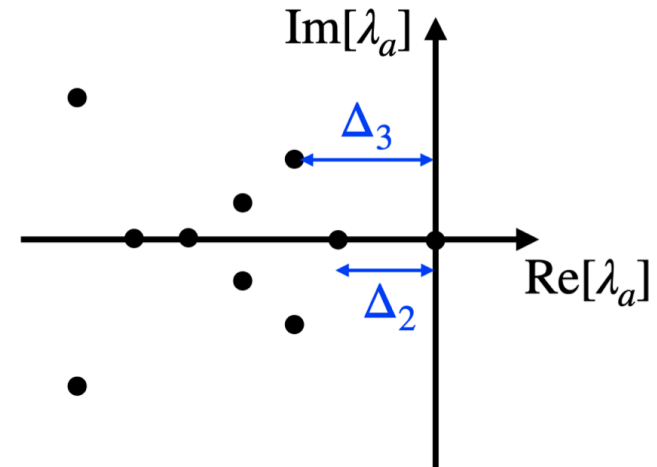
$$\mathcal{L}^{\dagger}[\hat{A}] = i[\hat{H}, \hat{A}] + \sum_{\eta} \hat{L}_{\eta}^{\dagger} \hat{A} \hat{L}_{\eta} - \frac{1}{2} \{ \hat{L}_{\eta}^{\dagger} \hat{L}_{\eta}, \hat{A} \} \quad \text{Heisenberg representation}$$

Due to non-Hermiticity, $\lambda_a \in \mathbb{C}$, $\hat{\phi}_a \neq \hat{\chi}_a$

Properties of spectral decomposition

$$\mathcal{L}[\hat{\phi}_a] = \lambda_a \hat{\phi}_a$$

$$\mathcal{L}^\dagger[\hat{\chi}_a] = \lambda_a^* \hat{\chi}_a$$



1. Eigenvalues are real or form complex-conjugate pairs
2. Zero eigenvalue $\lambda_1 = 0$ exists, and $\hat{\phi}_1 = \hat{\rho}_{ss}$ is a stationary state
3. All eigenvalues have zero or negative real parts

$$0 = \lambda_1 \geq \text{Re} [\lambda_2] \geq \text{Re} [\lambda_3] \geq \dots \geq \text{Re} [\lambda_{\dim[\mathcal{H}]^2}]$$

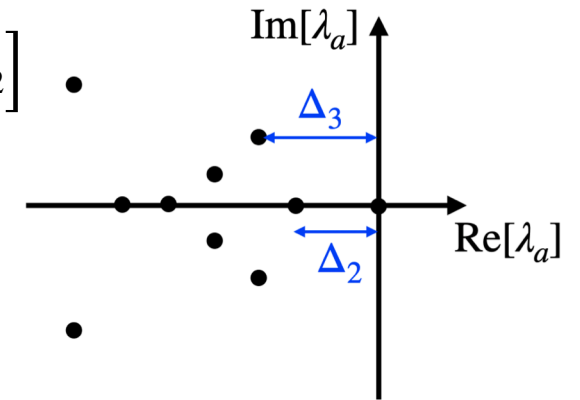
Properties of spectral decomposition

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The following spectral decomposition is possible



$$\hat{\rho}(t) = e^{\mathcal{L}t}[\hat{\rho}(0)] = \hat{\rho}_{\text{ss}} + \sum_{a=2}^{\dim[\mathcal{H}]^2} c_a e^{\lambda_a t} \hat{\phi}_a \quad c_a = \frac{\text{Tr}[\hat{\chi}_a^\dagger \hat{\rho}(0)]}{\text{Tr}[\hat{\chi}_a^\dagger \hat{\phi}_a]}$$

If λ_a had positive real parts, coefficients of $\hat{\phi}_a$ diverge in time, which contradicts the positivity of the state

Stationary states and relaxation in GKSL dynamics

Review articles:

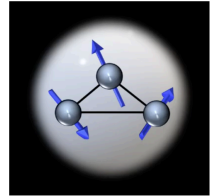
*[RH](#), K Mochizuki, H Oshima, Y Fuji, PTEP, ptag055 (2026).

*Z Gong, [RH](#),
International Journal of Modern Physics B 36 (31), 2230007 (2022).

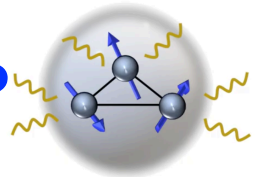
Fate of long-time dynamics of open systems?

Closed systems typically thermalize
in the long time due to the ETH

↔ It is broken by, e.g., conservation laws



How about long-time dynamics in open systems?



Assuming that the Hilbert-space dimension d is finite,
can we understand properties of infinitely long-time state?

$$\hat{\rho}_t \xrightarrow{(t \rightarrow \infty)} \hat{\rho}_t^* \quad \hat{\rho}_t = \mathcal{T} e^{\int_0^t d\tau \mathcal{L}_\tau} [\hat{\rho}_0]$$

Unlike the closed systems, we here consider dynamics at the level of
density matrices, without considering, e.g., locality of observables

e.g.) Is the steady state unique, irrespective of $\hat{\rho}_0$?

What is the speed of relaxation to steady states?

Fate of long-time dynamics of open systems?

C
in
↔

What is the relation with Liouvillian spectrum?

How to characterize it from symmetry?

How about long-time dynamics in open systems?

Assuming that the Hilbert-space dimension d is finite, can we understand properties of infinitely long-time state?

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Unlike the closed systems, we here consider dynamics at the level of density matrices, without considering, e.g., locality of observables

e.g.) Is the steady state unique, irrespective of $\hat{\rho}_0$?

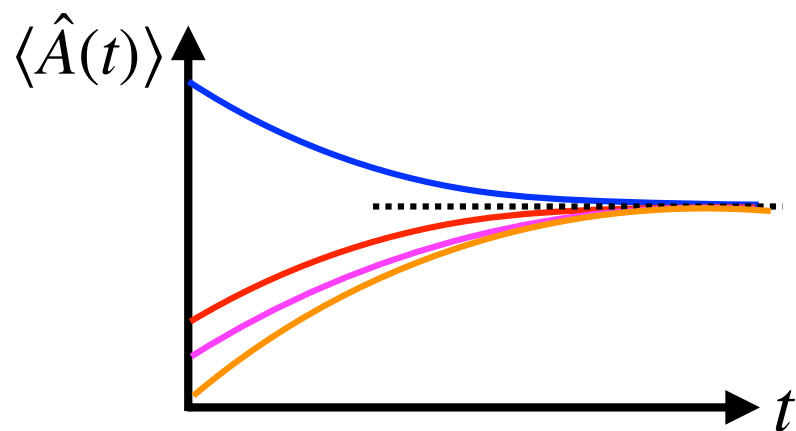
What is the speed of relaxation to steady states?

Setup and the definition of steady states

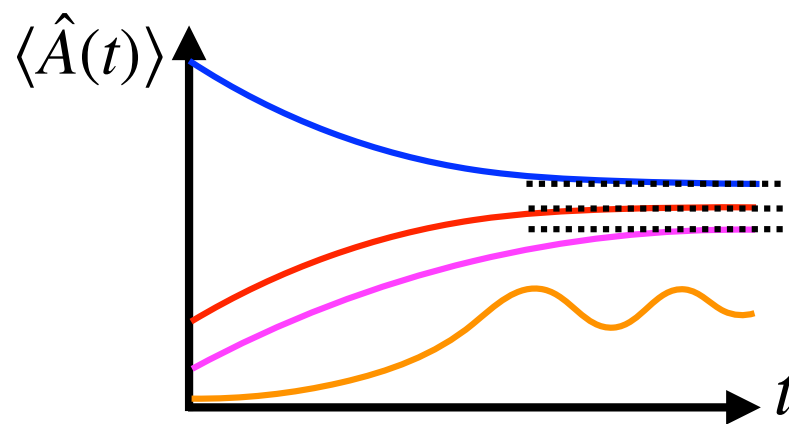
Time-independent GKSL equation

$$\frac{d\hat{\rho}_t}{dt} = -i[\hat{H}, \hat{\rho}_t] + \sum_{\mu} \hat{L}_{\mu} \hat{\rho}_t \hat{L}_{\mu}^{\dagger} - \frac{1}{2} \{ \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu}, \hat{\rho}_t \}$$

While there is at least one steady state, is it unique?



unique steady state



multiple steady states

Setup and the definition of steady states

Time-independent GKSL equation

$$\frac{d\hat{\rho}_t}{dt} = -i[\hat{H}, \hat{\rho}_t] + \sum_{\mu} \hat{L}_{\mu} \hat{\rho}_t \hat{L}_{\mu}^{\dagger} - \frac{1}{2} \{ \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu}, \hat{\rho}_t \}$$

While there is at least one steady state, is it unique?

$\langle \hat{A}(t) \rangle$ ↑

$\langle \hat{A}(t) \rangle$ ↑

While it is often just assumed for simplicity, there is a long history on *rigorous* criteria for the uniqueness

H. Spohn, Lett. Math. Phys. 2, 33 (1977), Rev. Mod. Phys. 52, 569 (1980).

A. Frigerio, Lett. Math. Phys. 2, 79 (1977), Commun. Math. Phys. 63, 269 (1978)

M.M. Wolf, Quantum channels and operations — guided tour (2012).

H. Yoshida, Phys. Rev. A 109, 022218 (2024).

Y. Zhang and T. Barthel, J. Phys. A: Math. Theor. 57, 115301 (2024).

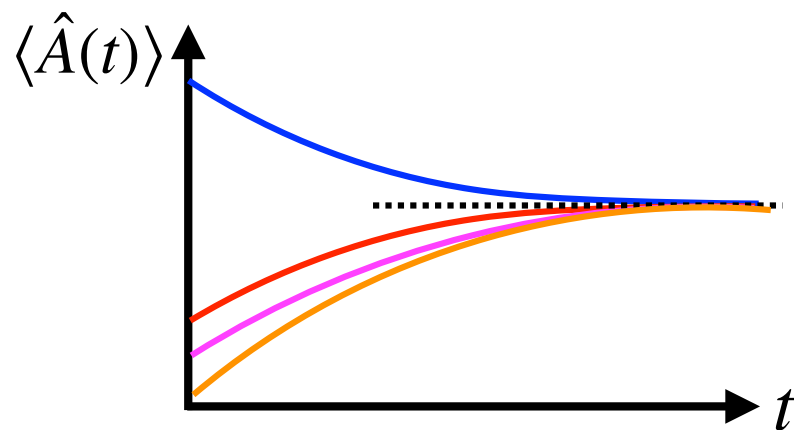
(Review) **RH** et al., PTEP ptag055 (2026).

Setup and the definition of steady states

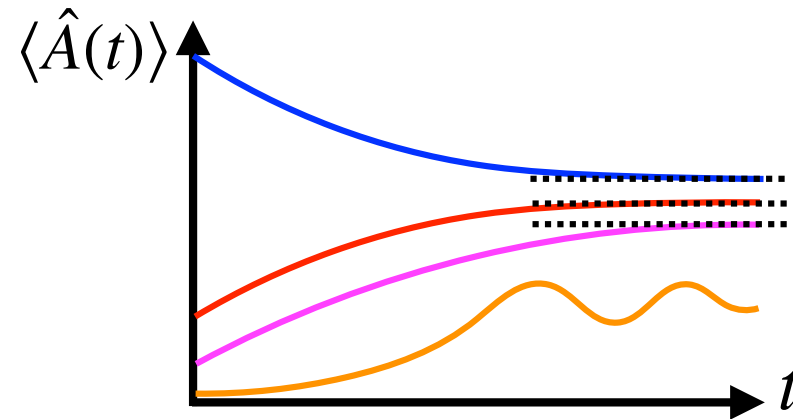
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While there is at least one steady state, is it unique?



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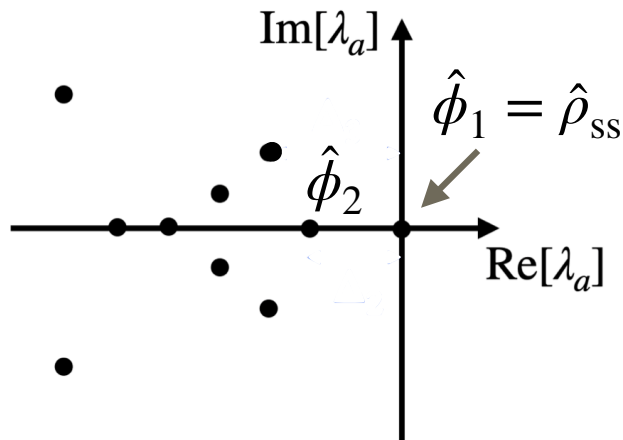
Here, we discuss the structure of steady state from the spectrum and symmetry of GKSL equation

Classification of dynamics: spectral viewpoint

Steady-state dynamics is classified by the eigenmodes satisfying $\text{Re}[\lambda_a] = 0 \rightarrow$ These modes are not decaying

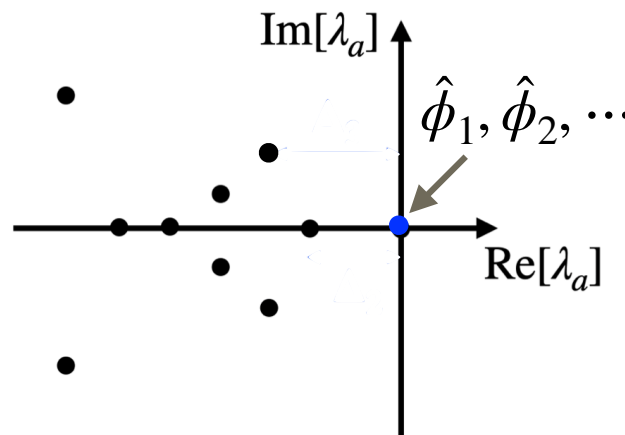
$$\text{Re}[\lambda_a] < 0 \\ (\forall a > 1)$$

Unique steady states



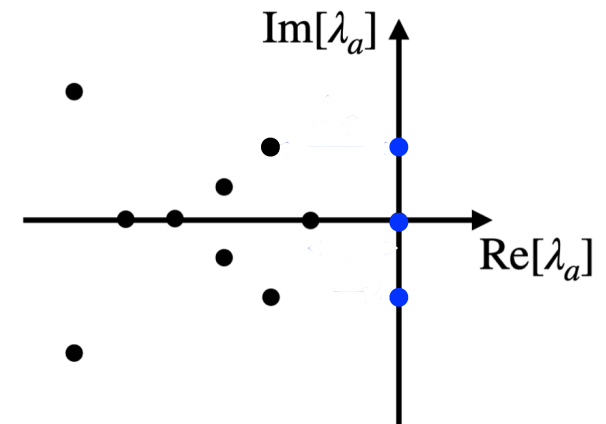
$$\lambda_a = 0 \\ (\exists a > 1)$$

Multiple time-independent steady states



$$\text{Re}[\lambda_a] = 0, |\text{Im}[\lambda_a]| > 0 \\ (\exists a > 1)$$

Multiple time-independent & time-dependent steady states



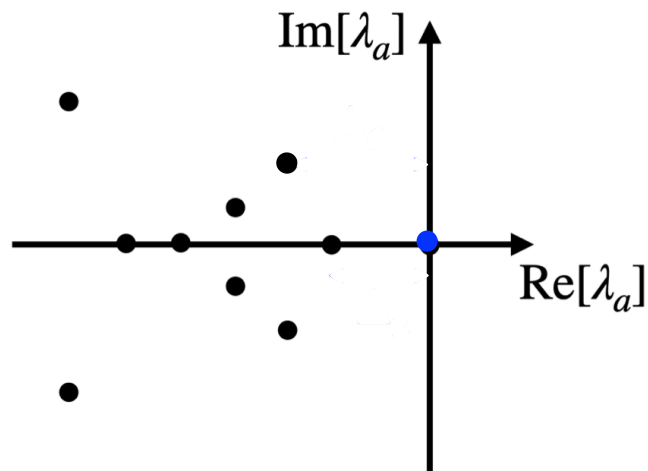
Sufficient condition for multiple steady states

Steady-state dynamics is classified by the eigenmodes satisfying $\text{Re}[\lambda_a] = 0 \rightarrow$ These modes are not decaying

$\lambda_a = 0$ ($\exists a > 1$) Sufficient condition: strong symmetry \hat{V}

Multiple time-independent steady states

$$[\hat{H}, \hat{V}] = [\hat{L}_\eta, \hat{V}] = [\hat{L}_\eta^\dagger, \hat{V}] = 0$$



$$\rightarrow \frac{d\langle \hat{V} \rangle}{dt} = \langle \hat{V}, \mathcal{L}[\hat{\rho}(t)] \rangle = \langle \mathcal{L}^\dagger[\hat{V}], \hat{\rho}(t) \rangle = 0$$

$$\mathcal{L}^\dagger[\hat{A}] = i[\hat{H}, \hat{A}] + \sum_\eta \hat{L}_\eta^\dagger \hat{A} \hat{L}_\eta - \frac{1}{2} \{ \hat{L}_\eta^\dagger \hat{L}_\eta, \hat{A} \}$$

\hat{V} is conserved \rightarrow multiple steady states depending on the initial state

cf) weak symmetry \hat{V}_w ,

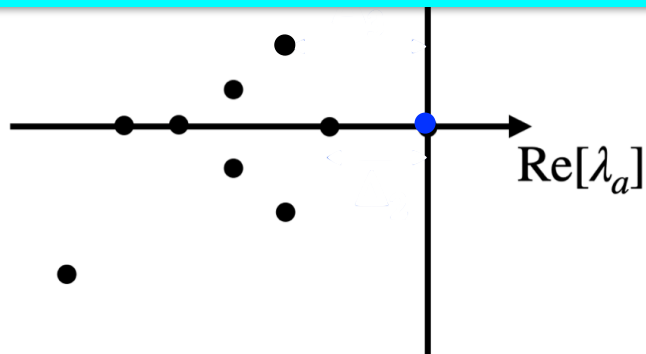
$\hat{V}_w \mathcal{L}[\hat{\rho}] \hat{V}_w^\dagger = \mathcal{L}[\hat{V}_w \hat{\rho} \hat{V}_w^\dagger]$, does not conserve

cf) J.Y. Lee's talk

Sufficient condition for multiple steady states

Steady-state dynamics is classified by the eigenmodes satisfying $\text{Re}[\lambda_a] = 0 \rightarrow$ These modes are not decaying

For Hermitian jump operators $\hat{L}_\eta = \hat{L}_\eta^\dagger$,
 Nontrivial strong symmetry \iff
 Multiple time-independent steady state



$$\mathcal{L}^\dagger[\hat{A}] = i[\hat{H}, \hat{A}] + \sum_{\eta} \hat{L}_\eta^\dagger \hat{A} \hat{L}_\eta - \frac{1}{2} \{ \hat{L}_\eta^\dagger \hat{L}_\eta, \hat{A} \}$$

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$\hat{V}_w \mathcal{L}[\hat{\rho}] \hat{V}_w^\dagger = \mathcal{L}[\hat{V}_w \hat{\rho} \hat{V}_w^\dagger]$, does not conserve

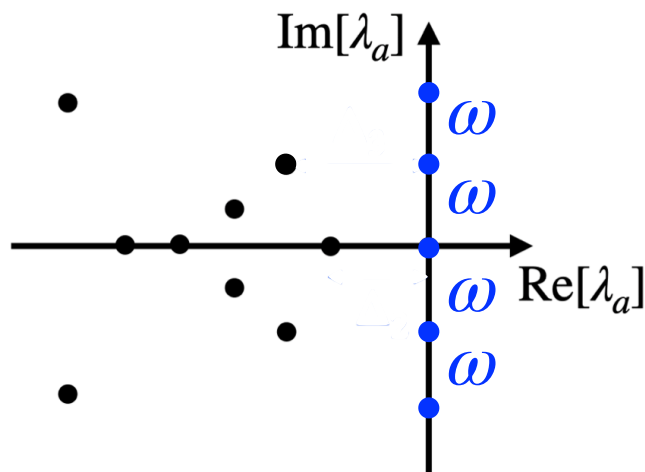
cf) J.Y. Lee's talk

Sufficient condition for time-dependent steady states

Steady-state dynamics is classified by the eigenmodes satisfying $\text{Re}[\lambda_a] = 0 \rightarrow$ These modes are not decaying

$$\text{Re}[\lambda_a] = 0, |\text{Im}[\lambda_a]| > 0 \\ (\exists a > 1)$$

Multiple time-independent & time-dependent steady states



Sufficient condition:
strong dynamical symmetry

$$[\hat{A}, \hat{H}] = \omega \hat{A}, [\hat{L}_\eta, \hat{A}] = [\hat{L}_\eta^\dagger, \hat{A}] = 0$$

$\hat{\rho}^{(nm)} \propto \hat{A}^n \hat{\rho}_{ss} (\hat{A}^\dagger)^m$ satisfies

$$\mathcal{L}[\hat{\rho}^{(nm)}] = i(m - n)\omega \hat{\rho}^{(nm)}$$

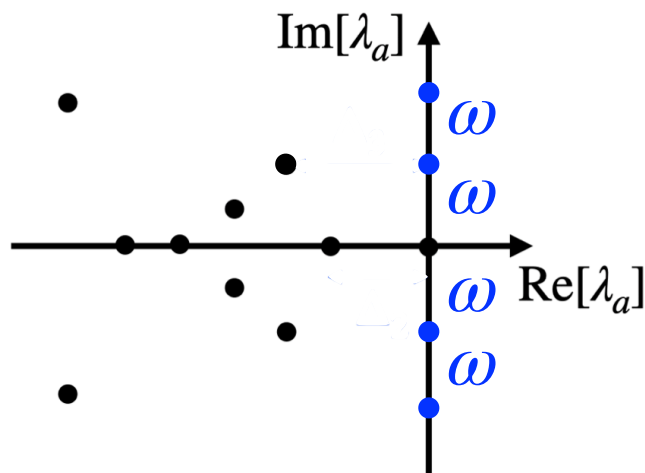
\rightarrow Oscillatory eigenmodes,
time-dependent steady states

cf) $\hat{A}^\dagger \hat{A}$ becomes the strong symmetry

Sufficient condition for time-dependent steady states

Steady-state dynamics is classified by the eigenmodes satisfying $\text{Re}[\lambda_a] = 0 \rightarrow$ These modes are not decaying

For Hermitian jump operators $\hat{L}_\eta = \hat{L}_\eta^\dagger$,
Nontrivial strong dynamical symmetry
 \iff Time-dependent steady state



$$\hat{\rho}^{(nm)} \propto \hat{A}^n \hat{\rho}_{ss} (\hat{A}^\dagger)^m \text{ satisfies}$$
$$\mathcal{L}[\hat{\rho}^{(nm)}] = i(m - n)\omega \hat{\rho}^{(nm)}$$

\rightarrow Oscillatory eigenmodes,
time-dependent steady states

cf) $\hat{A}^\dagger \hat{A}$ becomes the strong symmetry

Example: dissipative Hubbard model

$$\hat{H} = -\tau \sum_{\langle j,j' \rangle, s} \hat{c}_{j,s}^\dagger \hat{c}_{j',s} + \hat{c}_{j',s}^\dagger \hat{c}_{j,s} + \sum_j U \hat{n}_{j,\uparrow} \hat{n}_{j,\downarrow} + \epsilon_j \hat{n}_j + \frac{B}{2} (\hat{n}_{j,\uparrow} - \hat{n}_{j,\downarrow})$$

$$\hat{L}_j = \sqrt{\gamma_j} \hat{n}_j$$

Particle number $\sum_j \hat{n}_j$ & magnetization $\hat{S}^z = \sum_j \hat{S}_j^z = \sum_j \frac{\hat{n}_{j,\uparrow} - \hat{n}_{j,\downarrow}}{2}$ satisfy

$[\hat{H}, \hat{V}] = [\hat{L}_j, \hat{V}] = [\hat{L}_j^\dagger, \hat{V}] = 0 \rightarrow$ Time-independent steady states

SU(2) ladder operators $\hat{S}^\pm = \sum_j \hat{S}_j^\pm$ (with $\hat{S}_j^+ = (\hat{S}_j^-)^\dagger = \hat{c}_{j,\uparrow}^\dagger \hat{c}_{j,\downarrow}$) satisfy

$[\hat{A}, \hat{H}] = \pm B\hat{A}$, $[\hat{L}_j, \hat{A}] = [\hat{L}_j^\dagger, \hat{A}] = 0 \rightarrow$ Oscillatory steady states

Beyond time-independent GKSL equations

Time-dependent GKSL equation

$$\frac{d\hat{\rho}_t}{dt} = -i[\hat{H}_t, \hat{\rho}_t] + \sum_{\mu} \hat{L}_{\mu,t} \hat{\rho}_t \hat{L}_{\mu,t}^{\dagger} - \frac{1}{2} \{ \hat{L}_{\mu,t}^{\dagger} \hat{L}_{\mu,t}, \hat{\rho}_t \}$$

Spectral decomposition is not generally useful

e.g., spin 1/2 $\hat{H}_t = \frac{\omega}{2} \hat{\sigma}^z, \hat{L}_t = \sqrt{\gamma} (\hat{\sigma}^x \cos \omega t + \hat{\sigma}^y \sin \omega t)$

At each time t , there is

*Only one zero eigenvalue for the Liouvillian

*No nontrivial operator satisfying $[\hat{H}_t, \hat{O}] = [\hat{L}_t, \hat{O}] = 0$

Still, multiple time-dependent steady states exist!

A set of periodic steady states depending on $a = \langle \hat{\sigma}^x \rangle_0$: $\hat{\rho}_t \longrightarrow \hat{\rho}_t^* = \frac{\hat{\mathbb{1}} + a \hat{\sigma}^x \cos \omega t + a \hat{\sigma}^y \sin \omega t}{2}$

Two distinct strong symmetries

Focusing on the Hermitian jump operators $\hat{L}_{\mu,t}^\dagger = \hat{L}_{\mu,t}$, two distinct strong symmetries characterize the dynamics

Strong symmetry in the Schrödinger picture

$$\mathcal{C}^{\text{Sch}} = \{ \hat{O} : [\hat{H}_t, \hat{O}] = [\hat{L}_{k,t}, \hat{O}] = 0 \quad \text{for all } k, t \}$$

Strong symmetry in the interaction picture

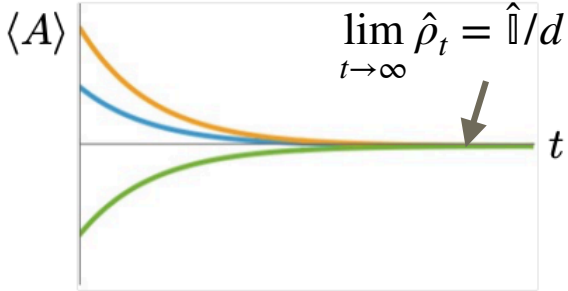
$$\mathcal{C}^{\text{Int}} = \{ \hat{O} : [\hat{\tilde{L}}_{k,t}, \hat{O}] = 0 \quad \text{for all } k, t \}$$

$$\hat{\tilde{L}}_{k,t} = U_t^\dagger \hat{L}_{k,t} U_t, \quad U_t = \mathbb{T} \exp \left[-i \int_0^t d\tau \hat{H}_\tau \right]$$

We find a hierarchy: $\mathcal{C}^{\text{Sch}} \subseteq \mathcal{C}^{\text{Int}}$

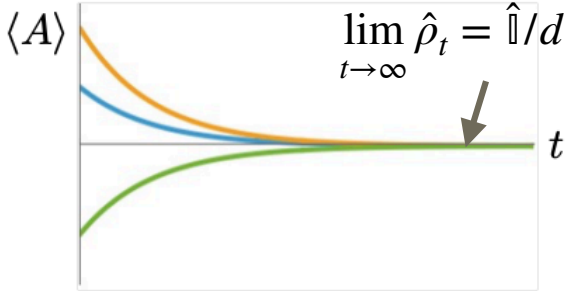
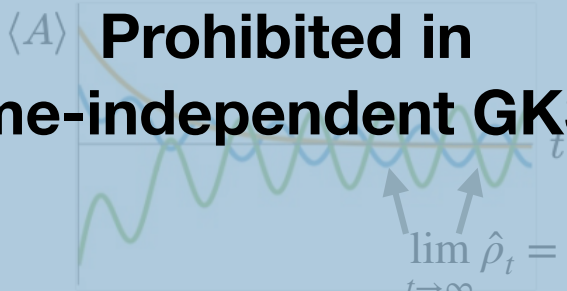
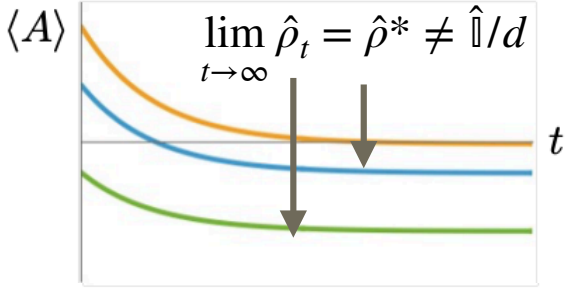
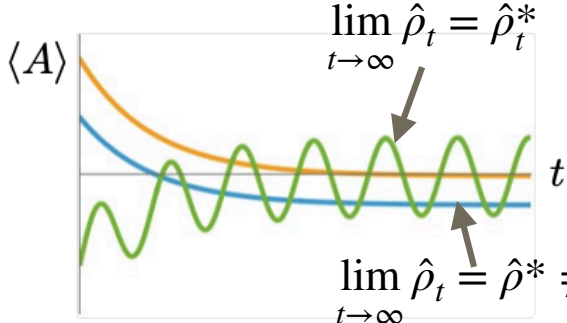
Two distinct strong symmetries

For “recurrent” GKSL equations, we have the following:

		$\mathcal{C}^{\text{Int}} \setminus \mathcal{C}^{\text{Sch}}$	
		$= \emptyset$	$\neq \emptyset$
\mathcal{C}^{Sch}	$= \{c\mathbb{I}\}$	<p>(i) Unique steady state</p> 	
	$\neq \{c\mathbb{I}\}$		

Two distinct strong symmetries

For “recurrent” GKSL equations, we have the following:

		$\mathcal{C}^{\text{Int}} \setminus \mathcal{C}^{\text{Sch}}$	
		$= \emptyset$	$\neq \emptyset$
\mathcal{C}^{Sch}	$= \{c\mathbb{I}\}$	<p>(i) Unique steady state</p> 	<p>(iv) Time-dependent steady states without non-trivial time-independent ones</p> <p>Prohibited in time-independent GKSL</p> 
	$\neq \{c\mathbb{I}\}$	<p>(ii) Multiple time-independent steady states without time-dependent ones</p> 	<p>(iii) Multiple time-independent and time-dependent steady states</p> 

Two distinct strong symmetries

For “recurrent” GKSL equations, we have the following:

		$\mathcal{C}^{\text{Int}} \setminus \mathcal{C}^{\text{Sch}}$	
		$= \emptyset$	$\neq \emptyset$
		(i) Unique steady state	(iv) Time-dependent steady states without non-trivial time-independent ones

\mathcal{C}^{Sch} determines *time-independent* steady states, and \mathcal{C}^{Int} determines *time-dependent* steady states

		(ii) Multiple time-independent steady states without time-dependent ones	(iii) Multiple time-independent and time-dependent steady states
	$\neq \{c\mathbb{I}\}$		

Fate of long-time dynamics of open systems?

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What is the relation with Liouvillian spectrum?

How to characterize it from symmetry?

How about long-time dynamics in open systems?

Assuming that the Hilbert-space dimension d is finite, can we understand properties of infinitely long-time state?

$$\hat{\rho}_t \xrightarrow{(t \rightarrow \infty)} \hat{\rho}_t^* \quad \hat{\rho}_t = \mathcal{T} e^{\int_0^t d\tau \mathcal{L}_\tau} [\hat{\rho}_0]$$

Unlike the closed systems, we here consider dynamics at the level of density matrices, without considering, e.g., locality of observables

e.g.) Is the steady state unique, irrespective of $\hat{\rho}_0$?

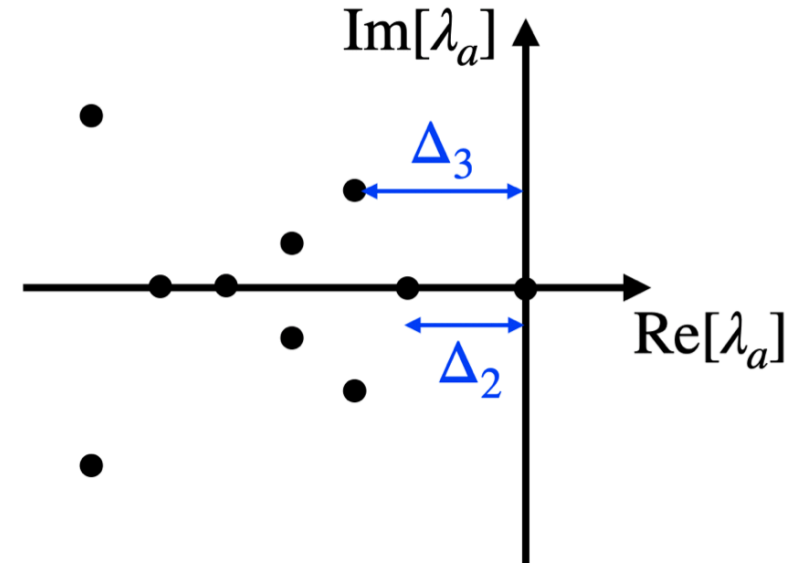
What is the speed of relaxation to steady states?

Relaxation timescale

Assume time-independent GKSL with a unique steady state

$$\hat{\rho}(t) = e^{\mathcal{L}t}[\hat{\rho}(0)] = \hat{\rho}_{\text{ss}} + \sum_{a=2}^{\dim[\mathcal{H}]^2} c_a e^{\lambda_a t} \hat{\phi}_a$$

$$c_a = \frac{\text{Tr}[\hat{\chi}_a^\dagger \hat{\rho}(0)]}{\text{Tr}[\hat{\chi}_a^\dagger \hat{\phi}_a]}$$



Setting $\Delta_a := |\text{Re}[\lambda_a]|$, we have $\|e^{\lambda_a t} \hat{\phi}_a\| \ll 1$ for $\Delta_a^{-1} \ll t$

Then, when $\Delta_2^{-1} \ll t$, we expect $\hat{\rho}(t) \simeq \hat{\rho}_{\text{ss}}$

Δ_2 : Liouvillian gap

Gap of \mathcal{L} determines the relaxation timescale τ_{relax} ?

Relaxation timescale

$$\hat{\rho}(t) = e^{\mathcal{L}t}[\hat{\rho}(0)] = \hat{\rho}_{\text{ss}} + \sum_{a=2}^{\dim[\mathcal{H}]^2} c_a e^{\lambda_a t} \hat{\phi}_a \quad c_a = \frac{\text{Tr}[\hat{\chi}_a^\dagger \hat{\rho}(0)]}{\text{Tr}[\hat{\chi}_a^\dagger \hat{\phi}_a]}$$

$\tau_{\text{relax}} \sim \Delta_2^{-1}$ indeed holds true in many cases! M. Žnidarič, PRE (2015)

Looking at Δ_2 and other low-lying modes, we can discuss long-time dynamics towards the steady state

e.g.) [RH](#) et al., PTEP ptag055 (2026).

*Dissipative phase transitions F. Minganti et al., PRA 98, 042118 (2018).

*Quantum Mpemba effects F. Carollo et al., PRL 127, 060401 (2021).

*Metastability K. Macieszczak et al., PRL 116, 240404 (2016)

*Relaxation in isolated systems (Ruelle-Pollicot resonances)

T Mori, PRB 109 (6), 064311 (2024) [cf. T. Prosen 2002 J. Phys. A: Math. Gen. (2002)]

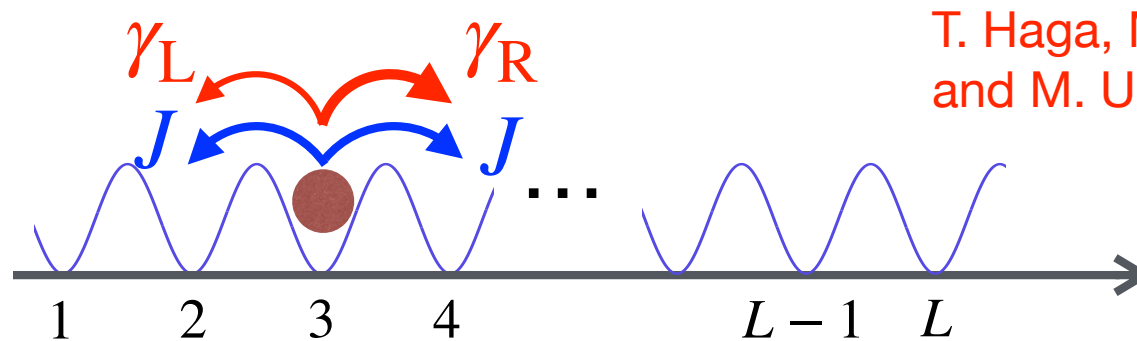
Discrepancy between relaxation timescale and gap

$$\hat{\rho}(t) = e^{\mathcal{L}t}[\hat{\rho}(0)] = \hat{\rho}_{\text{ss}} + \sum_{a=2}^{\dim[\mathcal{H}]^2} c_a e^{\lambda_a t} \hat{\phi}_a \quad c_a = \frac{\text{Tr}[\hat{\chi}_a^\dagger \hat{\rho}(0)]}{\text{Tr}[\hat{\chi}_a^\dagger \hat{\phi}_a]}$$

$\tau_{\text{relax}} \sim \Delta_2^{-1}$ becomes invalid in some cases

1D bosonic system with incoherent hopping (OBC)

$$\hat{H} = -J \sum_{l=1}^L \left(\hat{b}_{l+1}^\dagger \hat{b}_l + \hat{b}_l^\dagger \hat{b}_{l+1} \right) \quad \hat{L}_{\text{R},l} = \sqrt{\gamma_{\text{R}}} \hat{b}_{l+1}^\dagger \hat{b}_l$$
$$\hat{L}_{\text{L},l} = \sqrt{\gamma_{\text{L}}} \hat{b}_l^\dagger \hat{b}_{l-1}$$



T. Haga, M. Nakagawa, [RH](#),
and M. Ueda, PRL (2021)

Relaxation time τ_{relax} grows with L

Discrepancy between relaxation timescale and gap

$$\hat{\rho}(t) = e^{\mathcal{L}t}[\hat{\rho}(0)] = \hat{\rho}_{\text{ss}} + \sum_{a=2}^{\dim[\mathcal{H}]^2} c_a e^{\lambda_a t} \hat{\phi}_a \quad c_a = \frac{\text{Tr}[\hat{\chi}_a^\dagger \hat{\rho}(0)]}{\text{Tr}[\hat{\chi}_a^\dagger \hat{\phi}_a]}$$

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T. Haga, M. Nakagawa, **RH**,
and M. Ueda, PRL (2021)

For $J = 0$ and the single-particle case,

the corresponding \mathcal{L} has a gap $\Delta_2 = (\sqrt{\gamma_{\text{R}}} - \sqrt{\gamma_{\text{L}}})^2$

→ For $\gamma_{\text{R}} \neq \gamma_{\text{L}}$, we have $\Delta_2^{-1} = \mathcal{O}(L^0) \neq \tau_{\text{relax}} = \mathcal{O}(L)$

Discrepancy between relaxation timescale and gap

$$\hat{\rho}(t) = e^{\mathcal{L}t}[\hat{\rho}(0)] = \hat{\rho}_{\text{ss}} + \sum_{a=2}^{\dim[\mathcal{H}]^2} c_a e^{\lambda_a t} \hat{\phi}_a \quad c_a = \frac{\text{Tr}[\hat{\chi}_a^\dagger \hat{\rho}(0)]}{\text{Tr}[\hat{\chi}_a^\dagger \hat{\phi}_a]}$$

$\tau_{\text{relax}} \sim \Delta_2^{-1}$ becomes invalid in some cases

c_a can be significantly large (exponentially small denominator)

T. Haga, M. Nakagawa, **RH**, and M. Ueda, PRL (2021); T. Mori and T. Shirai, PRL (2020)

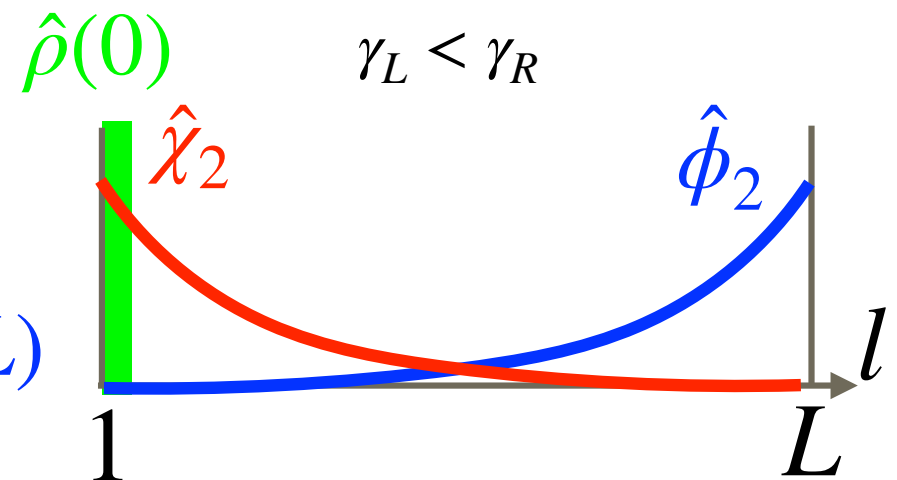
In the previous example,

we have $c_2 \sim e^{O(L/\xi)}$,

→ Consistent timescale

$$\tau_{\text{relax}} \sim (\log c_a) / |\text{Re}[\lambda_a]| = O(L)$$

ξ : localization length of $\hat{\phi}_2$

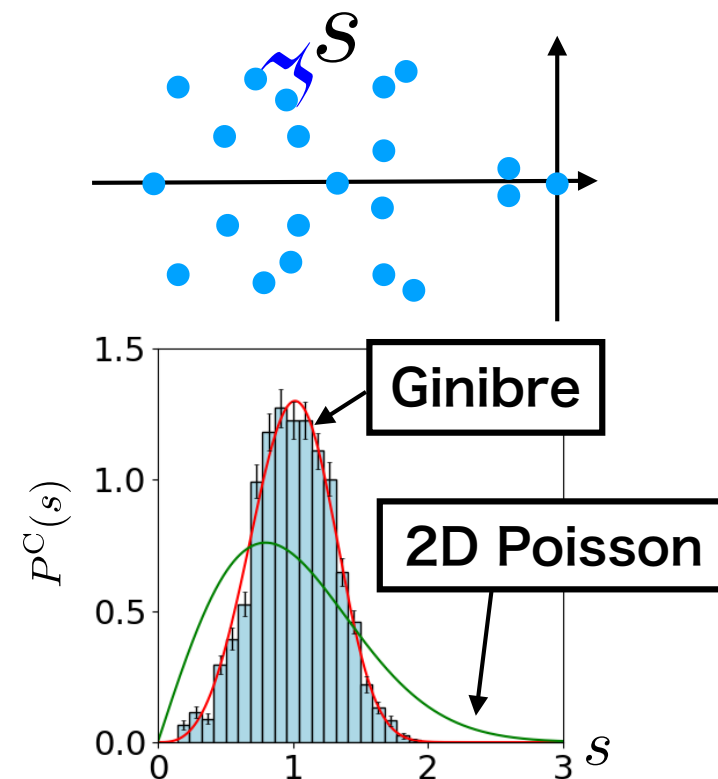


Remark: universality of spectrum in dissipative quantum chaos

Universality of middle of the spectrum?

Spectral statistics of complicated open systems are described by non-Hermitian random-matrix theory!

R. Grobe et al., PRL (1988); **RH** et al., PRL (2019);
G. Akemann et al., PRL (2019); AB. Jaiswal EPL (2019);
RH et al., PRResearch (2020); L. Sá et al., PRX (2020).



RH et al., PRL 123 (9),
090603 (2019)

Since middle of the spectrum corresponds to fast decaying modes, they can be related with complicated (chaotic) behavior in *transient* dynamics

RH et al., arXiv:2206.02984 (2022);
D. Mondal Phys. Rev. Lett. 136, 040401 (2026).

Summary: steady states and relaxation speeds

Steady states of time-independent GKSL equations are classified by their spectra & strong (dynamical) symmetries

Even for time-dependent GKSL equations, under certain assumptions, steady-state structures can be classified by two types of strong symmetries

Spectral gap of the Liouvillian often captures the timescale for relaxation, but exceptional cases also exist

Measurement-induced transitions and Lyapunov spectrum

Review articles:

(Typicality of quantum trajectories, Lyapunov exponents, MIPT)

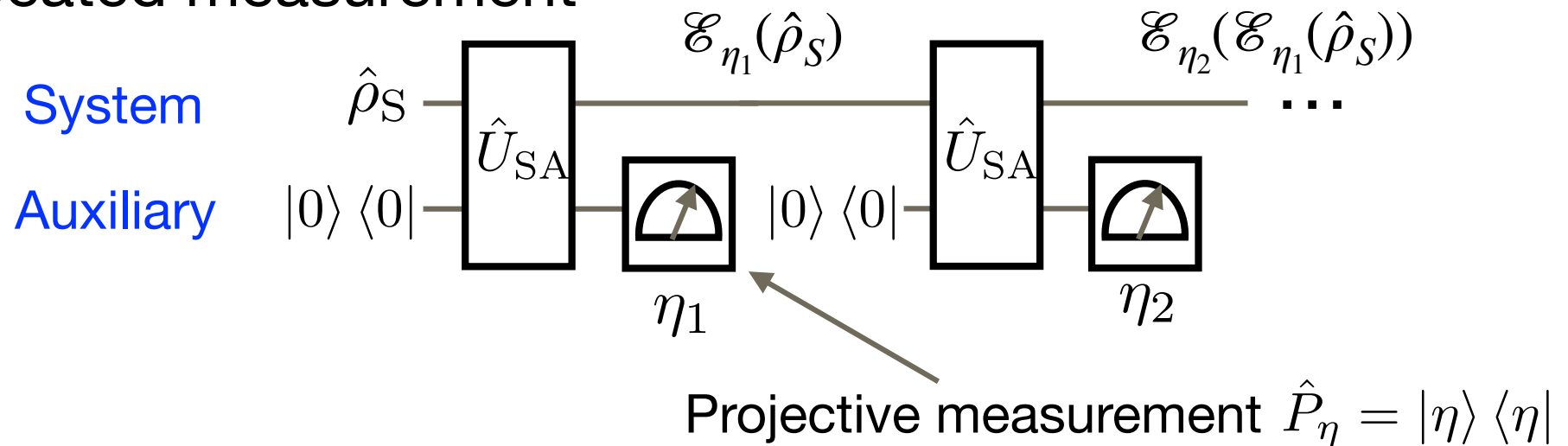
*[RH](#), K Mochizuki, H Oshima, Y Fuji, PTEP, ptag055 (2026).

(Measurement-induced phase transitions and quantum circuits)

*MPA. Fisher et al., Annual Review of Condensed Matter Physics, 14(1), 335 (2023).

Repeated measurement and quantum trajectories

Repeated measurement



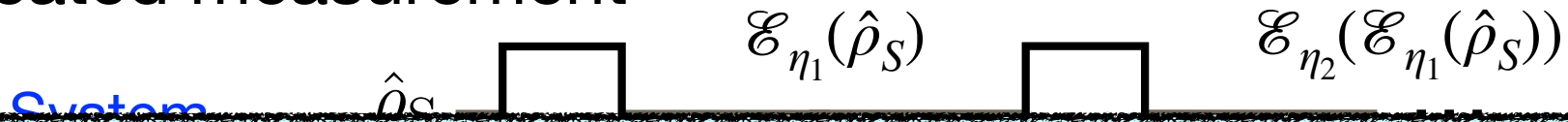
If the measurement outcomes are $\eta_1, \eta_2, \dots, \eta_n$
 we obtain a state conditioned on measurement outcomes

$$\hat{\rho}_S(n; \eta_1, \dots, \eta_n) \propto \mathcal{E}_{\eta_n} \circ \dots \circ \mathcal{E}_{\eta_2} \circ \mathcal{E}_{\eta_1}(\hat{\rho}_S)$$

Quantum trajectories (initially pure states remain pure)

Repeated measurement and quantum trajectories

Repeated measurement



Quantum trajectories can exhibit interesting physics that is hidden in dynamics averaged over outcomes (like GKSL equation)

If the measurement outcomes are $\eta_1, \eta_2, \dots, \eta_n$ we obtain a state conditioned on measurement outcomes

$$\hat{\rho}_S(n; \eta_1, \dots, \eta_n) \propto \mathcal{E}_{\eta_n} \circ \dots \circ \mathcal{E}_{\eta_2} \circ \mathcal{E}_{\eta_1}(\hat{\rho}_S)$$

Quantum trajectories (initially pure states remain pure)

Quantum trajectories vs averaged dynamics

Quantum trajectories

$$\hat{\rho}(n; \eta_1, \dots, \eta_n) \propto \mathcal{E}_{\eta_n} \circ \dots \circ \mathcal{E}_{\eta_2} \circ \mathcal{E}_{\eta_1}(\hat{\rho})$$

Averaged dynamics

$$\mathcal{E} = \sum_{\eta} \mathcal{E}_{\eta}$$

$$\hat{\rho}(n) = \mathbb{E}[\hat{\rho}(n; \eta_1, \dots, \eta_n)] = \sum_{\eta_1 \dots \eta_n} \mathcal{E}_{\eta_n} \circ \dots \circ \mathcal{E}_{\eta_2} \circ \mathcal{E}_{\eta_1}(\hat{\rho}) = \mathcal{E}^n(\hat{\rho})$$

↑
Average over outcomes

*Linear observables basically do not make distinctions

*Nonlinear observables can:

e.g., quantum trajectories purify the mixed state,
while the average dynamics mixes the pure state

Linear observables basically do not make distinctions

Compare linear observables for two states

$$\langle \hat{A}(n) \rangle_{\text{QT}} = \text{Tr}[\hat{\rho}(n; \eta_1, \dots, \eta_n) \hat{A}] \rightarrow \mathbb{E}[\langle \hat{A}(n) \rangle_{\text{QT}}] = \langle \hat{A}(n) \rangle$$
$$\langle \hat{A}(n) \rangle = \text{Tr}[\hat{\rho}(n) \hat{A}]$$

Even without ensemble average,
we have “*ergodicity*” of typical quantum trajectories,
if \mathcal{E} has a unique steady state $\hat{\rho}_{\text{SS}}$:

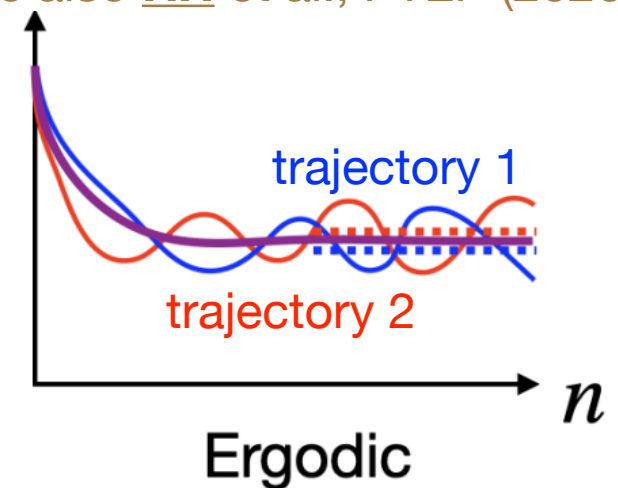
For almost all measurement outcomes,

$$\overline{\langle \hat{A}(n) \rangle_{\text{QT}}} = \overline{\langle \hat{A}(n) \rangle}$$

or, equivalently, $\overline{\hat{\rho}(n; \eta_1, \dots, \eta_n)} = \hat{\rho}_{\text{SS}}$

Long-time average = Ensemble average

B. Kümmerer and H. Maassen,
J. Phys. A: Math. Gen. (2004);
see also **RH** et al., PTEP (2026).



Nonlinear observables: purification of quantum trajectories

Measurement leads to purification of mixed states

Extreme case: projective measurement

For $\hat{P}_k = |k\rangle\langle k|$, we have a pure state after measurement $\frac{\hat{P}_k \hat{\rho} \hat{P}_k}{p_k} = |k\rangle\langle k|$

H. Maassen and B. Kümmerer,
Lecture Notes-Monograph Series (2006);
see also [RH](#) et al., PTEP (2026).

Even for indirect measurement,
we have the following result under certain assumptions:

For almost all measurement outcomes,

$$\lim_{n \rightarrow \infty} \mathcal{P}_n = \lim_{n \rightarrow \infty} \text{Tr}[\hat{\rho}(n; \eta_1, \dots, \eta_n)^2] = 1$$

Almost all states are purified!

Nonlinear observables: purification of quantum trajectories

Measurement leads to purification of mixed states

Is purification dynamics captured by spectral property?

Usual spectrum cannot be applied:

$$\hat{\rho}(n; \eta_1, \dots, \eta_n) \propto \mathcal{E}_{\eta_n} \circ \dots \circ \mathcal{E}_{\eta_2} \circ \mathcal{E}_{\eta_1}(\hat{\rho}) \propto \hat{M}_{\eta_n} \dots \hat{M}_{\eta_1} \hat{\rho} \hat{M}_{\eta_1}^\dagger \dots \hat{M}_{\eta_n}^\dagger$$

highly time-dependent due to random measurement outcomes!

For almost all measurement outcomes,

$$\lim_{n \rightarrow \infty} \mathcal{P}_n = \lim_{n \rightarrow \infty} \text{Tr}[\hat{\rho}(n; \eta_1, \dots, \eta_n)^2] = 1$$

Almost all states are purified!

Lyapunov exponents

Considering a maximal mixed initial state, we have

$$\hat{\rho}(n; \eta_1, \dots, \eta_n) \propto \hat{M}_{\eta_n} \cdots \hat{M}_{\eta_1} \hat{M}_{\eta_1}^\dagger \cdots \hat{M}_{\eta_n}^\dagger = e^{-\hat{H}_{\text{eff}}^{(n)}(\vec{\eta})n}$$

$$\begin{aligned} \hat{H}_{\text{eff}}^{(n)}(\vec{\eta}) &= -\frac{1}{2n} \log \hat{M}_{\eta_n} \cdots \hat{M}_{\eta_1} \hat{M}_{\eta_1}^\dagger \cdots \hat{M}_{\eta_n}^\dagger \\ &= \sum_i \varepsilon_{n,i}(\vec{\eta}) |\Psi_{n,i}(\vec{\eta})\rangle \langle \Psi_{n,i}(\vec{\eta})| \end{aligned}$$

Under some assumptions, $\varepsilon_{t,i}(\vec{\eta})$ converges in the long time irrespective of initial states & measurement outcomes, for almost all measurement outcomes

$$\lim_{n \rightarrow \infty} \varepsilon_{n,i}(\vec{\eta}) = \varepsilon_i \quad \text{Lyapunov exponents}$$

Lyapunov exponents

Considering a maximal mixed initial state, we have

$$\hat{\rho}(n; \eta_1, \dots, \eta_n) \propto \hat{M} \dots \hat{M} \hat{M}^\dagger \dots \hat{M}^\dagger = e^{-\hat{H}_{\text{eff}}^{(n)}(\vec{\eta})n}$$

In long times, we have

$$\hat{\rho}(n; \eta_1, \dots, \eta_n) \sim |\Psi_{n,1}(\vec{\eta})\rangle\langle\Psi_{n,1}(\vec{\eta})| + ce^{-\Delta n} |\Psi_{n,2}(\vec{\eta})\rangle\langle\Psi_{n,2}(\vec{\eta})|$$

$$\Delta = \varepsilon_2 - \varepsilon_1: \text{Lyapunov gap}$$

Δ controls the rate for the relaxation to steady states
(e.g., in purification dynamics)

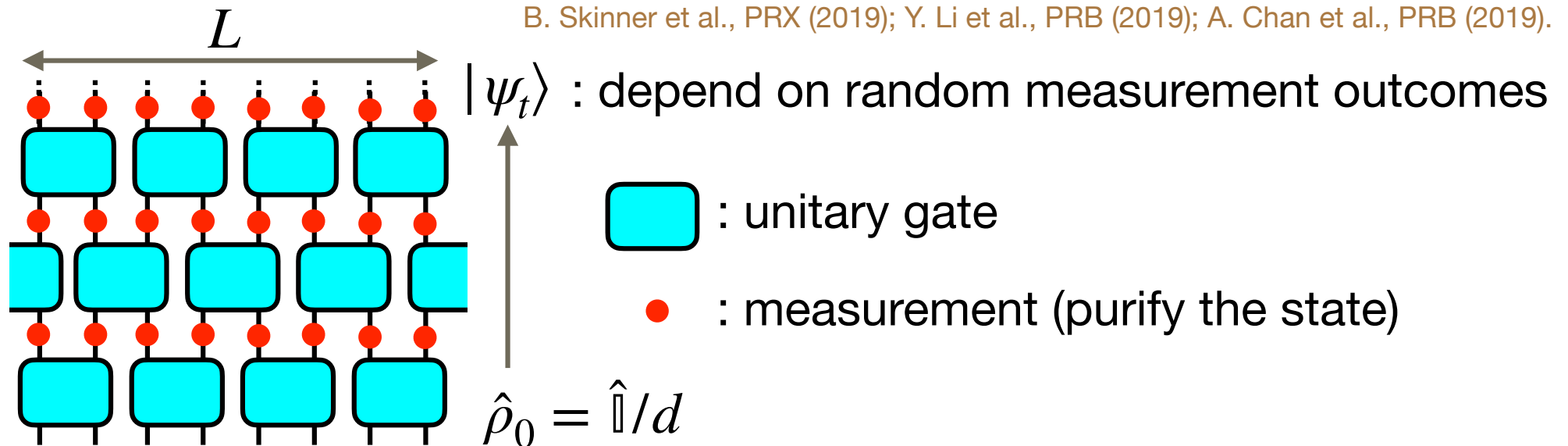
for almost all measurement outcomes

$$\lim_{n \rightarrow \infty} \varepsilon_{n,i}(\vec{\eta}) = \varepsilon_i \quad \text{Lyapunov exponents}$$

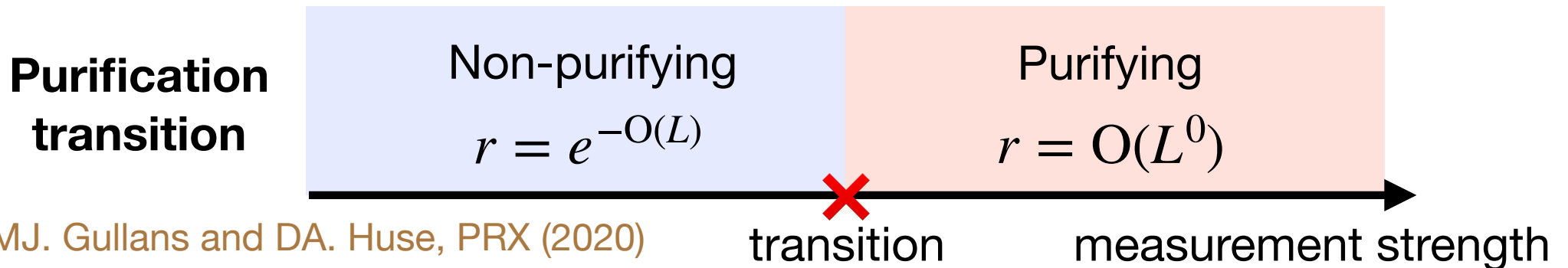
Purification transitions in many-body systems

Many-body dynamics vs measurement \rightarrow novel transition

B. Skinner et al., PRX (2019); Y. Li et al., PRB (2019); A. Chan et al., PRB (2019).



Almost all trajectories $\hat{\rho}(n; \eta_1, \dots, \eta_n)$ purify in the infinitely long time, but what is the rate r for purification? $\mathcal{P}_n \sim 1 - ce^{-rn}$



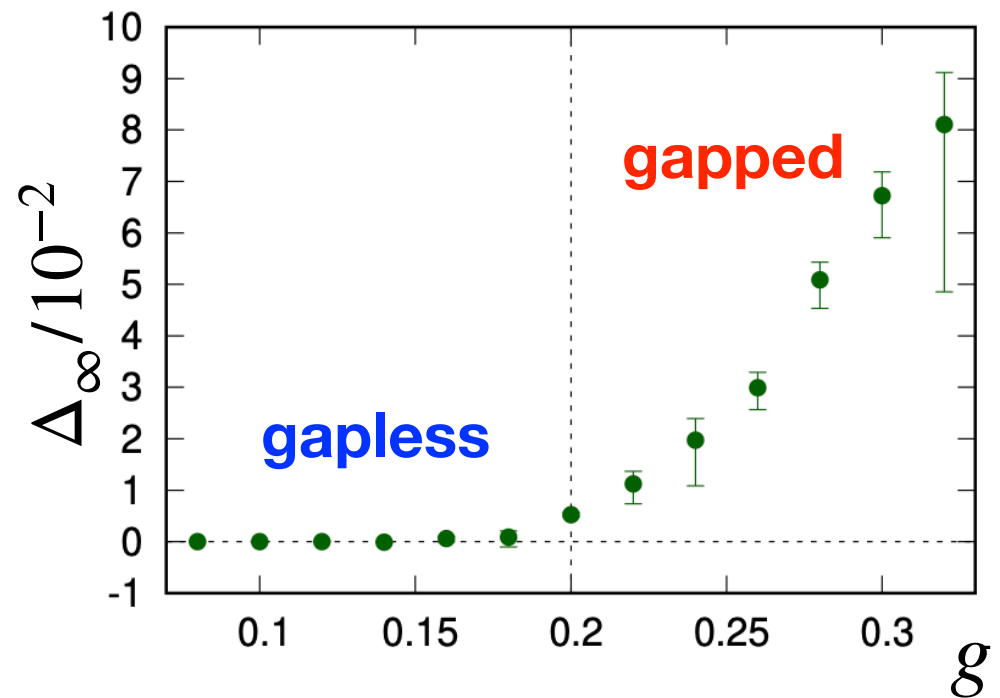
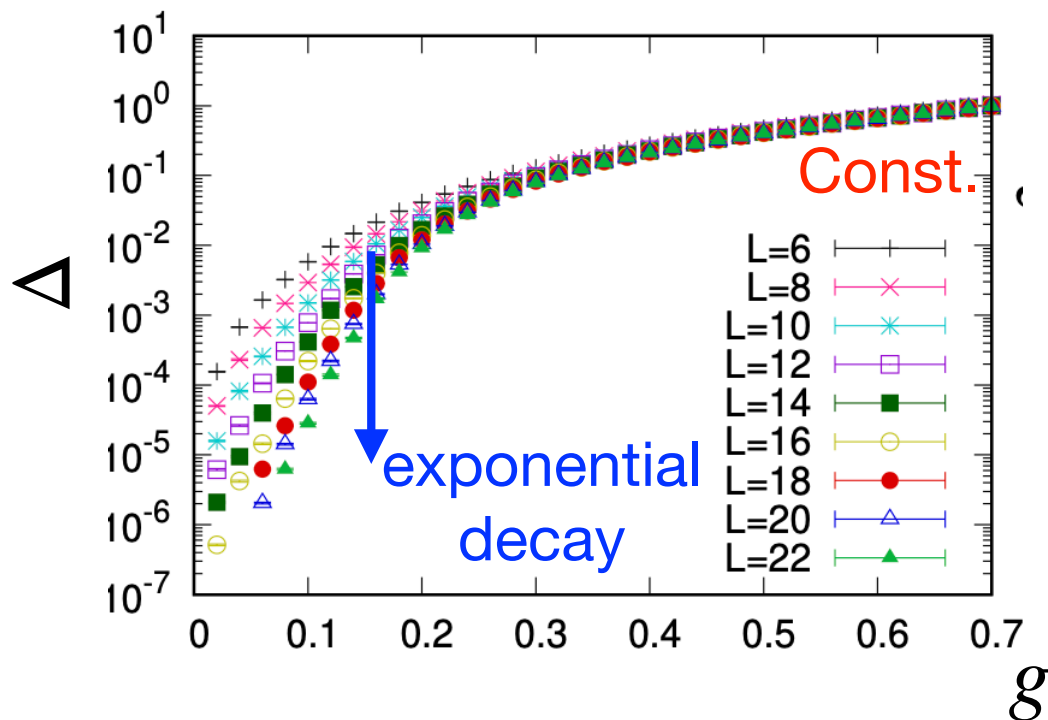
Measurement-induced spectral transitions

[K. Mochizuki and R. Hamazaki, PRL, 134, 010410 (2025).]

Since $r \sim \Delta$, can we find a transition for the Lyapunov gap?

Yes! Consider gap in the thermodynamic limit and change the measurement strength g

$$\Delta_\infty = \lim_{L \rightarrow \infty} \Delta$$



g small $\rightarrow \Delta_\infty = 0$ (gapless phase)

g large $\rightarrow \Delta_\infty \neq 0$ (gapped phase)


Spectral transition
at $g_c \simeq 0.2$

MIPT for entanglement, spectrum and purification

[K. Mochizuki and R. Hamazaki, PRL, 134, 010410 (2025).]

We also find measurement-induced phase transitions for entanglement of pure states at the same transition point

(Half-chain) Entanglement entropy	Volume law $S = O(L)$	Area law $S = O(L^0)$
Lyapunov spectrum	Gapless $\Delta = e^{-O(L)}$	Gapped $\Delta = O(L^0)$
Purification	Non-purifying $r = e^{-O(L)}$	Purifying $r = O(L^0)$


transition measurement strength

MIPT for entanglement, spectrum and purification

[K. Mochizuki and R. Hamazaki, PRL, 134, 010410 (2025).]

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(Half-chain) Entanglement entropy	Volume law $S = O(L)$	Area law $S = O(L^0)$
Lyapunov spectrum	Gapless $\Delta = e^{-O(L)}$	Gapped $\Delta = O(L^0)$

The correspondence between entanglement and spectral transitions makes an interesting analogy to ground-state quantum phase transitions in isolated systems

Summary: measurement-induced transitions

Quantum trajectories can exhibit interesting physics absent in averaged dynamics, especially for nonlinear quantities

Almost all quantum trajectories purify,
whose rate is captured by the Lyapunov exponents

For quantum many-body systems under measurement,
measurement-induced transitions for purification rates,
Lyapunov exponents, and entanglement can appear

Overall summary

Closed quantum systems

ETH provides a standard mechanism for thermalization, but it can be broken by e.g., integrability, symmetries, scars, and fragmentation

Thermalization is not the whole story:

second law of thermodynamics raises another fundamental question

Open quantum systems

Liouvillian spectra and strong symmetries give crucial information at and around steady states, while spectra can sometimes be insufficient

Measurement and many-body interaction lead to nontrivial transition of purification, entanglement, and the Lyapunov spectrum