測定下の量子系における複雑性とスペクトル

Ken Mochizuki and Ryusuke Hamazaki,
Physical Review Research **5**, 013177 (2023).

Ken Mochizuki and Ryusuke Hamazaki,
Physical Review Letters **134**, 010410 (2025).

科研費 KAKENHI

ERATO 沙川情報エネルギー変換プロジェクト

Outline

- Introduction
- Parity-Time symmetry breaking spectral transition and the boson sampling complexity
- Measurement-induced spectral transition and entanglement entropy
- Summary

Complexity of quantum states

$|\psi\rangle$: complex?

fundamental!

low complexity

high complexity

computational complexity

efficient simulation

superlong computational time

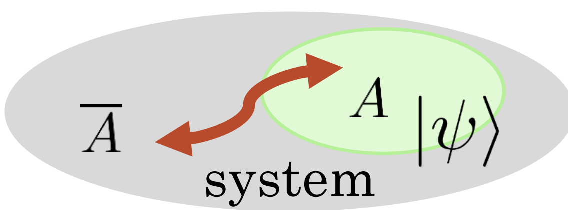
entanglement

small $S_A(\psi)$

c.f. $|\psi\rangle = |\uparrow_1\rangle \otimes |\downarrow_2\rangle$

large $S_A(\psi)$

c.f. $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow_1\downarrow_2\rangle + |\downarrow_1\uparrow_2\rangle)$



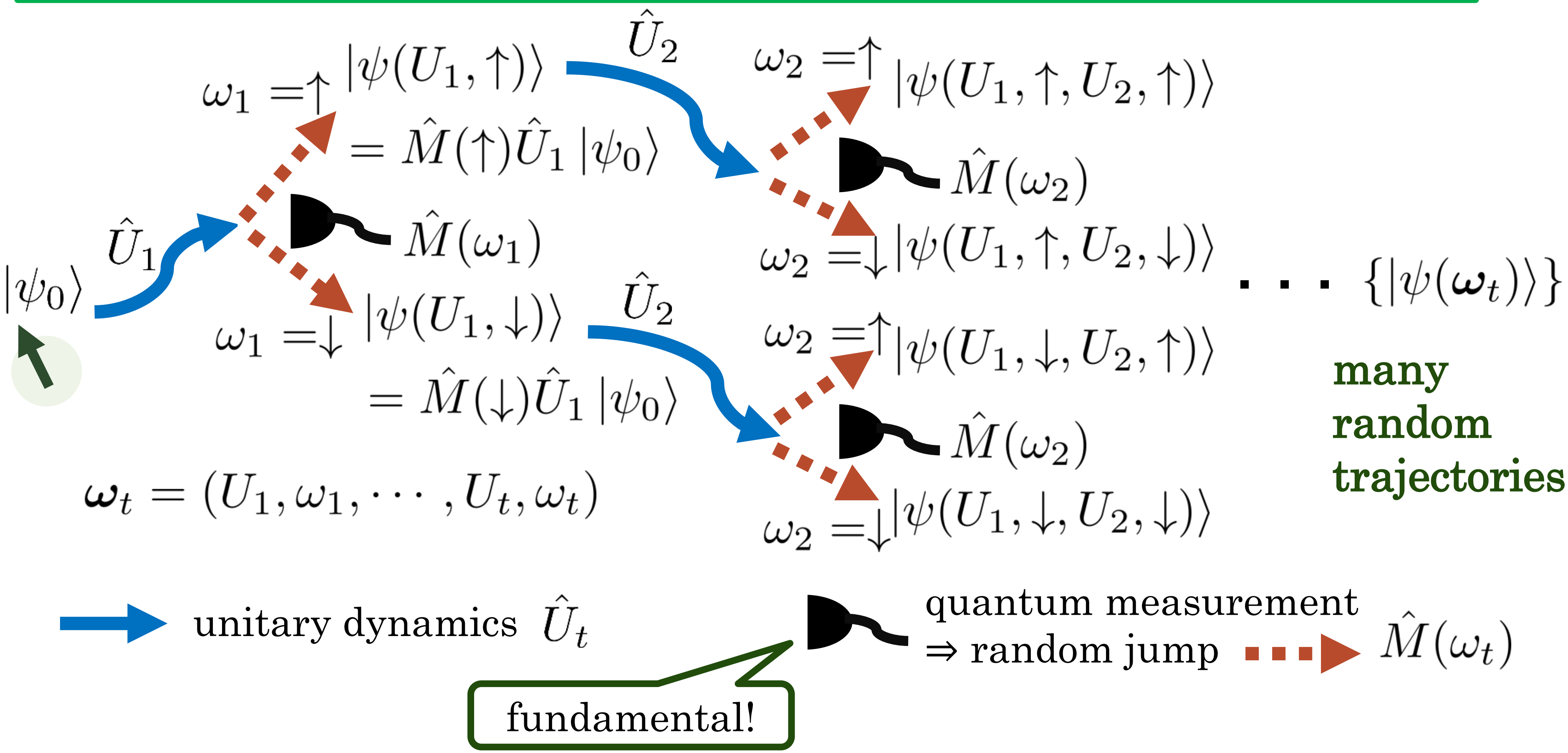
$$\rho_A(\psi) = \mathrm{tr}_{\bar{A}}\left(|\psi\rangle\langle\psi|\right)$$

$$S_A(\psi) = -\mathrm{tr}_A\left(\rho_A(\psi)\log\left[\rho_A(\psi)\right]\right)$$

Quantum measurement

monitored dynamics

$$|\psi(\boldsymbol{\omega}_t)\rangle \propto \hat{V}(\boldsymbol{\omega}_t)\,|\psi_0\rangle \neq e^{-i\hat{H}t}\,|\psi_0\rangle\,, \quad \hat{V}(\boldsymbol{\omega}_t) = \hat{M}(\omega_t)\hat{U}_t \cdots \hat{M}(\omega_1)\hat{U}_1$$



Monitored systems and complexity

monitored dynamics

$$|\psi(\boldsymbol{\omega}_t)\rangle \propto \hat{V}(\boldsymbol{\omega}_t)\,|\psi_0\rangle \neq e^{-i\hat{H}t}\,|\psi_0\rangle\,, \quad \hat{V}(\boldsymbol{\omega}_t) = \hat{M}(\omega_t)\hat{U}_t \cdots \hat{M}(\omega_1)\hat{U}_1$$

unitary dynamics $\hat{U}_t$ ⟷ quantum measurement $\hat{M}(\omega_t)$

increasing complexity

c.f.

$$\hat{C}_{\mathrm{NOT}}\frac{1}{\sqrt{2}}(|{\uparrow_1}\rangle + |{\downarrow_1}\rangle)\otimes|{\uparrow_2}\rangle = \frac{1}{\sqrt{2}}(|{\uparrow_1\uparrow_2}\rangle + |{\downarrow_1\downarrow_2}\rangle)$$

decreasing complexity

c.f.

$$\hat{M}(\uparrow_1)\frac{1}{\sqrt{2}}(|{\uparrow_1\uparrow_2}\rangle + |{\downarrow_1\downarrow_2}\rangle) \propto |{\uparrow_1}\rangle\otimes|{\uparrow_2}\rangle$$

$$\hat{M}(\downarrow_1)\frac{1}{\sqrt{2}}(|{\uparrow_1\uparrow_2}\rangle + |{\downarrow_1\downarrow_2}\rangle) \propto |{\downarrow_1}\rangle\otimes|{\downarrow_2}\rangle$$

competition ⇒ intriguing phase transitions in many-body systems

our work:
revealing measurement-induced transitions related to quantum complexity and the **spectrum** of $\hat{V}(\boldsymbol{\omega}_t)$

Why we consider the spectrum

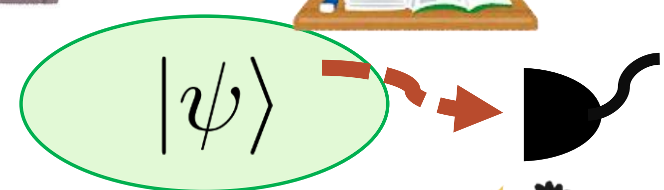
isolated quantum systems
governed by Hamiltonians

monitored quantum systems
not governed by Hamiltonians

$$\hat{H}\,|\Psi_i\rangle = \varepsilon_i\,|\Psi_i\rangle$$

$$|\psi(t)\rangle = e^{-i\hat{H}t}\,|\psi_0\rangle$$

intriguing complexity transitions
owing to measurements (discussed later)



$|\psi\rangle$

spectrum,
ground state,
topology,
dynamics,
...

our work

same concepts!

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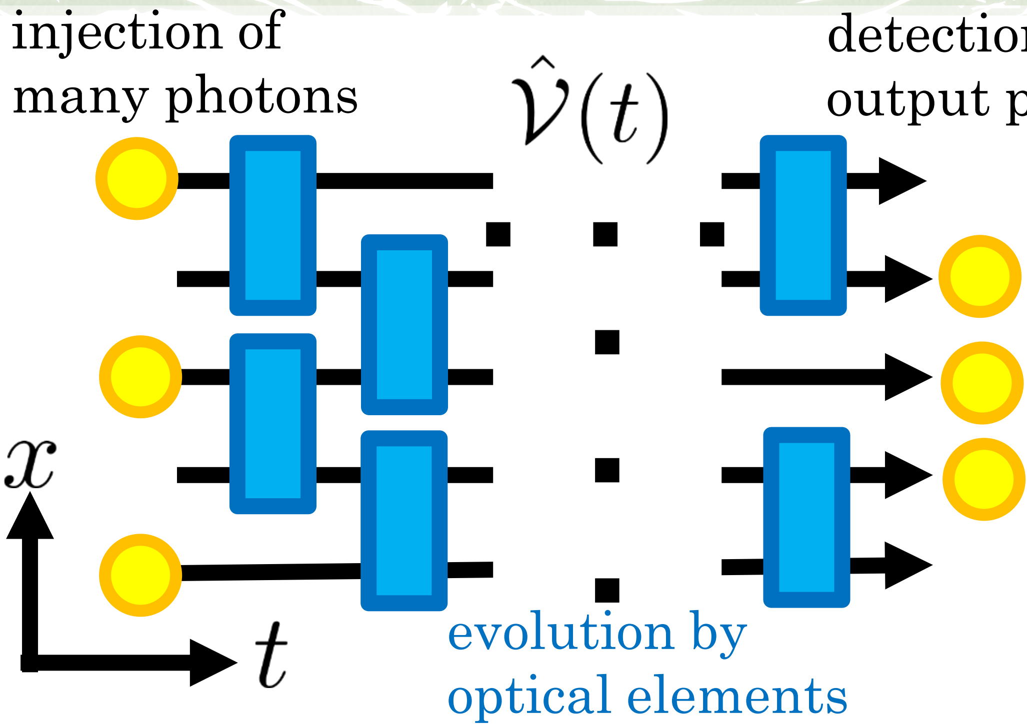
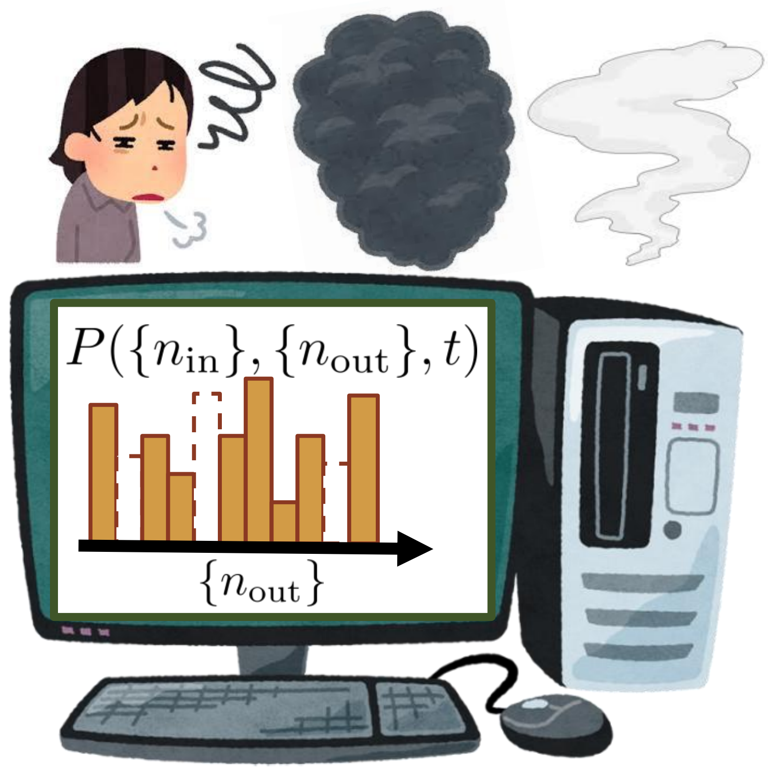
Boson sampling problem



$P(\{n_{\rm in}\},\{n_{\rm out}\},t)$??



$$\hat{\mathcal{V}}(t)\hat{b}_x^\dagger\hat{\mathcal{V}}^{-1}(t)=\sum_y \mathcal{V}_{yx}(t)\hat{b}_y^\dagger$$

Photon probabilities can be hard to compute by classical computers
⇒ computational complexity / quantum supremacy

Permanent of the matrix

fermion$\rightarrow$ $\det[\mathcal{V}_{\mathrm{out,in}}(t)]$: easy to compute

$$P(\{n_{\mathrm{in}}\},\{n_{\mathrm{out}}\},t) \propto |\mathrm{Per}[\mathcal{V}_{\mathrm{out,in}}(t)]|^2$$

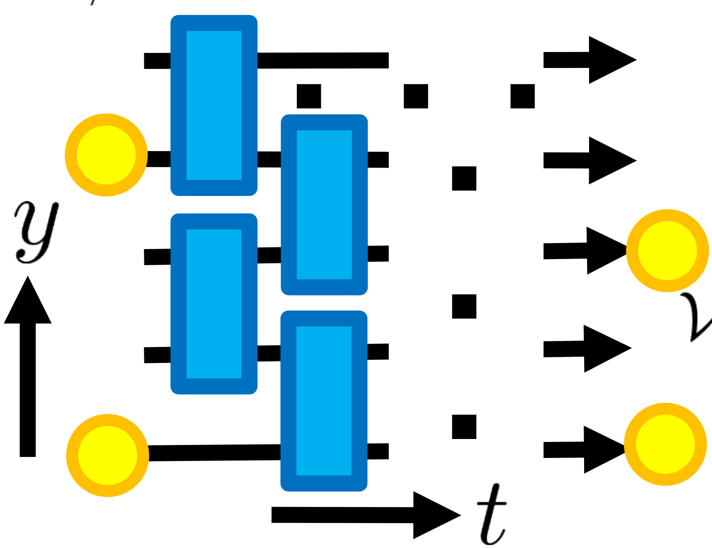
$$\mathrm{Per}[\mathcal{V}(t)] = \sum_{\sigma}\prod_{y}\mathcal{V}_{y\sigma(y)}(t)$$

typically hard to compute!

$$\mathcal{V}_{\mathrm{out,in}}(t) = \begin{pmatrix} \mathcal{V}_{y^1_{\mathrm{out}}y^1_{\mathrm{in}}}(t), \mathcal{V}_{y^2_{\mathrm{out}}y^1_{\mathrm{in}}}(t), \cdots \\ \mathcal{V}_{y^1_{\mathrm{out}}y^2_{\mathrm{in}}}(t), \mathcal{V}_{y^2_{\mathrm{out}}y^2_{\mathrm{in}}}(t), \cdots \\ \cdots, \cdots, \cdots \end{pmatrix}$$

$n \times n$

$y^p_{\mathrm{in/out}}$: input/output port of the pth photon



$$\begin{array}{c} \\ y^1_{\mathrm{in}} = 1 \\ \\ \\ y^2_{\mathrm{in}} = 4 \\ \\ \end{array} \quad \mathcal{V}(t) = \begin{array}{cccc} y^1_{\mathrm{out}} = 1 & & y^2_{\mathrm{out}} = 3 & \\ \end{array}$$

$$\mathcal{V}(t) = \begin{pmatrix} \mathcal{V}_{11}(t) & \mathcal{V}_{12}(t) & \mathcal{V}_{13}(t) & \cdots \\ \mathcal{V}_{21}(t) & \mathcal{V}_{22}(t) & \mathcal{V}_{23}(t) & \cdots \\ \mathcal{V}_{31}(t) & \mathcal{V}_{32}(t) & \mathcal{V}_{33}(t) & \cdots \\ \mathcal{V}_{41}(t) & \mathcal{V}_{42}(t) & \mathcal{V}_{43}(t) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

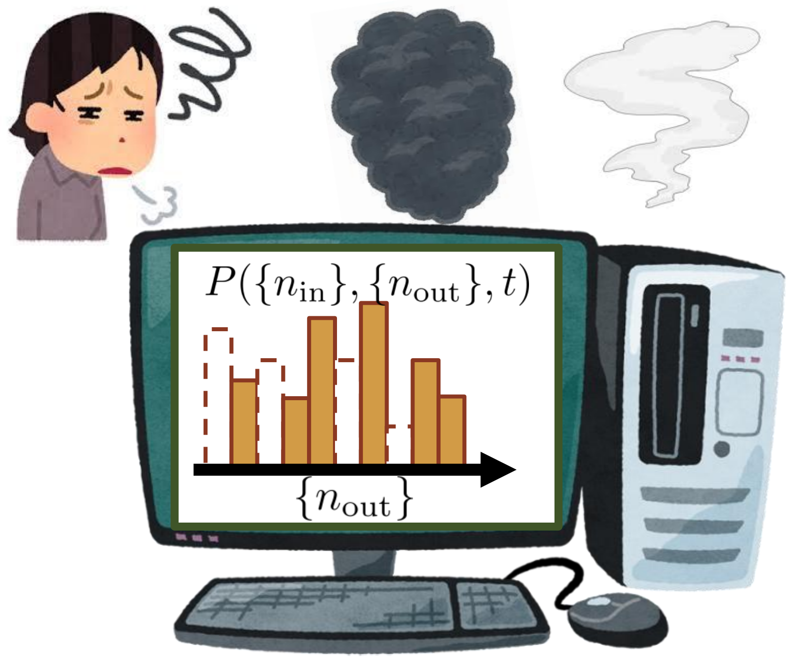
$\{n_{\mathrm{in/out}}\} = (n_1^{\mathrm{in/out}}, n_2^{\mathrm{in/out}}, \cdots)$

$\sum_y n_y^{\mathrm{in/out}} = n$: the number of photons

Computational complexity in various situations

$\mathcal{V}(t)$: random matrix
input : single photon
→ hard

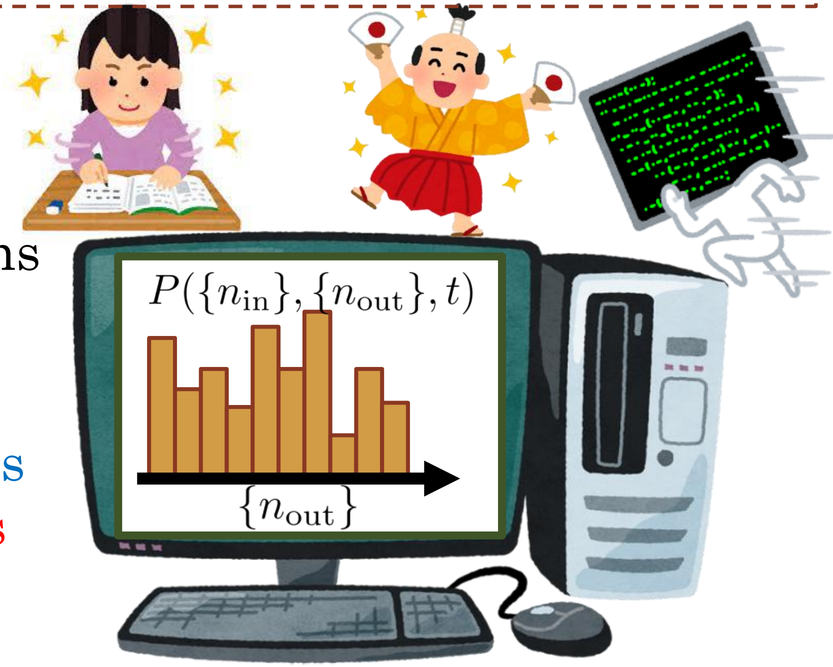
$\mathcal{V}(t)$: positive/shallow
input : coherent/thermal
→ easy



physical situations

⇕

hardness
easiness



S. Aaronson, A. Arkhipov, Theory Comput. **9**, 143 (2013).
M. Jerrum, *et.al.*, Journal of the ACM, **51**, 4, 671–697 (2004).
C. Oh, *et.al.*, Phys. Rev. A **104**, 022407 (2021).
S. R. Keshari, *et al*, Phys. Rev. Lett. **114**, 060501 (2015).

n/Y : the number of photons/states

$$\{n_{\rm in/out}\} = (n_1^{\rm in/out}, \cdots, n_Y^{\rm in/out})$$

$$\sum_{y=1}^{Y} n_y^{\rm in/out} = n$$

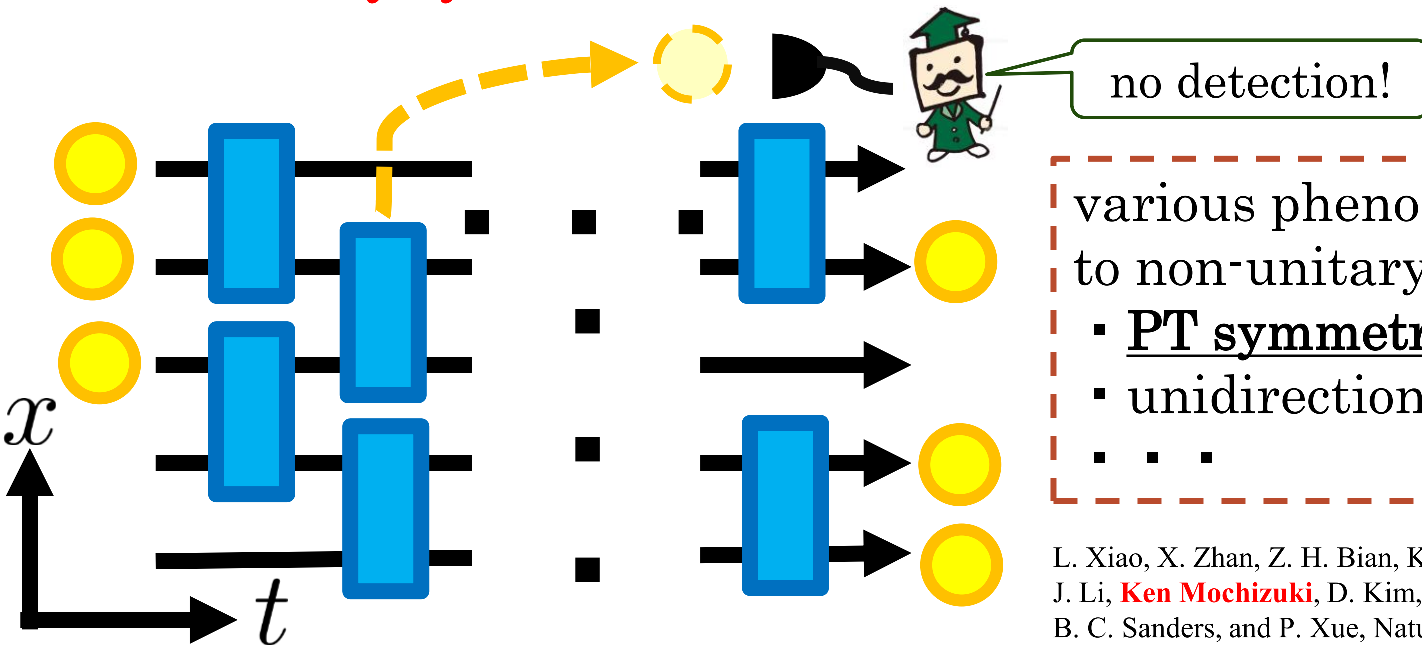
∗ hard/easy ⇔ exponential/polynomial computational time of n

characterization of dynamics/phase by computational complexity

Extension to monitored systems, Objective

photon measurement & no-detection postselection
⇒ non-unitary dynamics with number conservation

$$\mathcal{V}^\dagger(t) \neq \mathcal{V}^{-1}(t)$$



$$\hat{\mathcal{V}}(t)\hat{b}_x^\dagger\hat{\mathcal{V}}^{-1}(t) = \sum_y \mathcal{V}_{yx}(t)\hat{b}_y^\dagger$$

various phenomena unique to non-unitary dynamics:

- **PT symmetry breaking**
- unidirectional propagation
- . . .

L. Xiao, X. Zhan, Z. H. Bian, K. K. Wang, X. Zhang, X. P. Wang, J. Li, **Ken Mochizuki**, D. Kim, N. Kawakami, W. Yi, H. Obuse, B. C. Sanders, and P. Xue, Nature Physics **13**, 1117 (2017).

L. Xiao, T. Deng, K. Wang, G. Zhu, Z. Wang, W. Yi, and P. Xue, Nature Physics **16**, 761 (2020).

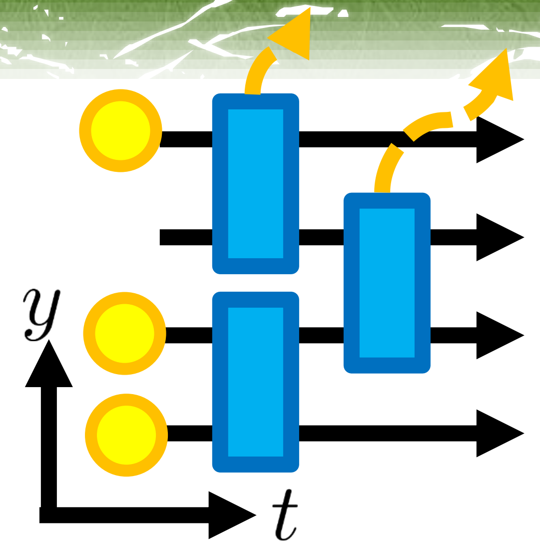
relation between the computational complexity and intriguing phenomena in nonunitary dynamics ??

Parity-Time symmetric Floquet model

$$V = C(\theta_1/2)SG(+\gamma)C(\theta_2)G(-\gamma)SC(\theta_1/2)$$

: unitary evolution

: non-unitary evolution due to measurement



$$G(\gamma) = \sum_{x=1}^{X} |x\rangle\langle x| \begin{pmatrix} e^{+\gamma} & 0 \\ 0 & e^{-\gamma} \end{pmatrix}$$

γ : strength of measurement

experimentally feasible

$$\mathcal{V}(t) = V^t$$

$y = (x, h/v)$

x : position

h/v : polarization

L. Xiao, X. Zhan, Z. H. Bian, K. K. Wang, X. Zhang, X. P. Wang, J. Li, **Ken Mochizuki**, D. Kim, N. Kawakami, W. Yi, H. Obuse, B. C. Sanders, and P. Xue, Nature Physics **13**, 1117 (2017).

Parity-Time (PT) symmetry : $\mathcal{P} : +\gamma \to -\gamma,\ \mathcal{T} : -\gamma \to +\gamma$

$$(\mathcal{PT})V(\mathcal{PT})^{-1} = V^{-1} \neq V^{\dagger}, \quad \mathcal{PT} = \sum_x |-x\rangle\langle x| \otimes \sigma_3 \mathcal{K}$$

Ken Mochizuki, D. Kim, H. Obuse, Physical Review A **93**, 062116 (2016).

$$P(\{n_{\mathrm{in}}\}, \{n_{\mathrm{out}}\}, t) \propto |\mathrm{Per}[\mathcal{V}_{\mathrm{out,in}}(t)]|^2$$

PT symmetry breaking

$\mathcal{PT}\,|\phi_k\rangle = |\phi_k\rangle$

PT symmetric phase

$\mathcal{PT}\,|\phi_k\rangle \neq |\phi_k\rangle$

PT broken phase

γ

weak loss

threshold
$e^{\gamma_{\mathcal{PT}}} \simeq 1.22$

strong loss

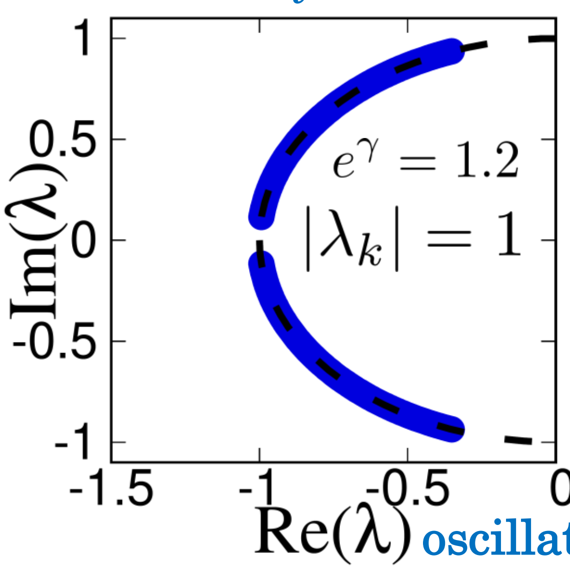
$$(\mathcal{PT})V(\mathcal{PT})^{-1} = V^{-1} \neq V^\dagger$$

$$V\,|\phi_k\rangle = \lambda_k\,|\phi_k\rangle$$

$$V^{-1}\mathcal{PT}\,|\phi_k\rangle = \lambda_k^*\mathcal{PT}\,|\phi_k\rangle$$

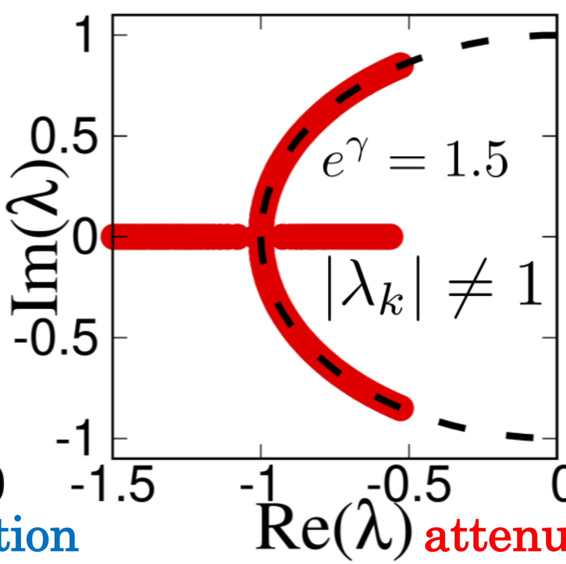
$$|\lambda_1| \geq |\lambda_2| \cdots \geq |\lambda_Y|$$

PT symmetric



oscillation

PT broken



attenuation/divergence

PT symmetry breaking

huge effects on complexity through distinguishability!

Ken Mochizuki and Ryusuke Hamazaki, Physical Review Research **5**, 013177 (2023).

$\theta_1 = 0.65\pi,\ \theta_2 = 0.25\pi$

Absorption to the classical distribution

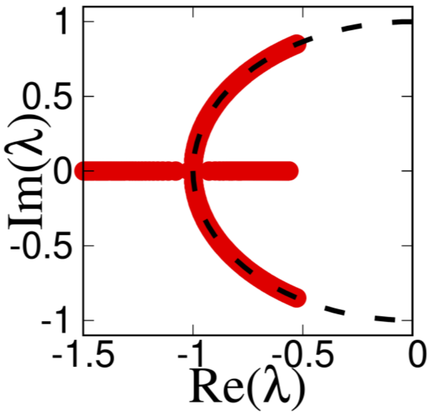
quantum

classical

easy to compute

$$D(t) = \sum_{\{n_{\rm out}\}} |P(\{n_{\rm in}\},\{n_{\rm out}\},t) - P_{\rm d}(\{n_{\rm in}\},\{n_{\rm out}\},t)|$$

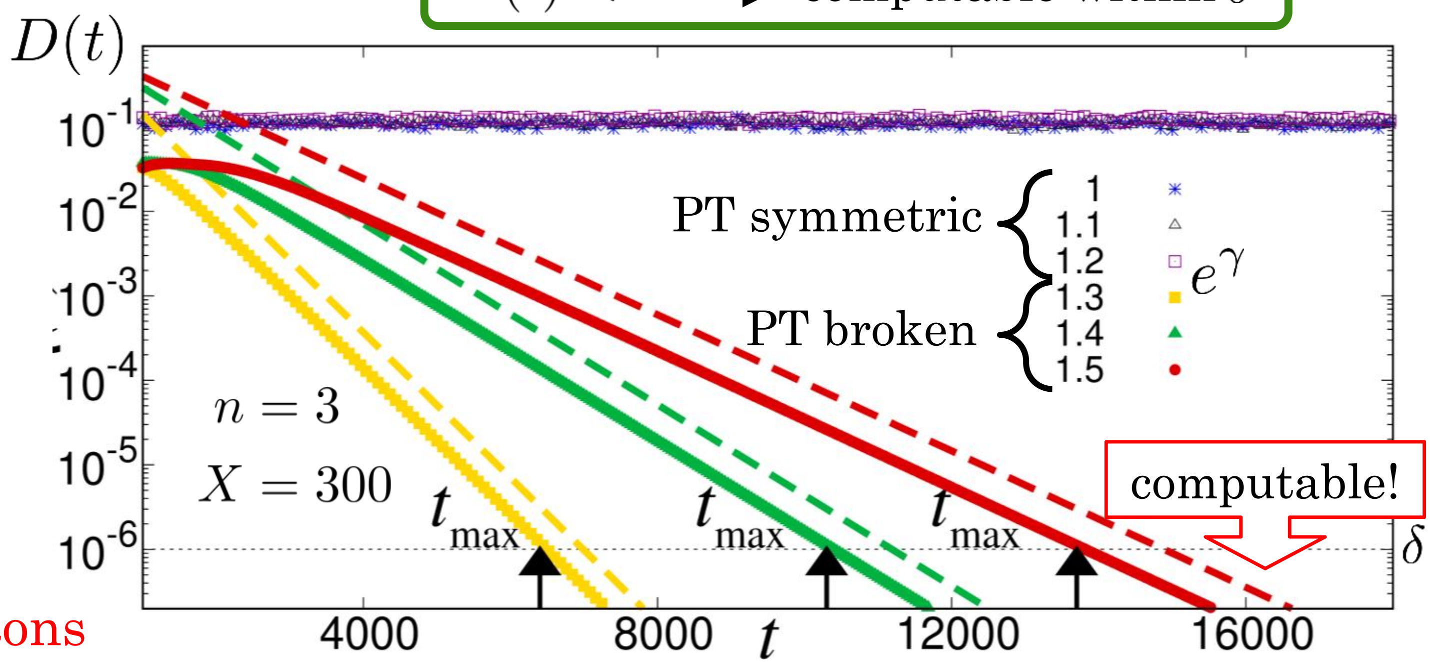
PT broken



$|\lambda_k| \neq 1$

bunching of photons

$D(t) < \delta$ → computable within δ



PT symmetry breaking ⇒ computable distribution in the long run

Ken Mochizuki and Ryusuke Hamazaki, Physical Review Research **5**, 013177 (2023).

Bunching state in the PT-broken phase

$$V\,|\phi_l^{\mathrm{R}}\rangle = \lambda_l\,|\phi_l^{\mathrm{R}}\rangle\,,\ \langle\phi_l^{\mathrm{L}}|\,V = \lambda_l\,\langle\phi_l^{\mathrm{L}}|$$

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_Y|$$

$$|\lambda_1|^t \gg |\lambda_2|^t$$

$$\boxed{|\Psi_n(t)\rangle \simeq \left(\tilde{b}_1^\dagger\right)^n |0\rangle\,/\sqrt{n!}}$$

$$\tilde{b}_j^\dagger = \sum_y \langle y|\phi_j^{\mathrm{R}}\rangle \hat{b}_y^\dagger$$

entanglement entropy

$$S_{L/2}(n) \propto \log(n)$$

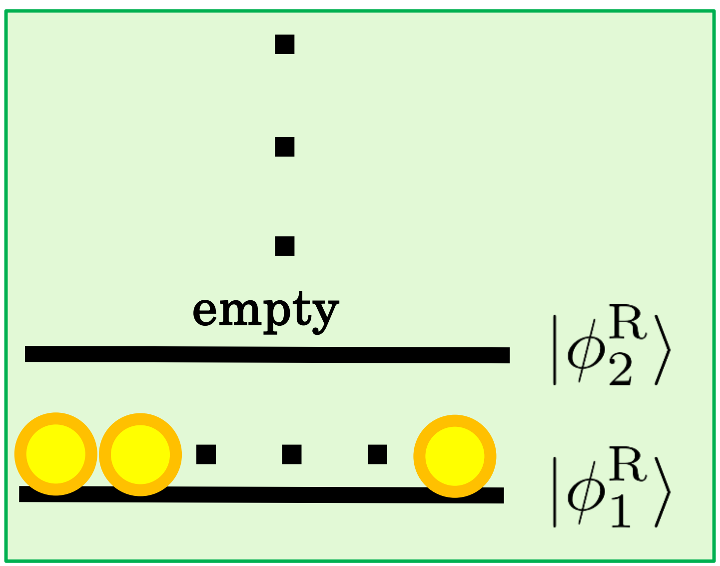
D. Kagamihara, R. Kaneko, K. Mochizuki, and I. Danshita, in preparation.

$$S_{L/2}(n) = -\log\left(\mathrm{tr}_A\left[\rho_A^2(n)\right]\right)$$

$$\rho_A(n) = \mathrm{tr}_B\left[|\Psi_n(t)\rangle\langle\Psi_n(t)|\right]$$

A, B : half chain

ubiquitous in non-unitary dynamics!



$$V^t = \sum_j \lambda_j^t\,|\phi_j^{\mathrm{R}}\rangle\langle\phi_j^{\mathrm{L}}| \simeq \lambda_1^t\,|\phi_1^{\mathrm{R}}\rangle\langle\phi_1^{\mathrm{L}}|$$

$$P(\{n_{\mathrm{in}}\},\{n_{\mathrm{out}}\},t) \simeq \frac{\prod_{p=1}^{n} |\langle y_p^{\mathrm{out}}|\phi_1^{\mathrm{R}}\rangle|^2}{N\prod_{y=1}^{Y} n_y^{\mathrm{out}}!}$$

$$\simeq P_{\mathrm{d}}(\{n_{\mathrm{in}}\},\{n_{\mathrm{out}}\},t)$$

$$\boxed{D(t) \simeq 0}$$

computable distribution!

Ken Mochizuki and Ryusuke Hamazaki, Physical Review Research **5**, 013177 (2023).

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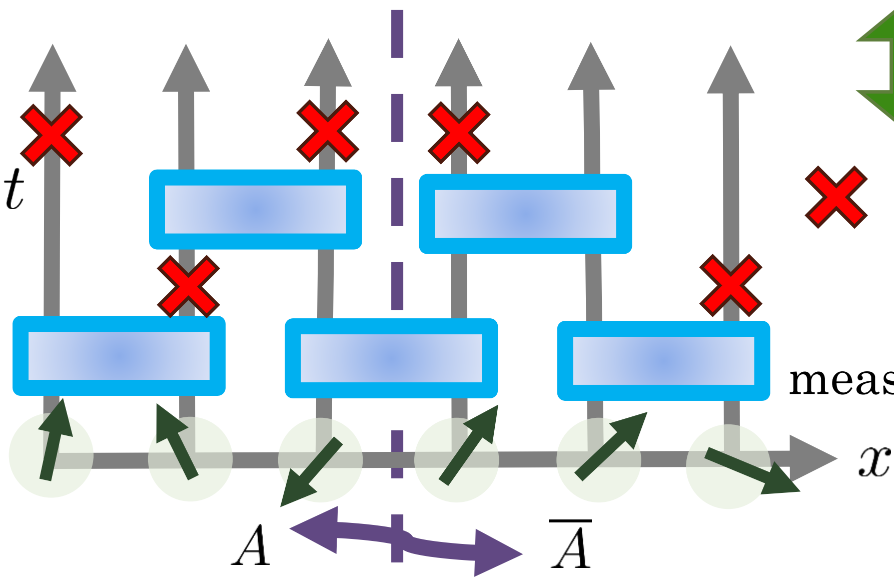
Monitored many-body spin systems

quantum gate $|\psi\rangle \to \hat{\mathsf{U}}_{x,x+1}|\psi\rangle$

competition ⇒ transition!

measurement on $\hat{Z}_x$ with probability p $\quad |\psi\rangle \to \dfrac{\hat{\mathsf{M}}(\omega_x)|\psi\rangle}{\sqrt{q_\psi(\omega_x)}}$

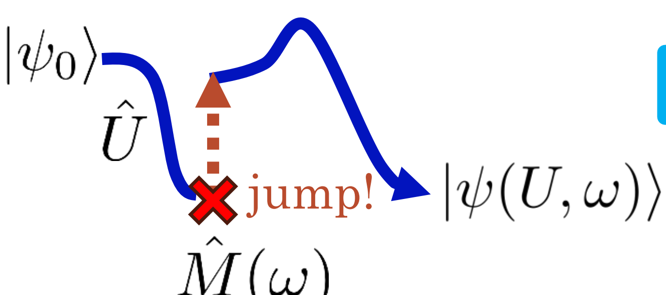


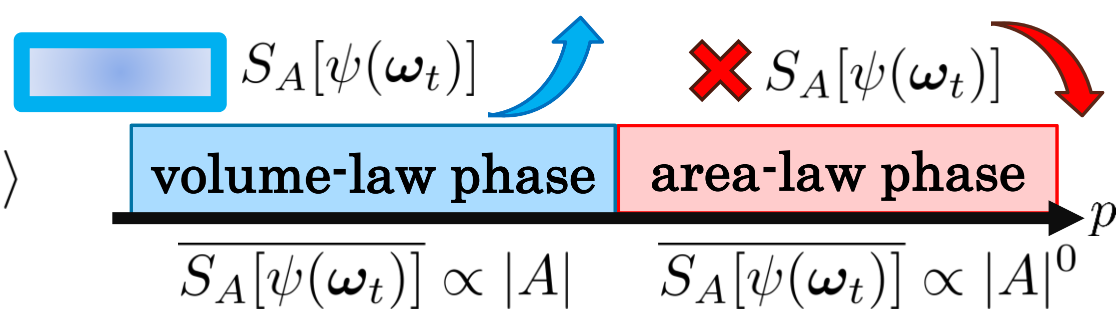
measurement operator: $\hat{\mathsf{M}}(\omega_x = \pm) = \dfrac{\hat{I} \pm \hat{Z}_x}{2}$

$\omega_x = \pm$: random measurement outcome

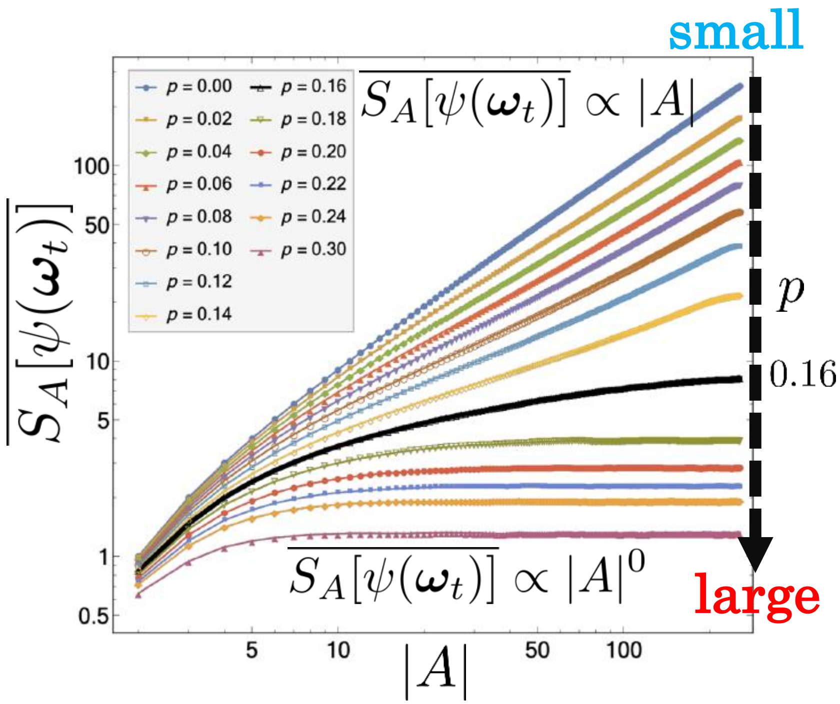
Born probability: $q_\psi(\omega_x) = \langle\psi|\hat{\mathsf{M}}^\dagger(\omega_x)\hat{\mathsf{M}}(\omega_x)|\psi\rangle$

1/2 spin





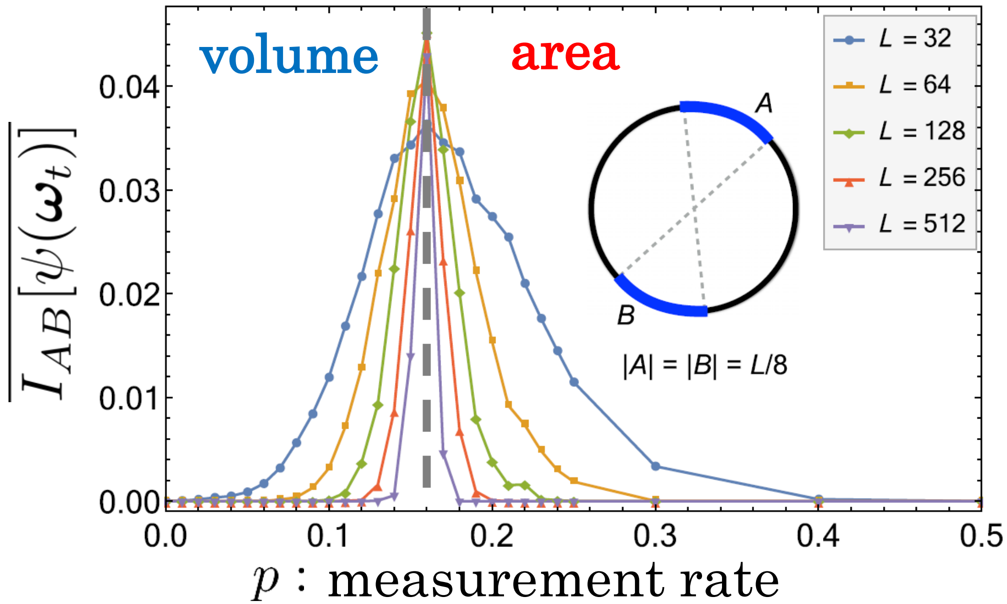
Measurement-induced entanglement transition



Y. Li et.al., PRB **100**, 134306 (2019).

$$I_{AB}(\psi) = S_A(\psi) + S_B(\psi) - S_{AB}(\psi)$$

$$\frac{|\langle O_A O_B \rangle_\psi - \langle O_A \rangle_\psi \langle O_B \rangle_\psi|^2}{2|O_A|^2|O_B|^2} \leq I_{AB}(\psi)$$

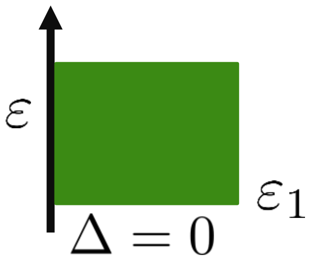


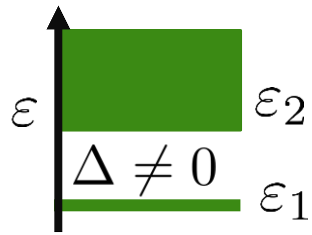
volume-area entanglement transition ⟺ peak of $\overline{I_{AB}[\psi(\boldsymbol{\omega}_t)]}$

Entanglement transition in isolated systems

the spectral gap $\Delta = \varepsilon_2 - \varepsilon_1, \quad \varepsilon_i \leq \varepsilon_{i+1}$

transition of the gap ⟷ **ground-state phase transition**





$$\hat{H} \left| \Psi_i \right\rangle = \varepsilon_i \left| \Psi_i \right\rangle$$

gapless phase logarithmic-law	gapped phase area-law

parameter J

$S_A(\Psi_1) \propto \log(|A|)$

$S_A(\Psi_1) \propto |A|^0$

e.g. XXZ chain

$$\hat{H} = \sum_x \hat{X}_x \hat{X}_{x+1} + \hat{Y}_x \hat{Y}_{x+1} + J \hat{Z}_x \hat{Z}_{x+1}$$

entanglement entropy

$$S_A(\psi) = -\mathrm{tr}_A \left(\rho_A \left(\psi \right) \log \left[\rho_A \left(\psi \right) \right] \right)$$

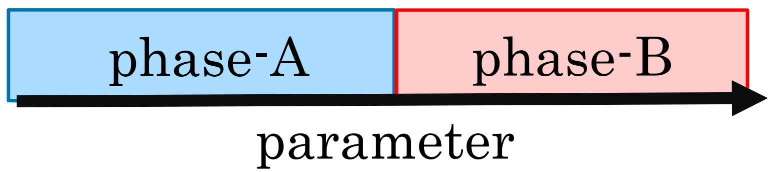
c.f. spectral gap ⇒ classification of quantum phases in isolated systems
Bei Zeng et.al. “Quantum Information Meets Quantum Mattes” (Springer)

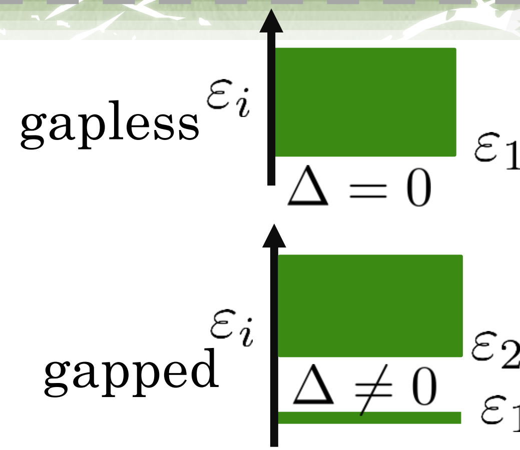
Comparison of isolated and monitored systems

isolated systems:

$$\hat{H}\,|\Psi_i\rangle = \varepsilon_i\,|\Psi_i\rangle$$

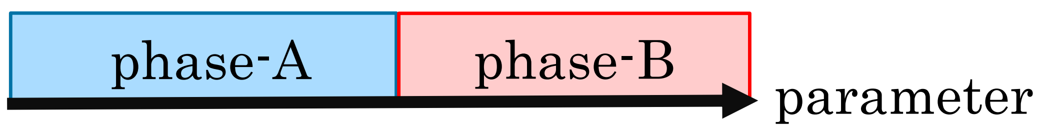
- ✅ entanglement scaling
- ✅ spectrum
- ✅ topological invariant
- …





monitored systems:

- ✅ entanglement scaling
- ? spectrum
- ? topological invariant



entanglement transition ⟷ spectrum/topology ??

🚫 **Conventional spectral analysis seems difficult**

temporal randomness due to measurement ❌

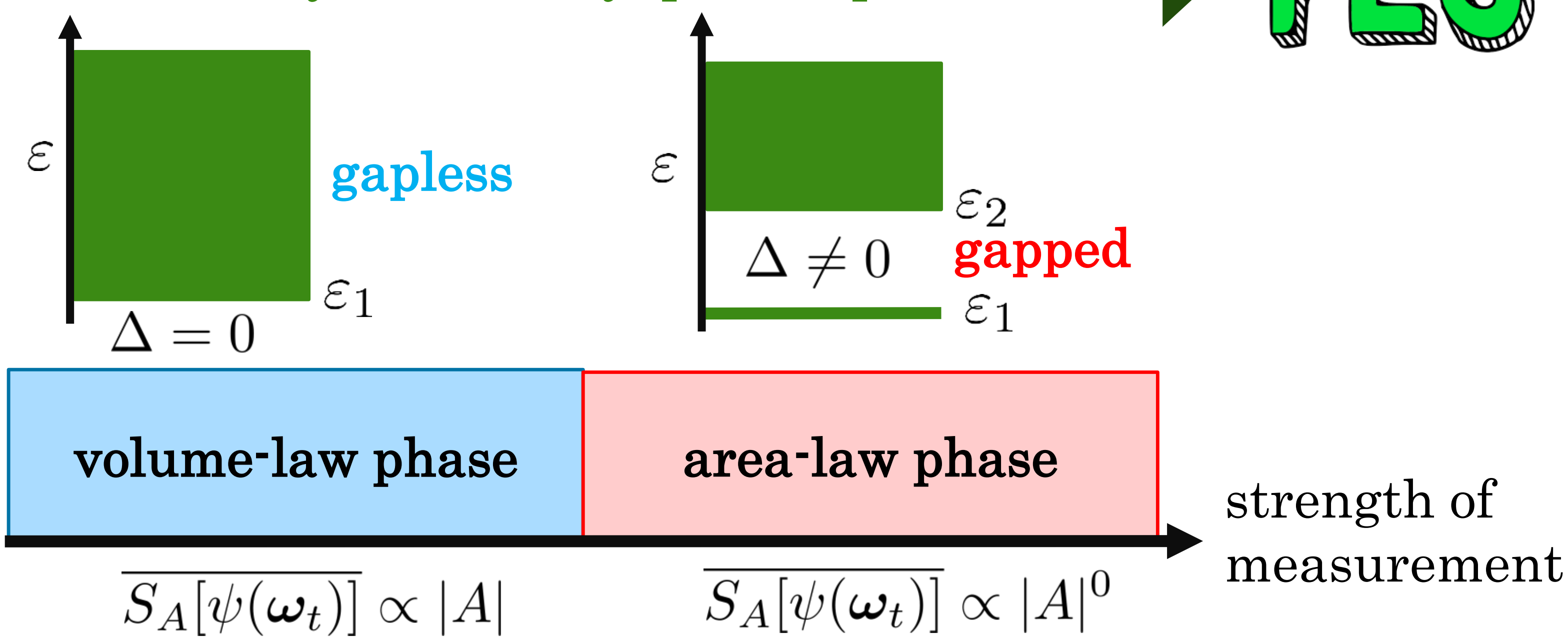
➡ **no static generator** $|\psi(\boldsymbol{\omega}_t)\rangle \neq e^{-i\hat{H}t}\,|\psi_0\rangle$

Objective, results

Ken Mochizuki and Ryusuke Hamazaki
Phys. Rev. Lett. **134**, 010410 (2025).

Measurement-induced entanglement transitions are related to some spectral/topological features??

analysis of the Lyapunov spectrum →



measurement-induced topological transition
Oshima, Mochizuki, Hamazaki, Fuji, PRL **134**, 240401 (2025).

Model: interacting hybrid circuit

Ken Mochizuki
Ryusuke Hamazaki
PRL **134**, 010410 (2025).

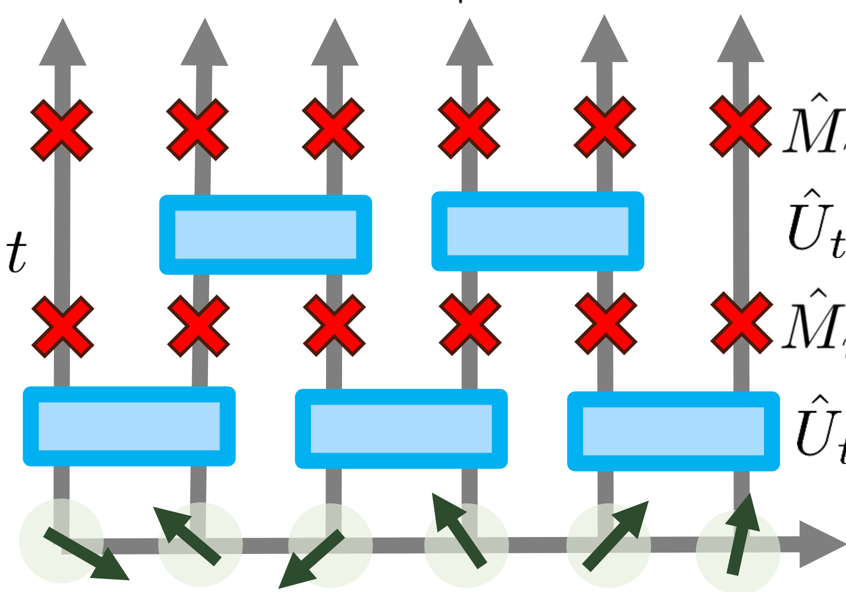
random nonunitary dynamics:
many trajectories conditioned on measurement outcomes

$$\boldsymbol{\omega}_t = (U_1, \omega_1, \cdots, U_t, \omega_t)$$

$$\boxed{|\psi(\boldsymbol{\omega}_t)\rangle \propto \hat{M}_\eta(\omega_t)\hat{U}_t \cdots \hat{M}_\eta(\omega_1)\hat{U}_1 |\psi_0\rangle}$$

Born probability:

$$P(\omega_t|\boldsymbol{\omega}_{t-1}) = \left|\hat{M}_\eta(\omega_t)\hat{U}_t |\psi(\boldsymbol{\omega}_{t-1})\rangle\right|^2$$



$\hat{M}_\eta(\omega_{t=2})$

$\hat{U}_{t=2}$

$\hat{M}_\eta(\omega_{t=1})$

$\hat{U}_{t=1}$

$x = 1, 2, \cdots, L$

generalized measurement

$$\hat{\mathsf{M}}_\eta(\omega_{t,x} = \pm) \propto \hat{I} \pm \eta \hat{Z}_x$$

η : strength of measurement

$\omega_{t,x} = \pm$: measurement outcomes

$$\hat{M}_\eta(\omega_t) = \prod_x \hat{\mathsf{M}}_\eta(\omega_{t,x})$$

$$\omega_t = (\omega_{t,x=1}, \cdots, \omega_{t,x=L})$$

quantum gate

$$\hat{U}_t = \prod_x \hat{\mathsf{U}}_t^{x,x+1}$$

Lyapunov analysis

Ken Mochizuki
Ryusuke Hamazaki
PRL **134**, 010410 (2025).

$$|\psi(\boldsymbol{\omega}_t)\rangle \propto \hat{V}(\boldsymbol{\omega}_t)\,|\psi_0\rangle\,, \quad \hat{V}(\boldsymbol{\omega}_t) = \hat{M}_\eta(\omega_t)\hat{U}_t\cdots\hat{M}_\eta(\omega_1)\hat{U}_1$$

effective "Hamiltonian": $\hat{H}(\boldsymbol{\omega}_t) = -\dfrac{1}{2t}\log\left[\hat{V}(\boldsymbol{\omega}_t)\hat{V}^\dagger(\boldsymbol{\omega}_t)\right]$

intuitive picture

imaginary-time evolution: $|\psi(\boldsymbol{\omega}_t)\rangle \sim \exp[-\hat{H}(\boldsymbol{\omega}_t)t]\,|\psi_0\rangle$

c.f. $\hat{U}(t) = \exp(-i\hat{H}t) \to \hat{H} = i\log[\hat{U}(t)]/t$

$$\hat{H}(\boldsymbol{\omega}_t)\,|\Psi_i(\boldsymbol{\omega}_t)\rangle = \varepsilon_i(\boldsymbol{\omega}_t)\,|\Psi_i(\boldsymbol{\omega}_t)\rangle\,, \quad \varepsilon_i(\boldsymbol{\omega}_t) \le \varepsilon_{i+1}(\boldsymbol{\omega}_t)$$

spectral gap : $\Delta = \varepsilon_2 - \varepsilon_1, \quad \varepsilon_i = \lim_{t\to\infty}\varepsilon_i(\boldsymbol{\omega}_t)$

ground state : $|\Psi_1(\boldsymbol{\omega}_t)\rangle$

analogy to ground-state transitions in isolated quantum systems

$$|\Phi_i(\boldsymbol{\omega}_t)\rangle \propto \hat{V}^\dagger(\boldsymbol{\omega}_t)\,|\Psi_i(\boldsymbol{\omega}_t)\rangle$$

$$\hat{V}(\boldsymbol{\omega}_t) \simeq \sum_i \exp(-\varepsilon_i t)\,|\Psi_i(\boldsymbol{\omega}_t)\rangle\langle\Phi_i(\boldsymbol{\omega}_t)| \simeq |\Psi_1(\boldsymbol{\omega}_t)\rangle\langle\Phi_1(\boldsymbol{\omega}_t)|$$

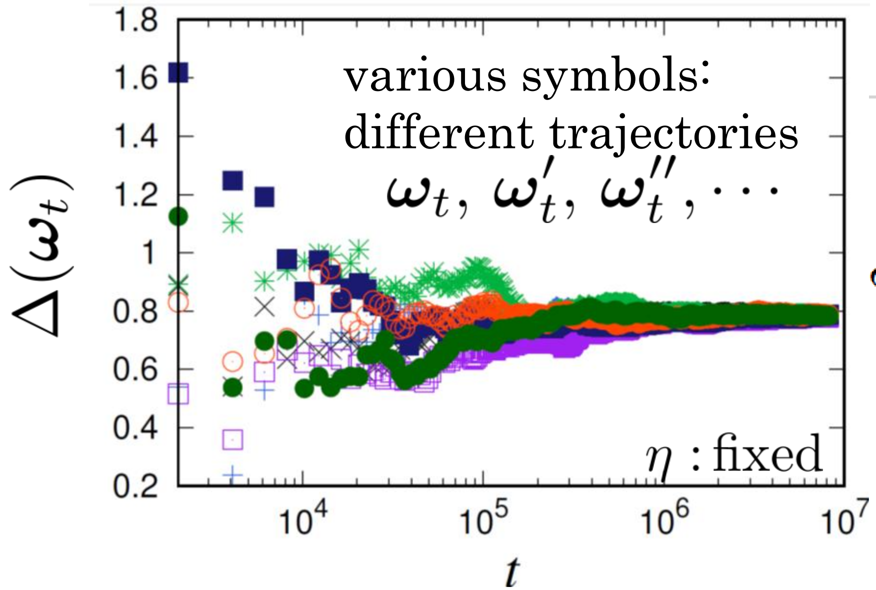
absorption to the ground state: $t \gg 1/\Delta \to |\psi(\boldsymbol{\omega}_t)\rangle \simeq |\Psi_1(\boldsymbol{\omega}_t)\rangle$

Spectral gap

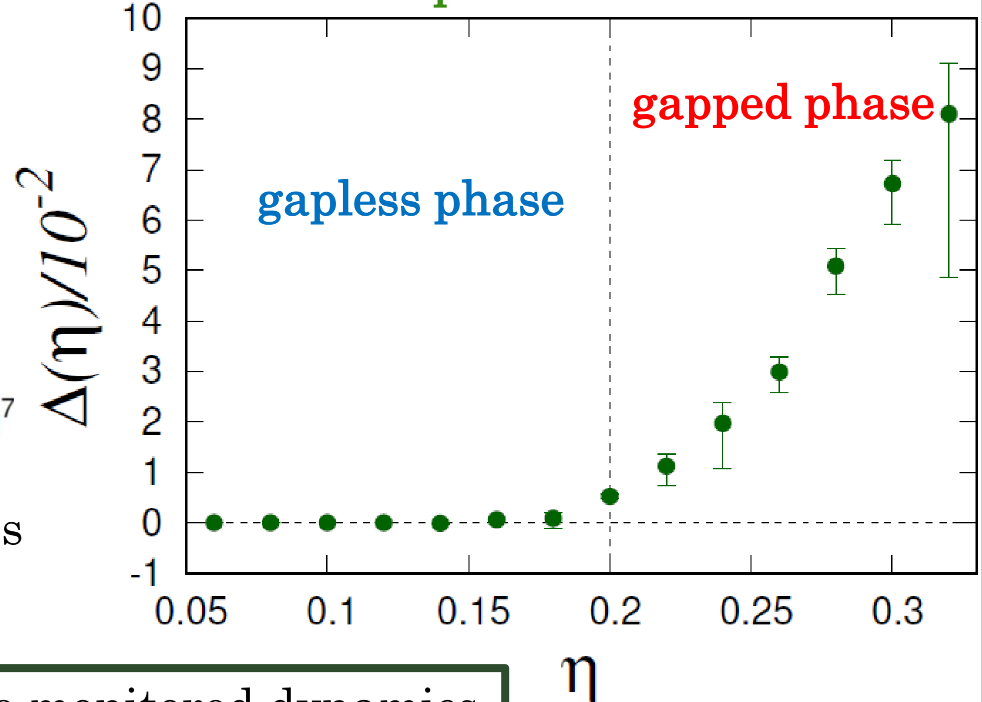
Ken Mochizuki
Ryusuke Hamazaki
PRL **134**, 010410 (2025).

$$\hat{H}(\boldsymbol{\omega}_t)\,|\Psi_i(\boldsymbol{\omega}_t)\rangle = \varepsilon_i(\boldsymbol{\omega}_t)\,|\Psi_i(\boldsymbol{\omega}_t)\rangle\,, \quad \varepsilon_i(\boldsymbol{\omega}_t) \leq \varepsilon_{i+1}(\boldsymbol{\omega}_t)$$

spectral gap: $\Delta = \lim_{t\to\infty} \Delta(\boldsymbol{\omega}_t),\ \Delta(\boldsymbol{\omega}_t) = \varepsilon_2(\boldsymbol{\omega}_t) - \varepsilon_1(\boldsymbol{\omega}_t)$



η : measurement strength
spectral transition!

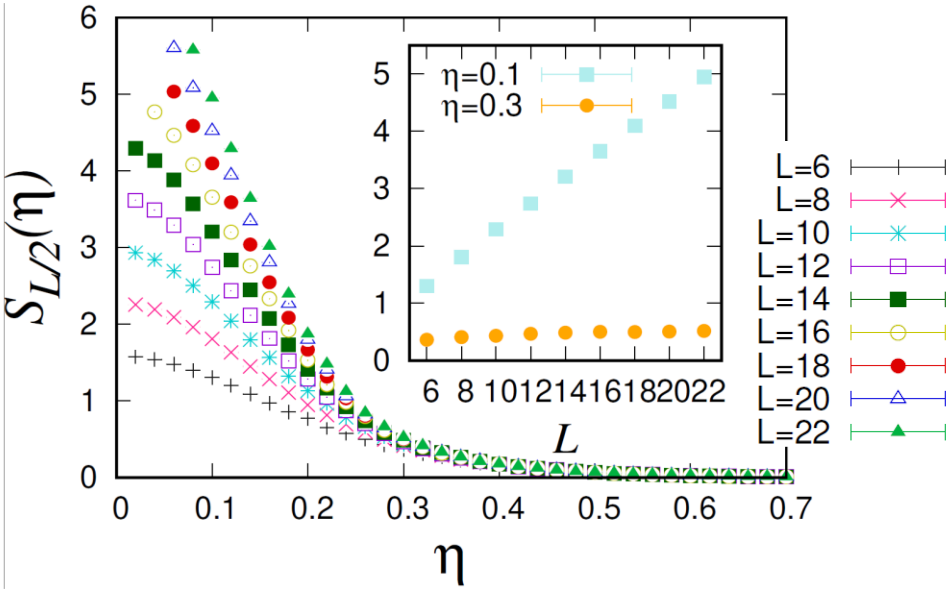


Δ : independent of trajectories

global quantity!

owing to ergodicity/irreducibility of the monitored dynamics
Hamazaki, Mochizuki, Oshima, Fuji, PTEP 052A01 (2026).

Ground-state entanglement entropy



$$S_{L/2} = \overline{\left(S_{L/2}\left[\Psi_1(\boldsymbol{\omega}_t)\right]\right)}$$

$$S_A(\psi) = -\mathrm{tr}_A\left(\rho_A(\psi)\log\left[\rho_A(\psi)\right]\right)$$

$$\rho_A(\psi) = \mathrm{tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$$

small η : volume law $S_{L/2}(\eta) \propto L$

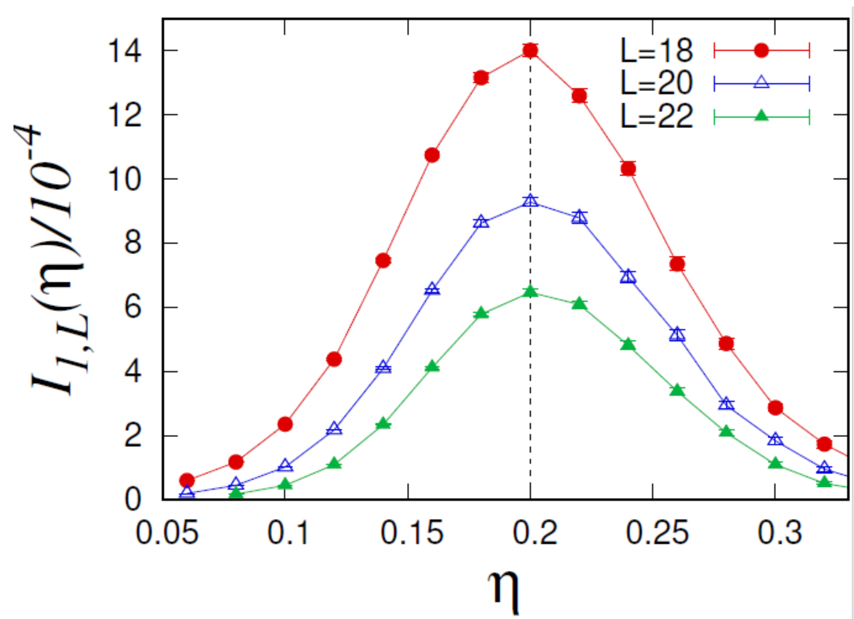
large η : area law $S_{L/2}(\eta) \propto L^0$

K. Mochizuki and R. Hamazaki
Phys. Rev. Lett. **134**, 010410 (2025).

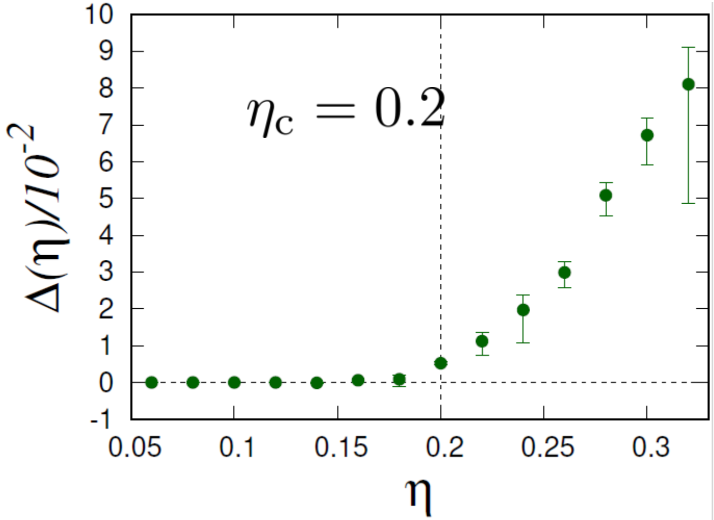
$$I_{AB}(\psi) = S_A(\psi) + S_B(\psi) - S_{AB}(\psi)$$

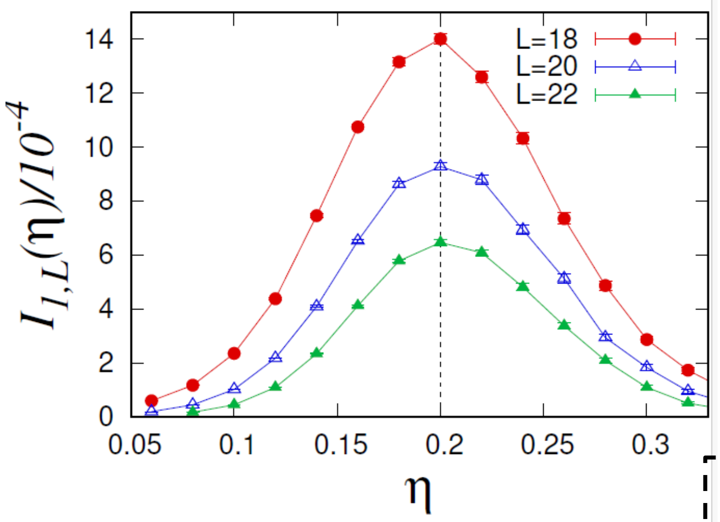
peak of $I_{1,L} = \overline{I_{1,L}\left[\Psi_1(\boldsymbol{\omega}_t)\right]}$

phase transition!



Coincidence of the thresholds





Ken Mochizuki, Ryusuke Hamazaki,
Physical Review Letters **134**, 010410 (2025).

$$\hat{H}(\boldsymbol{\omega}_t)\,|\Psi_i(\boldsymbol{\omega}_t)\rangle = \varepsilon_i(\boldsymbol{\omega}_t)\,|\Psi_i(\boldsymbol{\omega}_t)\rangle$$

gapless phase = volume-law phase

$\Delta(\eta) = 0 \qquad S_{L/2}(\eta) \propto L$

gapped phase = area-law phase

$\Delta(\eta) \neq 0 \qquad S_{L/2}(\eta) \propto L^0$

analogous to isolated quantum systems

concepts established in isolated systems can be extended to monitored systems

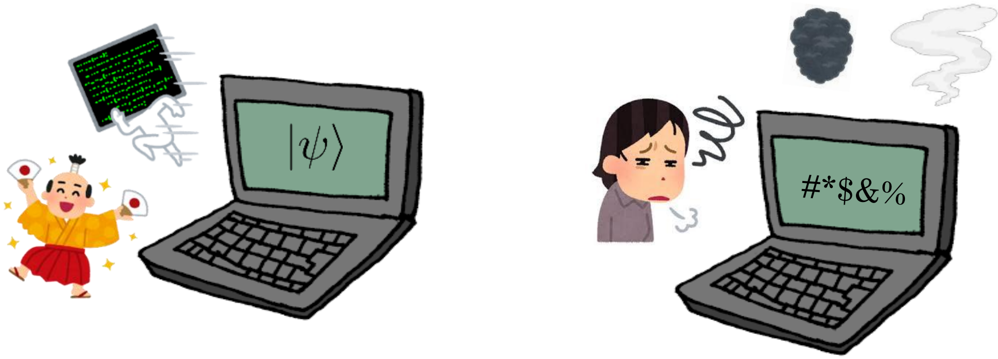
c.f. measurement-induced topological transition
Oshima, Mochizuki, Hamazaki, Fuji, PRL **134**, 240401 (2025).

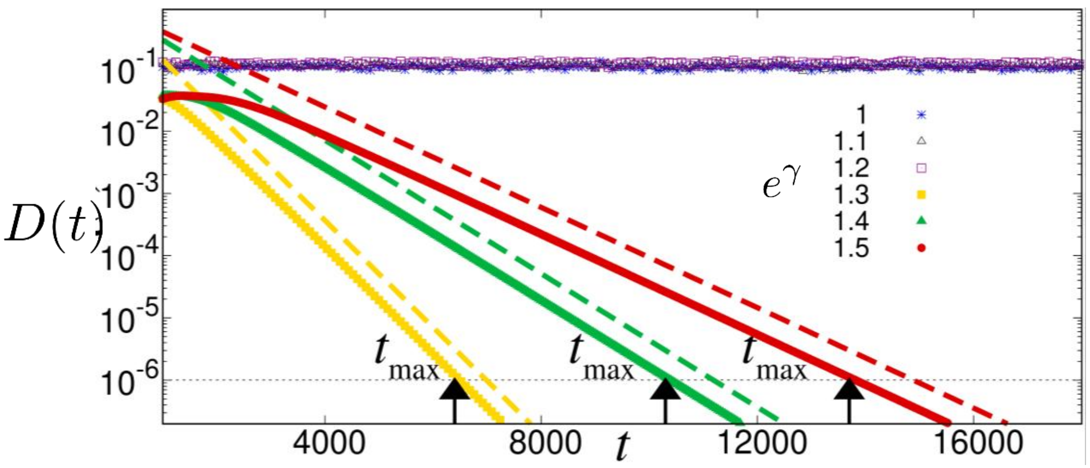
Outline

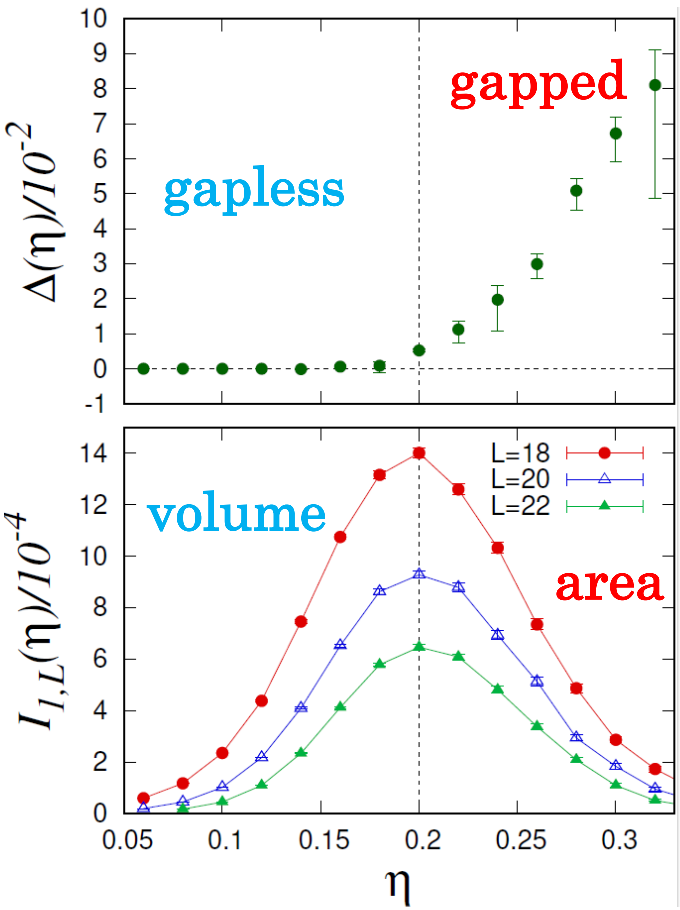
- Introduction
- Parity-Time symmetry breaking spectral transition and the boson sampling complexity
- Measurement-induced spectral transition and entanglement entropy
- Summary

Summary

Quantum measurements ⇒ transitions related to spectrum and complexity







Physical Review Research **5**, 013177 (2023).
Physical Review Letters **134**, 010410 (2025).

Parity-Time symmetric model

single-particle dynamics：

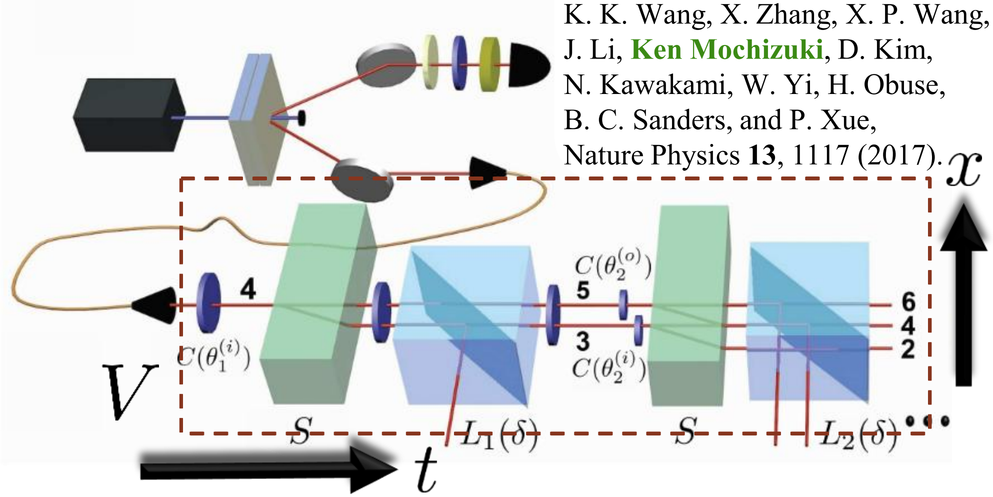
$$|\psi(t+1)\rangle = V\,|\psi(t)\rangle$$

$$|\psi(t)\rangle = \sum_{x=1}^{X} \begin{pmatrix} \psi_h(x,t) \\ \psi_v(x,t) \end{pmatrix} |x\rangle$$

$$\mathcal{V}(t) = V^t$$

$$V = C(\theta_1/2)SG(+\gamma)C(\theta_2)G(-\gamma)SC(\theta_1/2)$$

L. Xiao, X. Zhan, Z. H. Bian, K. K. Wang, X. Zhang, X. P. Wang, J. Li, **Ken Mochizuki**, D. Kim, N. Kawakami, W. Yi, H. Obuse, B. C. Sanders, and P. Xue, Nature Physics **13**, 1117 (2017).



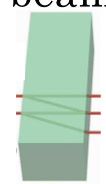
$$P(\{n_{\rm in}\},\{n_{\rm out}\},t) \propto |\mathrm{Per}[\mathcal{V}_{\rm out,in}(t)]|^2$$

$$\mathrm{Per}[\mathcal{V}(t)] = \textstyle\sum_{\omega}\prod_{y}\mathcal{V}_{\omega(y)y}(t)$$

wave plates：changes of polarizations

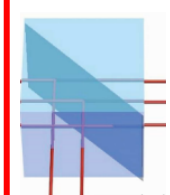
$$C(\theta) = \sum_{x=1}^{X} |x\rangle\langle x| \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

beam displacers：position shift of photons



$$S = \sum_{x=1}^{X} \begin{pmatrix} |x-1\rangle\langle x| & 0 \\ 0 & |x+1\rangle\langle x| \end{pmatrix}$$

partially polarizing beam splitters：
polarization dependent photon losses
⇒non-unitary dynamics $V^\dagger \neq V^{-1}$



$$G(\gamma) = \sum_{x=1}^{X} |x\rangle\langle x| \begin{pmatrix} e^{+\gamma} & 0 \\ 0 & e^{-\gamma} \end{pmatrix}$$

Parity-Time (PT) symmetry：

$$(\mathcal{PT})V(\mathcal{PT})^{-1} = V^{-1}, \quad \mathcal{PT} = \sum_{x} |-x\rangle\langle x| \otimes \sigma_3\mathcal{K}$$

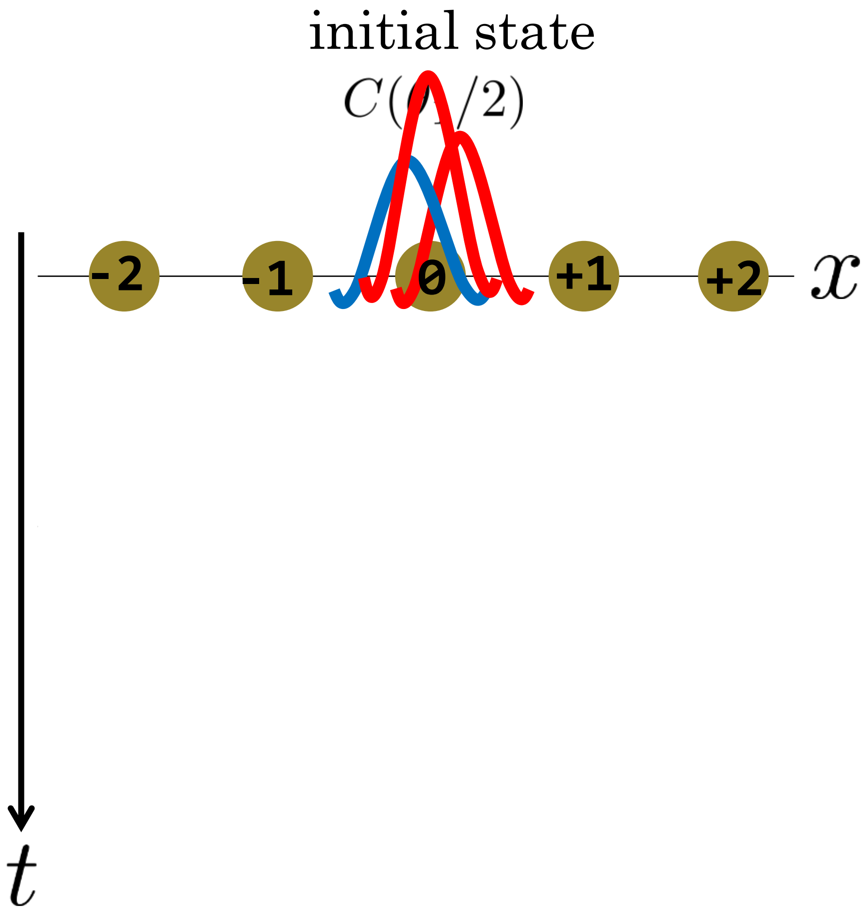
$$y = (x, h/v)$$

Single-particle dynamics

Ken Mochizuki and Ryusuke Hamazaki, Physical Review Research **5**, 013177 (2023).

$$|\psi(t+1)\rangle = V\,|\psi(t)\rangle$$

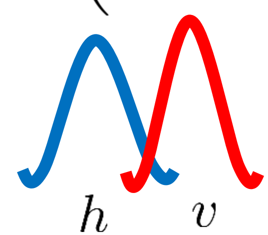
$$V = C(\theta_1/2)SG(+\gamma)C(\theta_2)G(-\gamma)SC(\theta_1/2)$$



$$C(\theta) = \sum_{x=1}^{X} |x\rangle\langle x| \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$S = \sum_{x=1}^{X} \begin{pmatrix} |x-1\rangle\langle x| & 0 \\ 0 & |x+1\rangle\langle x| \end{pmatrix}$$

$$G(\gamma) = \sum_{x=1}^{X} |x\rangle\langle x| \begin{pmatrix} e^{+\gamma} & 0 \\ 0 & e^{-\gamma} \end{pmatrix}$$



$$|\psi(t)\rangle = \sum_{x=1}^{X} \begin{pmatrix} \psi_h(x,t) \\ \psi_v(x,t) \end{pmatrix} |x\rangle$$

Behavior in the short-time regime

Ken Mochizuki and Ryusuke Hamazaki, Physical Review Research **5**, 013177 (2023).

actual distribution of many photons

probability distribution of distinguishable particles

easy to compute

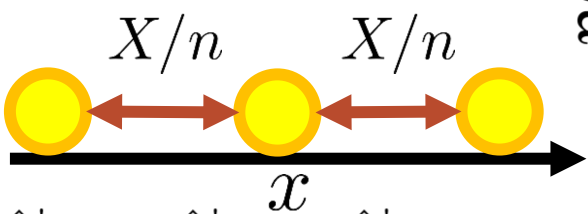
$$\eta(t) = \sum_{\{n_{\mathrm{out}}\}} |P(\{n_{\mathrm{in}}\}, \{n_{\mathrm{out}}\}, t) - P_{\mathrm{d}}(\{n_{\mathrm{in}}\}, \{n_{\mathrm{out}}\}, t)|$$

$\eta(t) < \delta$

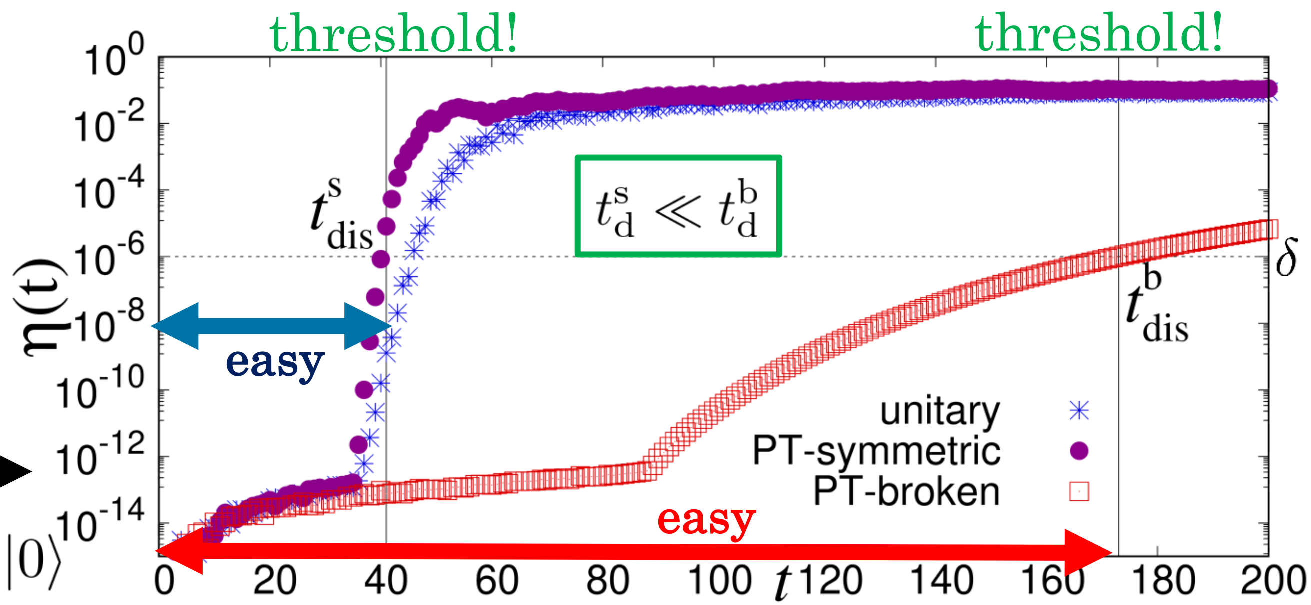
computable within δ

$n = 3, X = 300$

initial state：



$\hat{b}^\dagger_{x=\frac{X}{6},h} \hat{b}^\dagger_{x=\frac{X}{2},v} \hat{b}^\dagger_{x=\frac{5}{6}X,v} |0\rangle$



$t < t^{\mathrm{s/b}}_{\mathrm{dis}}$: neglectable overlap of single-particle distributions ⇒ distinguishable

$t \geq t^{\mathrm{s/b}}_{\mathrm{dis}}$: deviation from the computable distribution due to quantum interference

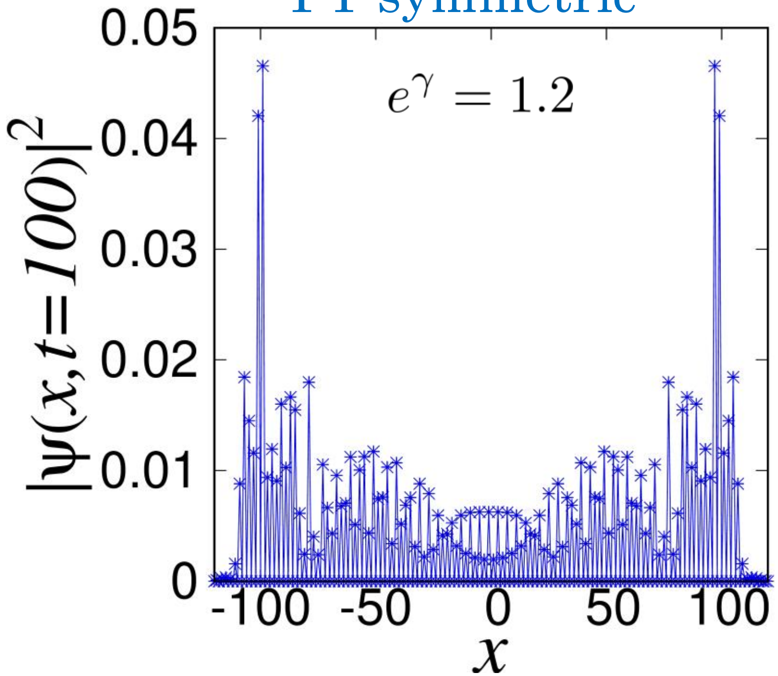
PT symmetry breaking enlarges the computable region

Origin of the enlargement for the computable region

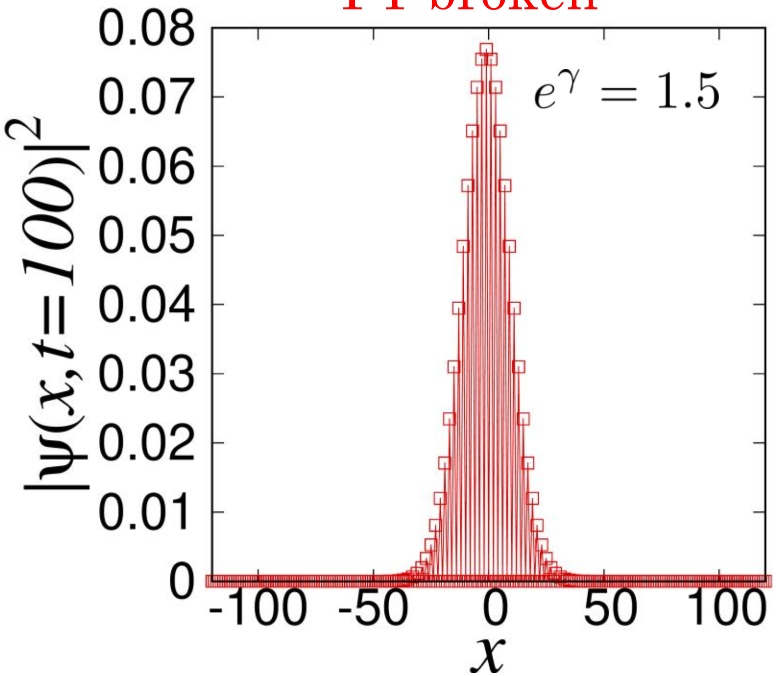
Ken Mochizuki and Ryusuke Hamazaki, Physical Review Research 5, 013177 (2023).

single-particle dynamics with $|\psi(t=0)\rangle = |x=0\rangle \otimes (|h\rangle + i\,|v\rangle)/\sqrt{2}$

PT symmetric



PT broken



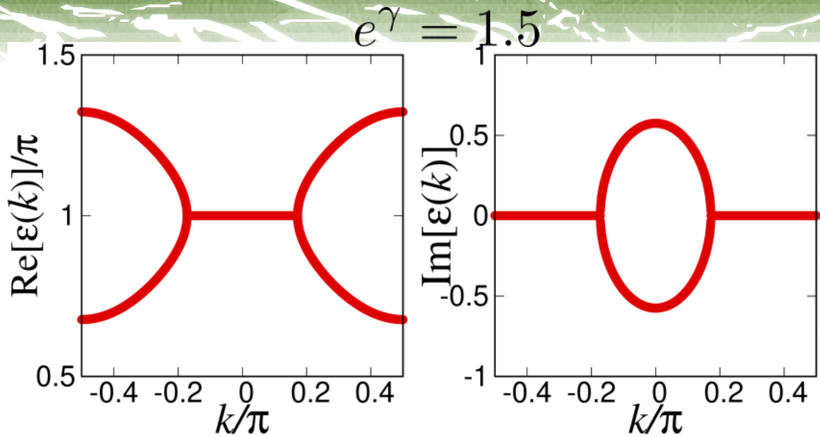
ballistic $\sqrt{\langle x^2(t)\rangle - \langle x(t)\rangle^2} \propto t$

diffusive $\sqrt{\langle x^2(t)\rangle - \langle x(t)\rangle^2} \propto \sqrt{t}$

PT symmetry breaking leads to diffusive dynamics
$\Rightarrow$ prolongation of threshold times $t^{\mathrm{s}}_{\mathrm{dis}} \ll t^{\mathrm{b}}_{\mathrm{dis}}$

Diffusive dynamics can be derived analytically.

Derivation of the diffusive dynamics



$$V^t(k) = \sum_s e^{-i\varepsilon_s(k)t} |\phi_s^{\mathrm{R}}(k)\rangle \langle \phi_s^{\mathrm{L}}(k)|$$

$$\varepsilon_\pm(k) \simeq \varepsilon_\pm(k=0) \pm i\frac{D}{2}k^2,\ D > 0$$

quadratic dispersion

$$|\psi(t=0)\rangle = |x=0\rangle |\sigma_0\rangle$$

$$N(t) = \int_{-\pi}^{\pi} dk \langle \sigma_0| [V^\dagger(k)]^t V^t(k) |\sigma_0\rangle$$

$$\langle x^r(t)\rangle = \frac{\sum_x x^r |\psi(x,t)|^2}{\sum_x |\psi(x,t)|^2} = \frac{(-i)^r}{N(t)} \int_{-\pi}^{\pi} dk \langle \sigma_0| [V^\dagger(k)]^t \frac{d^r}{dk^r} V^t(k) |\sigma_0\rangle$$

extraction of leading terms

$$\simeq \frac{(-i)^r \int_{-\infty}^{+\infty} dk e^{-\frac{Dk^2}{2}t} \frac{d^r}{dk^r} e^{-\frac{Dk^2}{2}t}}{\int_{-\infty}^{+\infty} dk e^{-Dk^2 t}} = \frac{i^r \int_{-\infty}^{+\infty} dk e^{-Dk^2 t} \left(\frac{Dt}{2}\right)^{\frac{r}{2}} H_r\left(\sqrt{\frac{Dt}{2}}k\right)}{\int_{-\infty}^{+\infty} dk e^{-Dk^2 t}}$$

extension of the integral range

$$\langle \exp[\xi x(t)]\rangle \simeq \frac{\int_{-\infty}^{+\infty} dk \exp\left(-Dk^2 t + \frac{Dt}{2}\xi^2 + i\xi Dtk\right)}{\int_{-\infty}^{+\infty} dk \exp(-Dk^2 t)} = \exp\left[\frac{\xi^2}{2}\left(\sqrt{\frac{Dt}{2}}\right)^2\right]$$

$H_r\left(\sqrt{\frac{Dt}{2}}k\right)$: Hermite Polynomial $\sum_{r=0}^{\infty} H_r(y) z^r / r! = \exp(2yz - z^2)$: generating function

Moment generating function corresponds to that of Gaussian distribution

PT symmetry breaking

Parity-Time symmetry (PT symmetry)

$$H \neq H^\dagger$$

PT symmetry of a Hamiltonian

$$(\mathcal{PT})H(\mathcal{PT})^{-1} = H$$

$$H\,|\phi_l\rangle = \varepsilon_l\,|\phi_l\rangle$$

C. M. Bender and S. Boettcher, Phys. Rev. Lett, **80**, 5243 (1998).

PT symmetry of eigenstates

preserved (oscillation)：

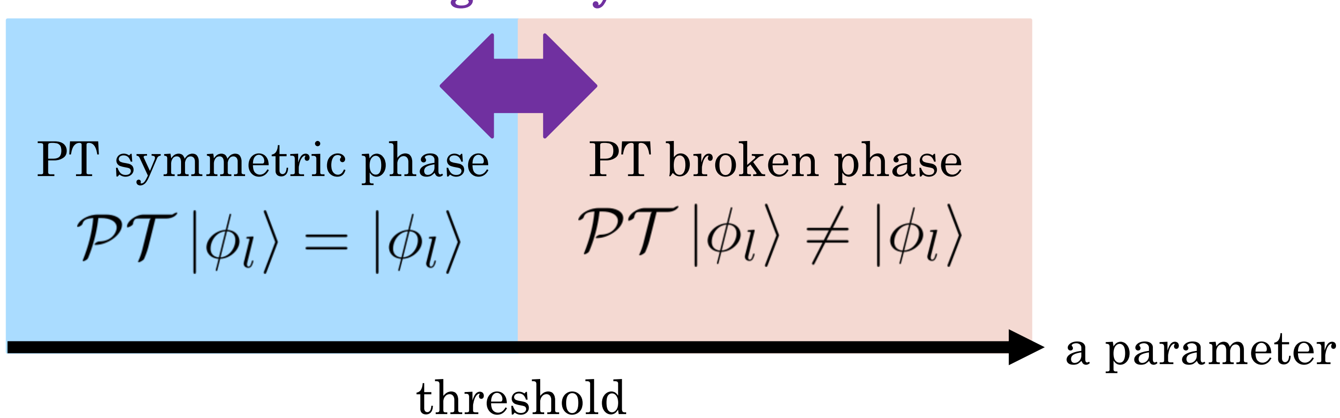
$$\mathcal{PT}\,|\phi_l\rangle = |\phi_l\rangle \rightarrow \varepsilon_l : \mathrm{real}$$

broken (attenuation/divergence)：

$$\mathcal{PT}\,|\phi_l\rangle \neq |\phi_l\rangle \rightarrow \varepsilon_l : \mathrm{complex}$$

typical behavior in PT symmetric systems: $\boxed{\text{unique transition in open systems!}}$

drastic change of dynamics!



Transition of purification timescale

mixed-state dynamics :

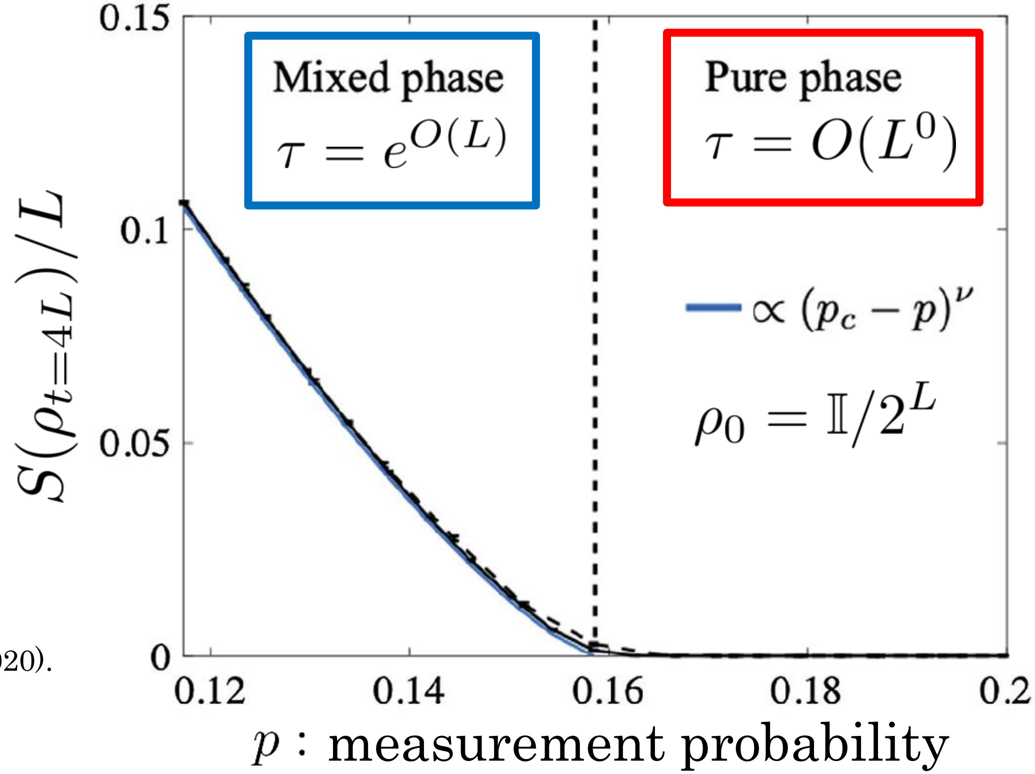
circuit $\rho \to \hat{U}\rho\hat{U}^\dagger$

measurement $\rho \to \dfrac{\hat{M}(\omega)\rho\hat{M}^\dagger(\omega)}{\mathrm{tr}\left[\hat{M}(\omega)\rho\hat{M}^\dagger(\omega)\right]}$

purification at τ : $S[\rho(\boldsymbol{\omega}_\tau)] \simeq 0$

$$S(\rho) = -\mathrm{tr}\left[\rho \log(\rho)\right]$$

purification transition in the mixed-state dynamics



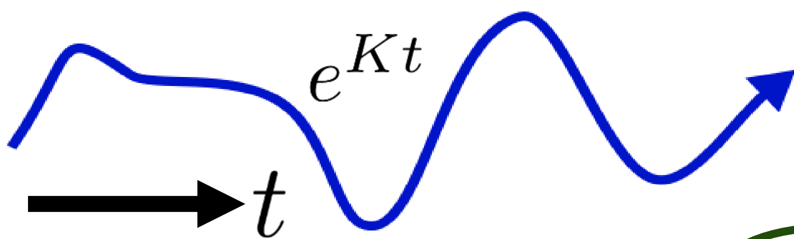
M. J. Gullans et.al., PRX **10**, 041020 (2020).

Spectrum in systems with no randomness

systems described by time-independent generators

$$\frac{d}{dt}\left|\psi(t)\right\rangle = -i\hat{H}\left|\psi(t)\right\rangle, \quad \frac{d}{dt}\hat{\rho}(t) = \mathcal{L}\hat{\rho}(t), \cdots$$

$\left|\psi(0)\right\rangle, \hat{\rho}(0)$

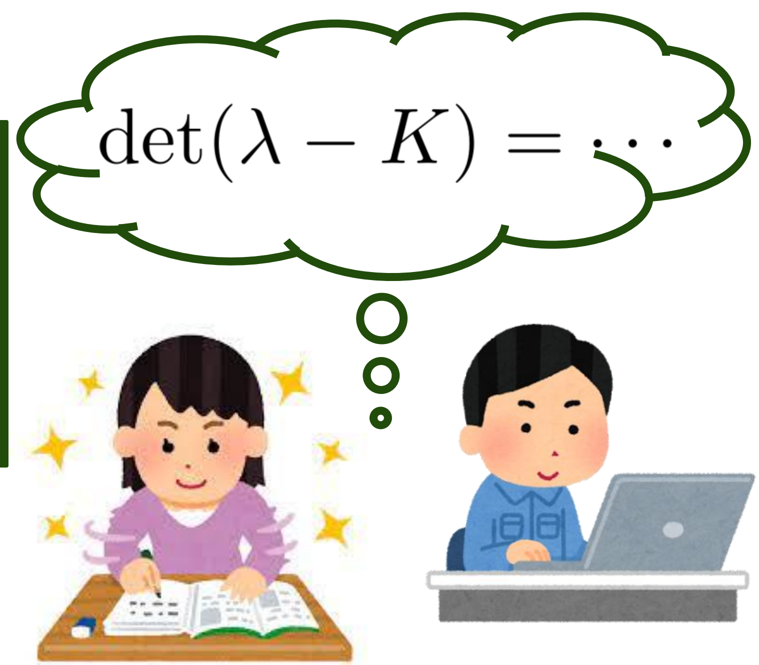


$\left|\psi(t)\right\rangle, \hat{\rho}(t)$

the spectrum of $K = -i\hat{H}, \mathcal{L}$

↓

essential information of the system



Spectral gap in open quantum systems

Markovian open quantum systems

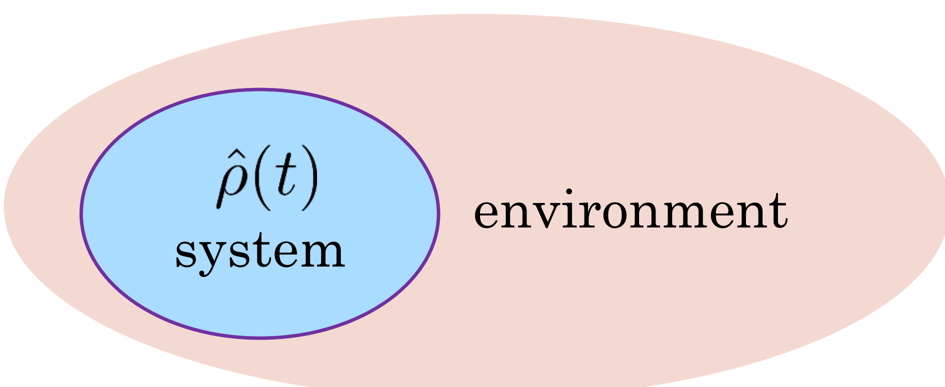
$$\frac{d}{dt}\hat{\rho}(t) = \mathcal{L}\hat{\rho}(t) \qquad \hat{\rho}(t) = e^{\mathcal{L}t}\hat{\rho}(0) = \sum_i c_i e^{\lambda_i t}\hat{\Psi}_i$$

the spectral gap $\Delta = |\mathrm{Re}(\lambda_2) - \mathrm{Re}(\lambda_1)|, \quad \mathrm{Re}(\lambda_j) \geq \mathrm{Re}(\lambda_{j+1})$

→ **the relaxation time to the stationary state** $1/\Delta$

$$\boxed{t \gg \frac{1}{\Delta} \to \hat{\rho}(t) \simeq \hat{\Psi}_1}$$

$$\mathcal{L}\hat{\Psi}_i = \lambda_i \hat{\Psi}_i$$



Averaged dynamics

dynamics averaged over measurement outcomes and random unitaries:

$$\rho_\eta(t+1) = \sum_\omega \int dU \, \hat{M}_\eta(\omega) \hat{U} \rho_\eta(t) \hat{U}^\dagger \hat{M}_\eta^\dagger(\omega)$$

trivial steady state $\rho_{\mathrm{ss}} = I/2^L$ **for any** η ➡ **no transition**

R. Hamazaki, K. Mochizuki, H. Oshima, and Y. Fuji, arXiv:2512.19922

$$\overline{\langle \psi_\eta(\boldsymbol{\omega}_t) | \hat{O} | \psi_\eta(\boldsymbol{\omega}_t) \rangle} = \sum_{\boldsymbol{\omega}_t} P_\eta(\boldsymbol{\omega}_t) \langle \psi_\eta(\boldsymbol{\omega}_t) | \hat{O} | \psi_\eta(\boldsymbol{\omega}_t) \rangle = \mathrm{tr}[\rho_\eta(t) \hat{O}]$$

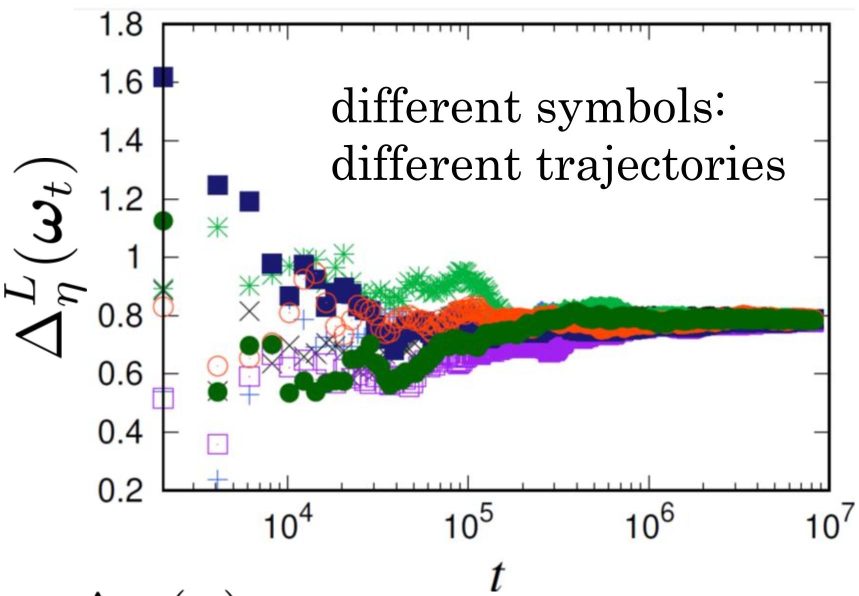
$$\boxed{\text{Measurement-induced transitions are unique to quantum trajectories}}$$

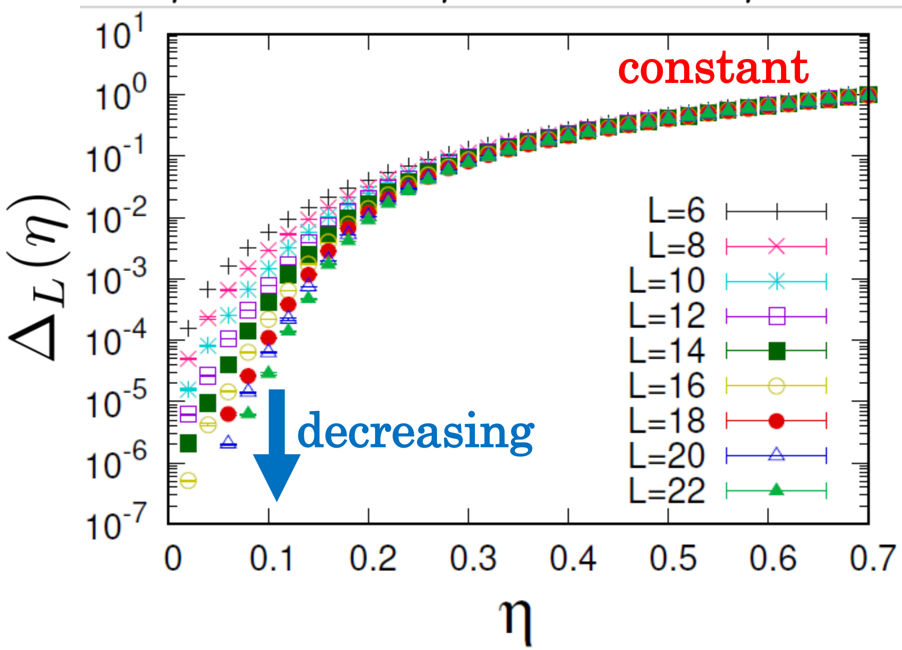
Spectral gap

Ken Mochizuki
Ryusuke Hamazaki
PRL **134**, 010410 (2025).

$$\hat{H}_\eta(\boldsymbol{\omega}_t)\,|\Psi^i_\eta(\boldsymbol{\omega}_t)\rangle = \varepsilon^{i,L}_\eta(\boldsymbol{\omega}_t)\,|\Psi^i_\eta(\boldsymbol{\omega}_t)\rangle\,,\quad \varepsilon^{i,L}_\eta(\boldsymbol{\omega}_t) \le \varepsilon^{i+1,L}_\eta(\boldsymbol{\omega}_t)$$

spectral gap: $\Delta_L(\eta) = \lim_{t\to\infty} \Delta^L_\eta(\boldsymbol{\omega}_t),\ \Delta^L_\eta(\boldsymbol{\omega}_t) = \varepsilon^{2,L}_\eta(\boldsymbol{\omega}_t) - \varepsilon^{1,L}_\eta(\boldsymbol{\omega}_t)$





$\Delta_L(\eta)$: independent of trajectories

global quantity!

η : measurement strength

L : system size

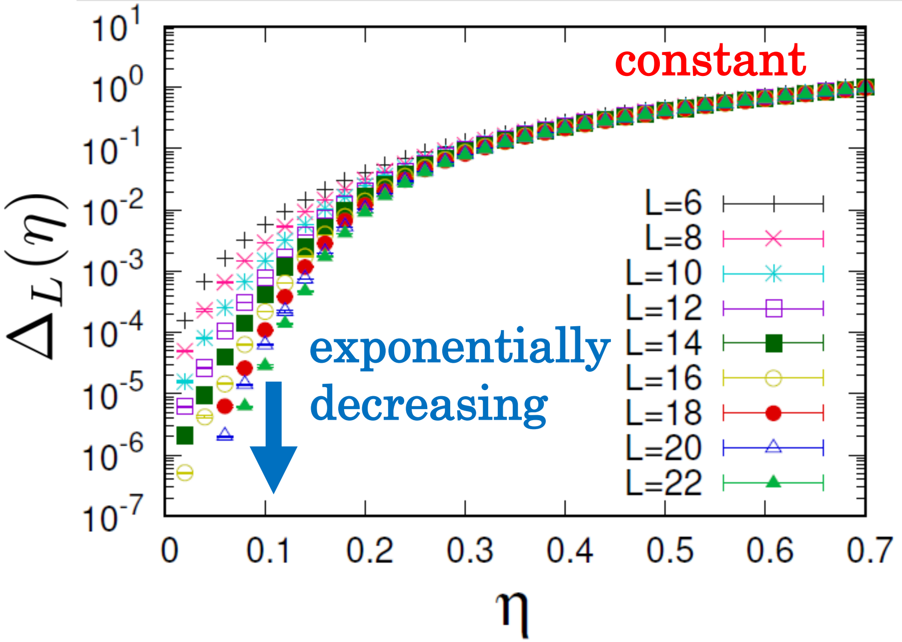
small η : $\lim_{L\to\infty} \Delta_L(\eta) = 0$

large η : $\lim_{L\to\infty} \Delta_L(\eta) \neq 0$

Spectral transition

Ken Mochizuki
Ryusuke Hamazaki
PRL **134**, 010410 (2025).

spectral gap: $\Delta_L(\eta) = \lim_{t\to\infty} \Delta_\eta^L(\boldsymbol{\omega}_t), \ \Delta_\eta^L(\boldsymbol{\omega}_t) = \varepsilon_\eta^{2,L}(\boldsymbol{\omega}_t) - \varepsilon_\eta^{1,L}(\boldsymbol{\omega}_t)$

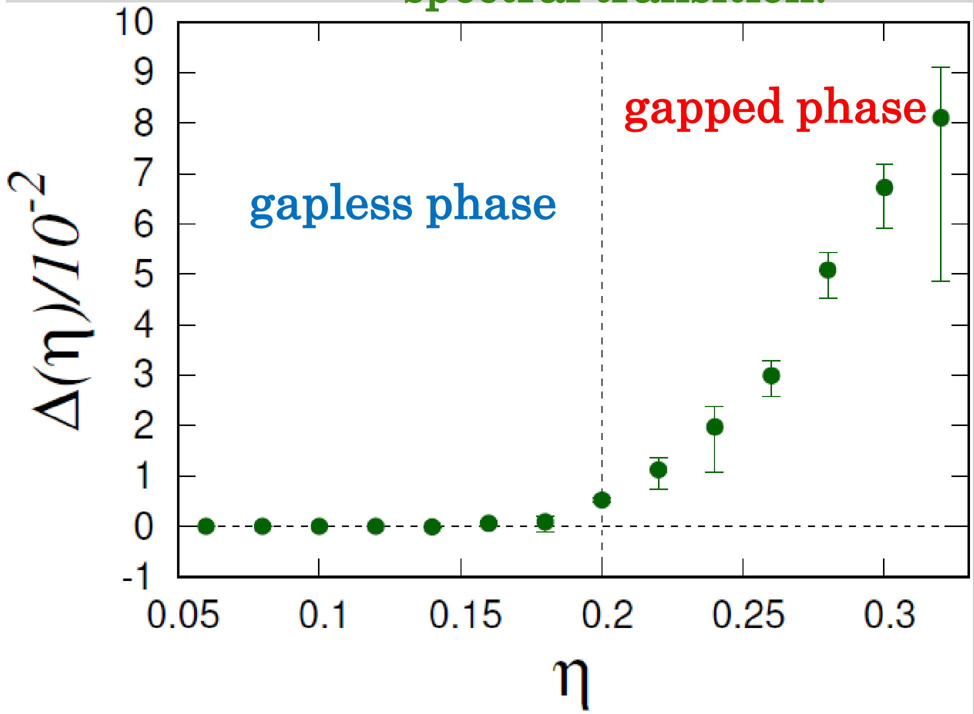


fitting:

$$\Delta_L^{\mathrm{fit}}(\eta) = \Delta(\eta) + \alpha(\eta)[\beta(\eta)]^{-L}$$

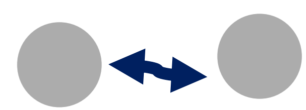
$$\Delta(\eta) = \lim_{L\to\infty} \Delta_L^{\mathrm{fit}}(\eta)$$

spectral transition!



Comparison of transitions

	Monitored dynamics where Hamiltonians can be non-local		Well-known equilibrium systems described by local Hamiltonians
Gapped phase	$S_{L/2} = O(L^0)$ $\Delta_L = O(L^0)$	same	$S_{L/2} = O(L^0)$ $\Delta_L = O(L^0)$
Gapless phase	$S_{L/2} = O(L)$ $\Delta_L = \exp[-O(L)]$	distinct	$S_{L/2} = O[\log(L)]$ $\Delta_L = O[1/\mathrm{poly}(L)]$



*1D systems
e.g. XXZ model

long-range interaction ⇒ volume law?

T. Udagawa and H. Katsura, J. Phys. A: Math. Theor. **50** (2017) 405002.

B. Zeng, X. Chen, D.-L. Zhou, and X.-G. Wen, Quantum information meets quantum matter (Springer, 2019).

Ken Mochizuki, Ryusuke Hamazaki, Physical Review Letters **134**, 010410 (2025).