Symmetric mass generation of 8 Majorana domain-wall fermions

Sho Araki -> Hidenori Fukaya
(U. Osaka)

work in progress

Ref. Sho Araki, HF, Tetsuya Onogi and Satoshi Yamaguchi, 2512.11424 to appear in PRD

= free 8 Majorana fermion on unoriented manifolds and Z_8 anomaly.

Chiral symmetry is difficult on the lattice

Nielsen-Ninomiya theorem 1981 : Chiral symmetry MUST BE BROKEN to avoid fermion doubling.

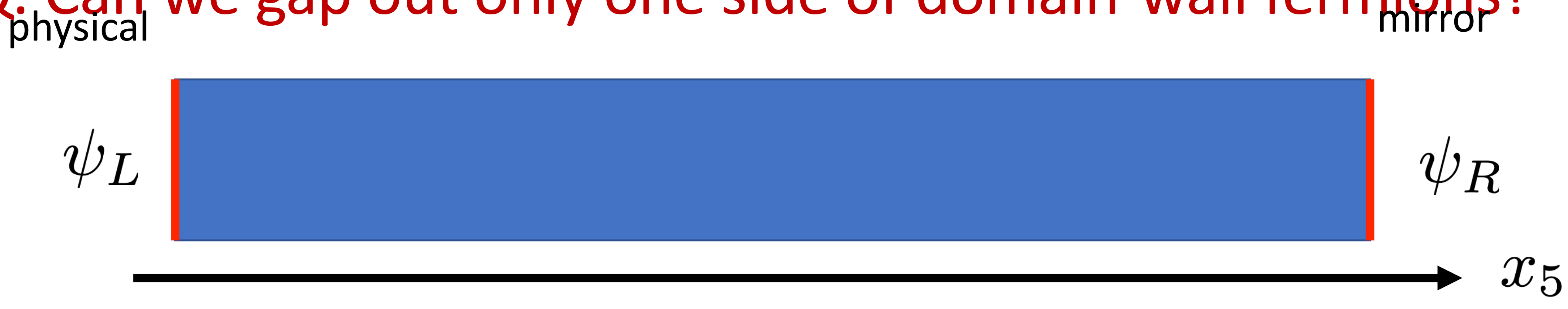
Domain-wall fermion [Kaplan 1992] works for vectorlike-gauge theory but chiral gauge symmetry is hard to maintain.

Symmetric mass generation (SMG) + domain-wall fermion

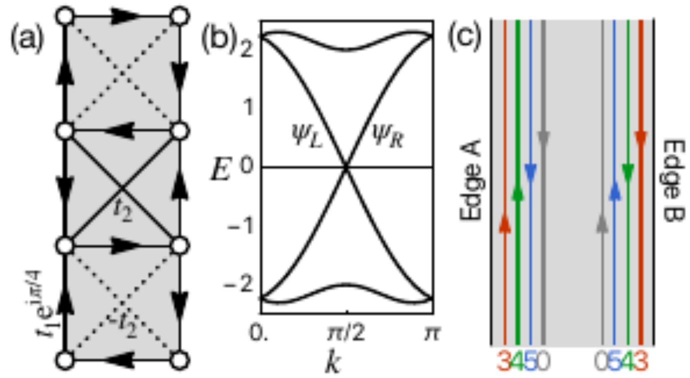
Symmetric mass generation :

4(or more)-Fermi interaction without symmetry breaking may gap out the fermions if they are ('t Hooft) anomaly free.

Q. Can we gap out only one side of domain-wall fermions?



So far no numerical study on SMG
+ domain-wall fermions



Cf) Zeng, Zhu, Wang, You, 2022

looks like domain-wall fermion:

physical wall on left + mirror wall on right.

But the extra-dimension has only one lattice spacing.

Still the symmetry is exact (not $\exp(-ML_5)$) .

We believe that the model is a variant of staggered fermions.

Our work

We simulate 2D 8 Majorana domain-wall fermions and try to gap out only mirror fermions with 4-Fermi interactions.

- This theory is the path-integral version of the Fidkowski-Kitaev 1D Majorana chain with 0D boundaries (= the first solid benchmark of SMG)
- Hybrid Monte Carlo w/ phase quenching+reweighting works.
- 2D Wilson Dirac spectrum clearly explains how SMG works in path-integral formalism.
- Show numerical evidence for SMG of mirror fermions.

Contents

Fidkowski-Kitaev model [2010]

1D Majorana chain Hamiltonian with 8 flavors

$$\hat{H} = \frac{i}{2}\sum_{x}\left[\sum_{j=1}^{8}\{u\hat{c}_j(x-a)\hat{c}_j(x) + v\hat{c}_j(x)\hat{c}_j(x+a)\} + W_{4\text{-Fermi}}\right]$$

$$\{\hat{c}_j(x), \hat{c}_k(x)\} = 2\delta_{jk} \qquad \hat{c}_j{}^{\dagger}(x) = \hat{c}_j(x)$$

T symmetry :

$$T\hat{c}_j(x_{\text{odd}})T^{-1} = -\hat{c}_j(x_{\text{odd}})$$

$$T\hat{c}_j(x_{\text{even}})T^{-1} = +\hat{c}_j(x_{\text{even}})$$

-> On-site quadratic terms $i\hat{c}_j(x)\hat{c}_k(x)$ are NOT allowed.

Fidkowski-Kitaev model [2010]

4 Fermi interaction which gives a mass without symmetry breaking =

$$
\begin{aligned}
W = {} & \hat{c}_1\hat{c}_2\hat{c}_3\hat{c}_4 + \hat{c}_5\hat{c}_6\hat{c}_7\hat{c}_8 + \hat{c}_1\hat{c}_2\hat{c}_5\hat{c}_6 + \hat{c}_3\hat{c}_4\hat{c}_7\hat{c}_8 - \hat{c}_2\hat{c}_3\hat{c}_6\hat{c}_7 \\
& - \hat{c}_1\hat{c}_4\hat{c}_5\hat{c}_8 + \hat{c}_1\hat{c}_3\hat{c}_5\hat{c}_7 + \hat{c}_3\hat{c}_4\hat{c}_5\hat{c}_6 + \hat{c}_1\hat{c}_2\hat{c}_7\hat{c}_8 - \hat{c}_2\hat{c}_3\hat{c}_5\hat{c}_8 \\
& - \hat{c}_1\hat{c}_4\hat{c}_6\hat{c}_7 + \hat{c}_2\hat{c}_4\hat{c}_6\hat{c}_8 - \hat{c}_1\hat{c}_3\hat{c}_6\hat{c}_8 - \hat{c}_2\hat{c}_4\hat{c}_5\hat{c}_7 \qquad (8)
\end{aligned}
$$

$$= -\frac{1}{24}\sum_{\alpha=1}^{7}\Phi_\alpha^2$$

Great simplification by You and Xu 2015 where SO(7) symmetry is manifest.

$$\Phi^\alpha = c_i c_j \Gamma^\alpha_{ij}$$

$$\sigma^{ijk} = \sigma^i \otimes \sigma^j \otimes \sigma^k$$

$$\Gamma = \left\{\sigma^{123}, \sigma^{203}, \sigma^{323}, \sigma^{211}, \sigma^{021}, \sigma^{231}, \sigma^{002}\right\} \qquad \sigma^0 = 1_{2\times 2}$$

FK model expressed by 2-component spinors

$c_i^e = \eta_i + \bar{\eta}_i$ for even sites

$c_i^o = \eta_i - \bar{\eta}_i$ for odd sites

$$W^e + W^o = -\frac{1}{6}\sum_{\alpha=1}^{7}\sum_{i,j=1}^{8}\left(\eta_i\eta_j\bar{\eta}_k\bar{\eta}_l\Gamma^\alpha_{ij}\Gamma^\alpha_{kl} - 2\eta_i\eta_k\bar{\eta}_j\bar{\eta}_l\Gamma^\alpha_{ij}\Gamma^\alpha_{kl}\right)$$

$$W^{\mathrm{so}(8)} = -\sum_{i=1}^{8}(c_i^e c_i^o)^2 = -4\left(\sum_{i=1}^{8}\eta_i\bar{\eta}_i\right)^2$$

$$C = i\sigma_2 \qquad \bar{\gamma} = \sigma_3$$

$$\chi = \begin{pmatrix}\eta \\ \bar{\eta}\end{pmatrix} \qquad W^{\mathrm{tot}} = -\frac{g_S^2}{2}\left(\frac{1}{2}\chi^T C\chi\right)^2 - \frac{g_P^2}{2}\sum_{\alpha=1}^{7}\left(\frac{1}{2}\chi^T C i\bar{\gamma}\Gamma^\alpha\chi\right)^2$$

assigning general values of coefficients.

Comparison of two SO(7) invariant expression

$$W^{\mathrm{tot}} = -\frac{g_S^2}{2}\left(\frac{1}{2}\chi^T C\chi\right)^2 - \frac{g_P^2}{2}\sum_{\alpha=1}^{7}\left(\frac{1}{2}\chi^T C i\bar{\gamma}\Gamma^\alpha\chi\right)^2$$

[You Xu 2015]

SO(8) invariant SO(7) invariant

$$W^{\mathrm{FK}} = -A\left(\sum_{j=1}^{7}\psi_j\bar{\psi}_j\right)^2 - B\left(\sum_{j=1}^{7}\psi_j\bar{\psi}_j\right)\psi_8\bar{\psi}_8.$$

* ψ is non-locally defined.

SO(7) invariant SO(7) invariant

$$g_P^2 = \frac{1}{4}A - \frac{1}{8}B,$$

$$g_S^2 = \frac{5}{4}A + \frac{3}{8}B$$

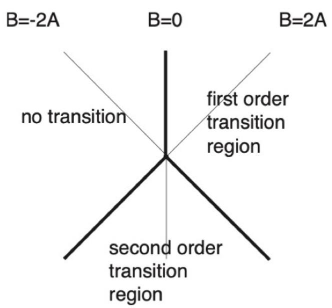


FIG. 5. Phase diagram for the transition from negative to positive m, indicating dependence on B (x axis) and A (y axis).

We focus on the

$$g_P^2 = \frac{1}{6}A$$ line

fixing

$$g_S^2 = 0, \quad B = -\frac{10}{3}A$$

FK claimed that A>0 allows SMG.

2-dimensional path integral version

$$S_E = \int dx^2 \left\{ \frac{1}{2}\chi^T C \left(\gamma^\mu \partial_\mu + m\right)\chi - \frac{g_S^2}{2}\left(\frac{1}{2}\chi^T C \chi\right)^2 - \frac{g_P^2}{2}\sum_{\alpha=1}^{7}\left(\frac{1}{2}\chi^T C i\bar{\gamma}\Gamma^\alpha \chi\right)^2 \right\} \qquad g_S^2 = 0$$

Notations:

$$\gamma_1 = \sigma_1, \quad \gamma_2 = \sigma_2, \quad C = i\sigma_2, \quad \bar{\gamma} = \sigma_3 \quad \text{acts on} \quad \chi = \begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix}$$

$$\sigma^{ijk} = \sigma^i \otimes \sigma^j \otimes \sigma^k$$

$$\Gamma = \left\{\sigma^{123}, \sigma^{203}, \sigma^{323}, \sigma^{211}, \sigma^{021}, \sigma^{231}, \sigma^{002}\right\} \qquad \sigma^0 = 1_{2\times 2}$$

T symmetry $\quad T\hat{c_a}(x_{\mathrm{odd}})T^{-1} = -\hat{c_a}(x_{\mathrm{odd}}) \quad T\hat{c_a}(x_{\mathrm{even}})T^{-1} = +\hat{c_a}(x_{\mathrm{even}})$

$$T\chi T^{-1} = T\begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix} T^{-1} = T\begin{pmatrix} (c^e + c^o)/2 \\ (c^e - c^o)/2 \end{pmatrix} T^{-1} = \sigma_1 \chi \qquad ???$$

is not respected. We need to switch from Minkowski to Euclidean.

Euclidean 2-dimensional path integral version

$$S_E = \int dx^2 \left\{ \frac{1}{2}\chi^T C \left(\gamma^\mu \partial_\mu + m\right)\chi - \frac{g_S^2}{2}\left(\frac{1}{2}\chi^T C \chi\right)^2 - \frac{g_P^2}{2}\sum_{\alpha=1}^{7}\left(\frac{1}{2}\chi^T C i\bar{\gamma}\Gamma^\alpha \chi\right)^2 \right\} \qquad g_S^2 = 0$$

Reflection symmetry (Rt) in Euclidean t direction is instead respected.

$$R_t \chi(x,t) R_t^{-1} = i\sigma_1 \chi(x,-t) \qquad \chi = \begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix}$$

Note : Rt is one element of $Pin^-(2)$ group. $\qquad i\sigma_1 = \sigma_2\sigma_3$

"Dirac" mass is allowed, $\quad m\chi^T C\chi = 2m\bar{\eta}\eta$

but Majorana mass is NOT : $\quad \eta_j\eta_k, \bar{\eta}_j\bar{\eta}_k \quad$ are Rt odd.

2-dimensional path integral version

$$S_E = \int dx^2 \left\{ \frac{1}{2}\chi^T C\left(\gamma^\mu \partial_\mu + m\right)\chi - \frac{g_S^2}{2}\left(\frac{1}{2}\chi^T C \chi\right)^2 - \frac{g_P^2}{2}\sum_{\alpha=1}^{7}\left(\frac{1}{2}\chi^T C i\bar{\gamma}\Gamma^\alpha \chi\right)^2 \right\} \qquad g_S^2 = 0$$

By Hubbard-Stratonovich transformation, the path-integral becomes

$$Z = \int \prod_{x,\alpha} d\pi^\alpha(x) d\chi(x) \exp\left[-\frac{1}{2}\int d^2x \chi^T(x) \left[C\left(\gamma^i \partial_i + m + \sqrt{g_P^2}\sum_{\alpha=1}^{7}\pi^\alpha(x) i\gamma_3 \Gamma^\alpha \right)\right] \chi(x) + \sum_{\alpha=1}^{7}(\pi^\alpha(x))^2 \right]$$

$$= \int \prod_{x,\alpha} d\pi^\alpha(x) \mathrm{Pf}\left[C\left(\gamma^i \partial_i + m + \sqrt{g_P^2}\sum_{\alpha=1}^{7}\pi^\alpha(x) i\gamma_3 \Gamma^\alpha \right)\right] \exp\left[-\frac{1}{2}\int d^2x \sum_{\alpha=1}^{7}(\pi^\alpha(x))^2 \right]$$

Yukawa interaction Auxiliary scalar fields

Our goal = To simulate this path integral on a 2-dim lattice and confirm

Real $\sqrt{g_P^2}$ -> mass gap Pure imaginary $\sqrt{g_P^2}$ -> gap closed

2-dimensional lattice simulation

We compute the two-dim lattice path integral

$$Z = \int \prod_{x,\alpha} d\pi^{\alpha}(x) \mathrm{Pf}\left[C\left(D_W + m + \sqrt{g_P^2}\sum_{\alpha=1}^{7}\pi^{\alpha}(x) i\gamma_3\Gamma^{\alpha}\right)\right] \exp\left[-\frac{1}{2}\sum_{x}\sum_{\alpha=1}^{7}(\pi^{\alpha}(x))^2\right]$$

by stochastically generating $\{\pi^{\alpha}(x)\}$ by Hybrid Monte Carlo algorithm with phase quenching of the Pfaffian.

For the Dirac operator, we use the Wilson Dirac operator

$$D_W = \sum_{\mu}\left[\gamma^{\mu}\frac{\nabla_{\mu}^{f} + \nabla_{\mu}^{b}}{2} - \frac{a}{2}\nabla_{\mu}^{f}\nabla_{\mu}^{b}\right]$$

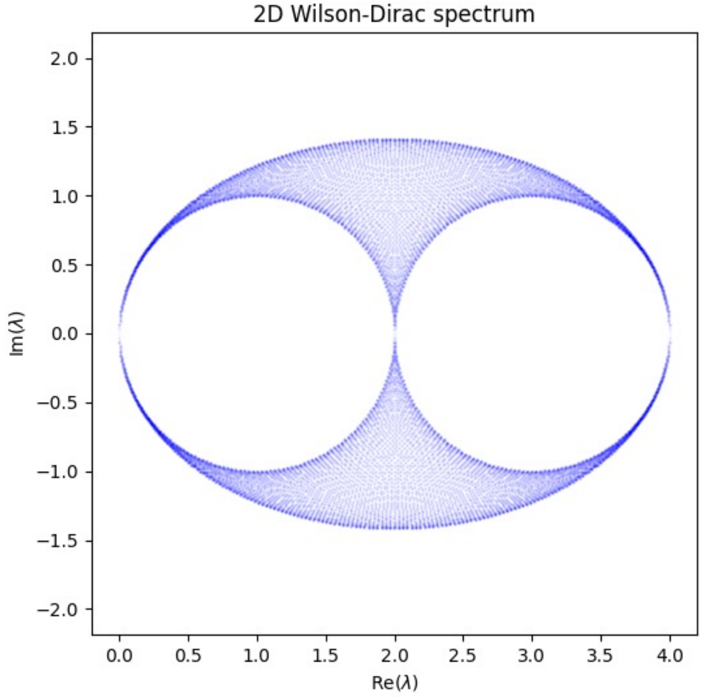
$$\nabla^{f}\psi(x) = \frac{\psi(x+\hat{\mu}a) - \psi(x)}{a}$$

$$\nabla^{b}\psi(x) = \frac{\psi(x) - \psi(x-\hat{\mu}a)}{a}$$

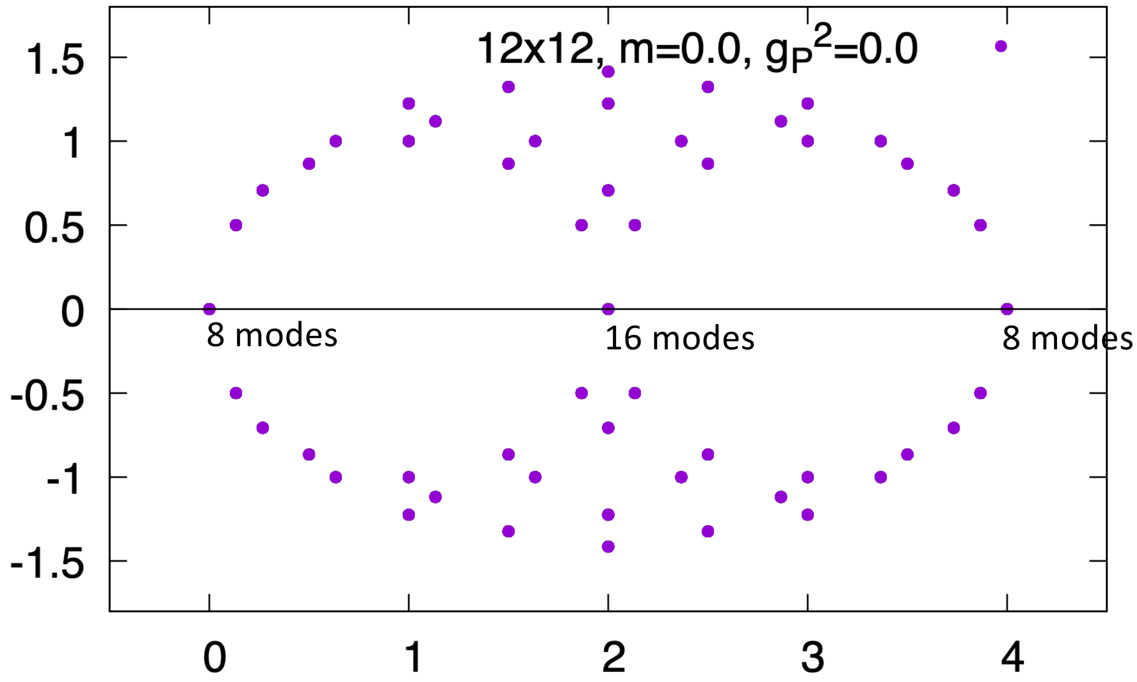
Contents

Free 8-flavor Wilson Dirac operator spectrum

Infinite V



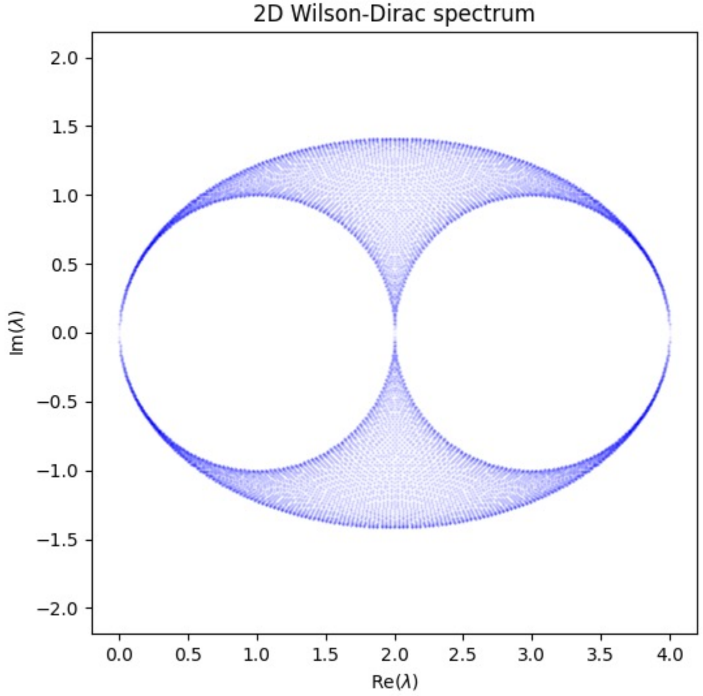
12x12 lattice



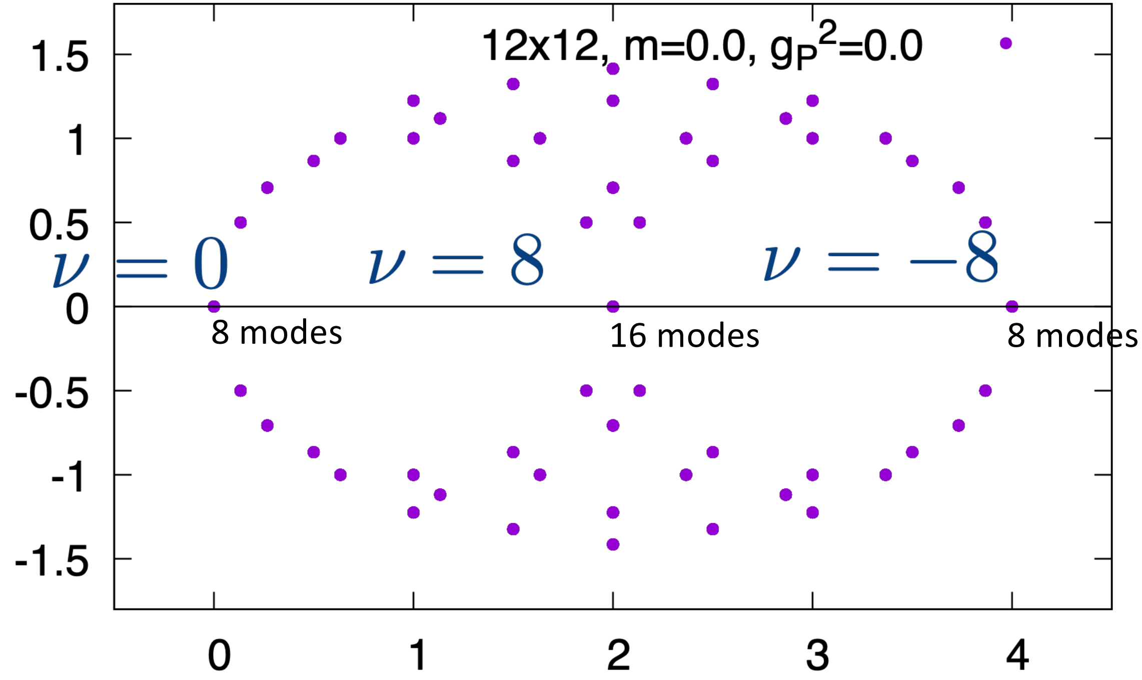
Adding mass : $D_W + m$ = shifting spectrum along real axis

Free 8-flavor Wilson Dirac operator spectrum

Infinite V



8x8 lattice with periodic boundary conditions



Adding mass : $D_W + m$ = shifting spectrum along real axis

Numerical simulation without domain-walls

$$Z^{\text{phasequenched}} = \int \prod_{x,\alpha} d\pi^\alpha(x) |\mathrm{Pf}\,[C(D+m)]\,| \exp\left[-\frac{1}{2}\sum_x \sum_{\alpha=1}^{7} (\pi^\alpha(x))^2\right]$$

$$D = D_W + \sqrt{g_P^2} \sum_{\alpha=1}^{7} \pi^\alpha(x) i\gamma_3 \Gamma^\alpha$$

We simulate the phase quenched theory by hybrid Monte Carlo and reweight the complex phase in the measurements.

$$\langle O \rangle = \frac{\langle O e^{i\theta} \rangle_{pq}}{\langle e^{i\theta} \rangle_{pq}}$$

Lattice size : 12x12, 12x24 (on going)

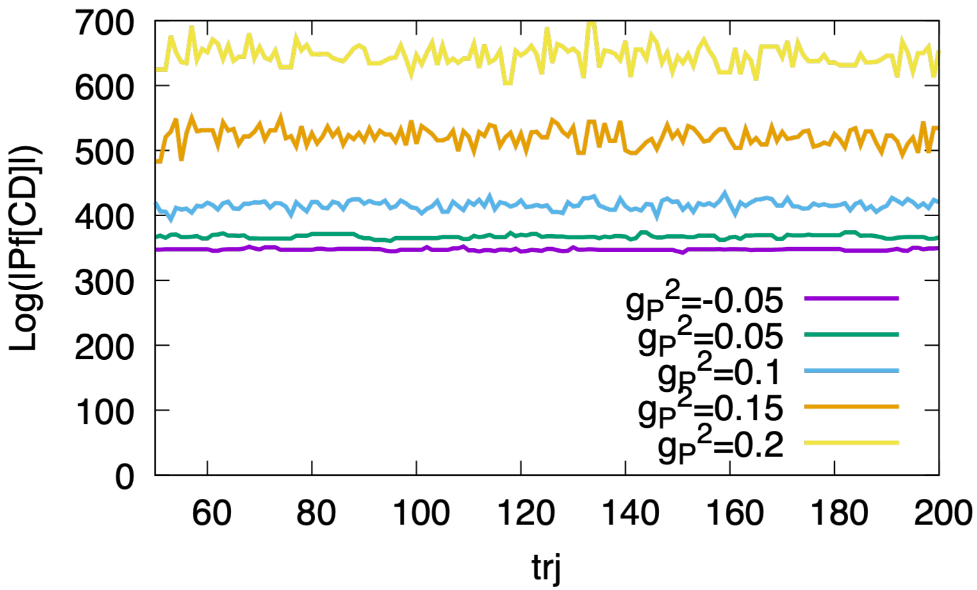
m = -2 (center-branch [Misumi 2020]), -1, 0

g_P^2 = -0.05, 0.02, 0.05, 0.1, 0.15, 0.2

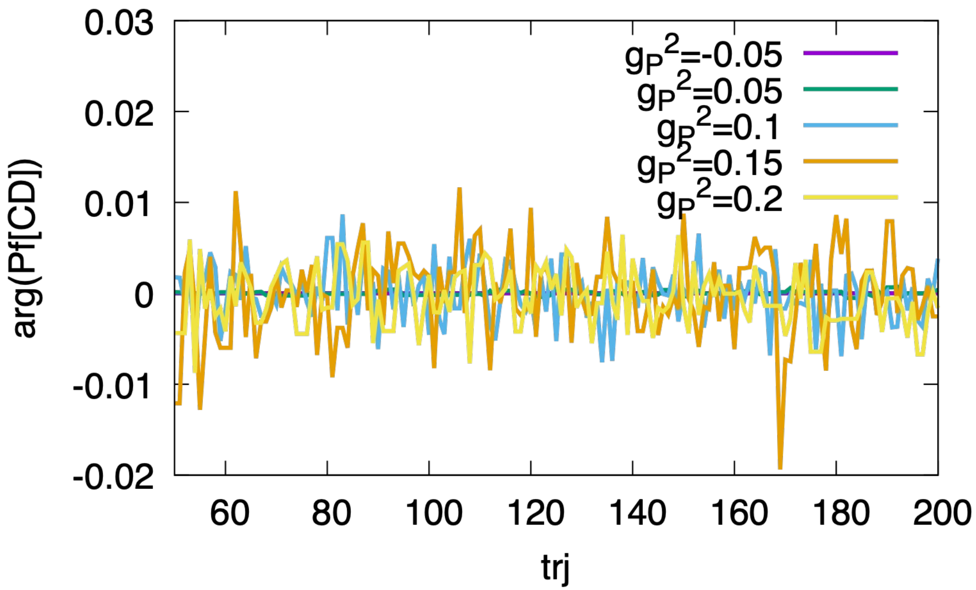
We modify the Julia QCD code [We thank Yuki Nagai and Akio Tomiya]

HMC history of pfaffian

Log of |Pfaffian| + const.



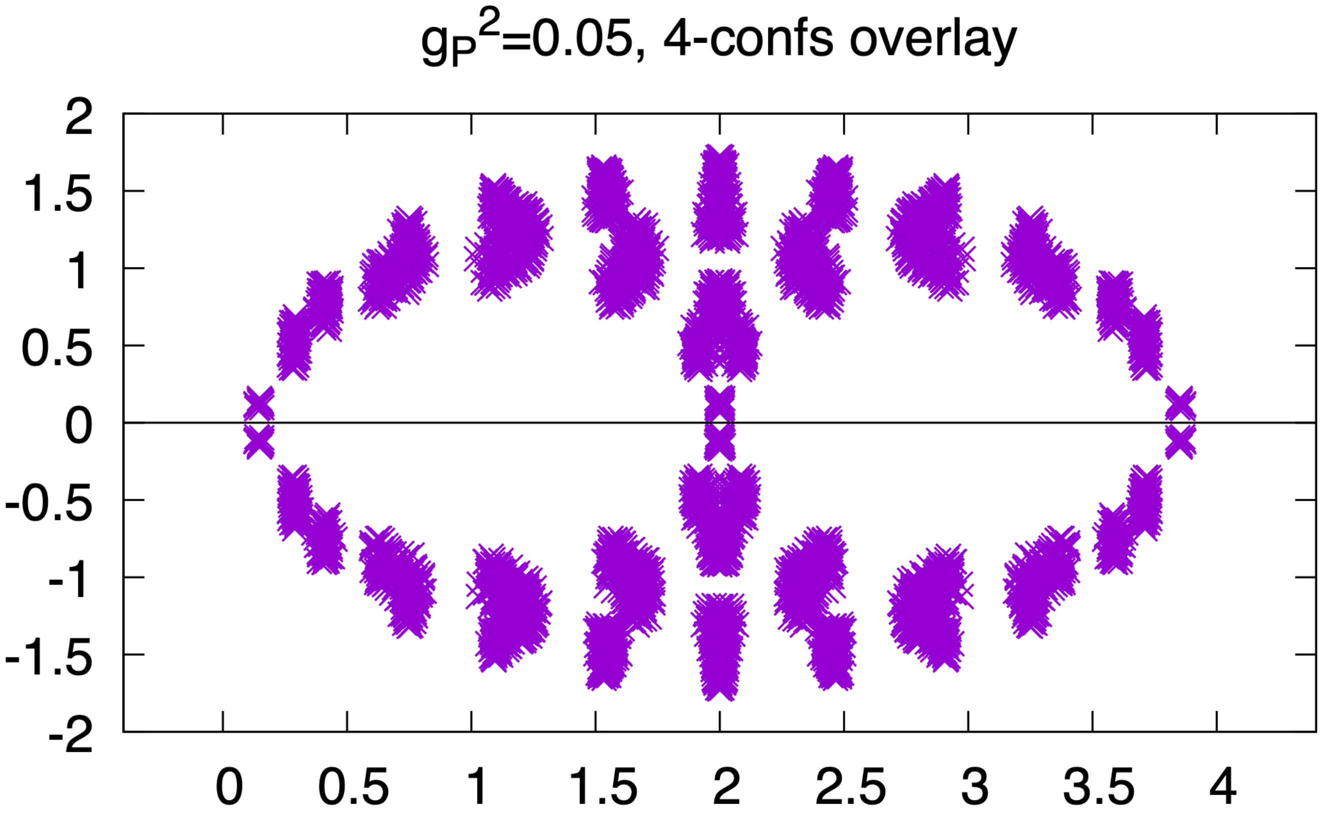
phase of Pfaffian (effect is < 1%)



We simply ignore the phase
in the preliminary analysis.

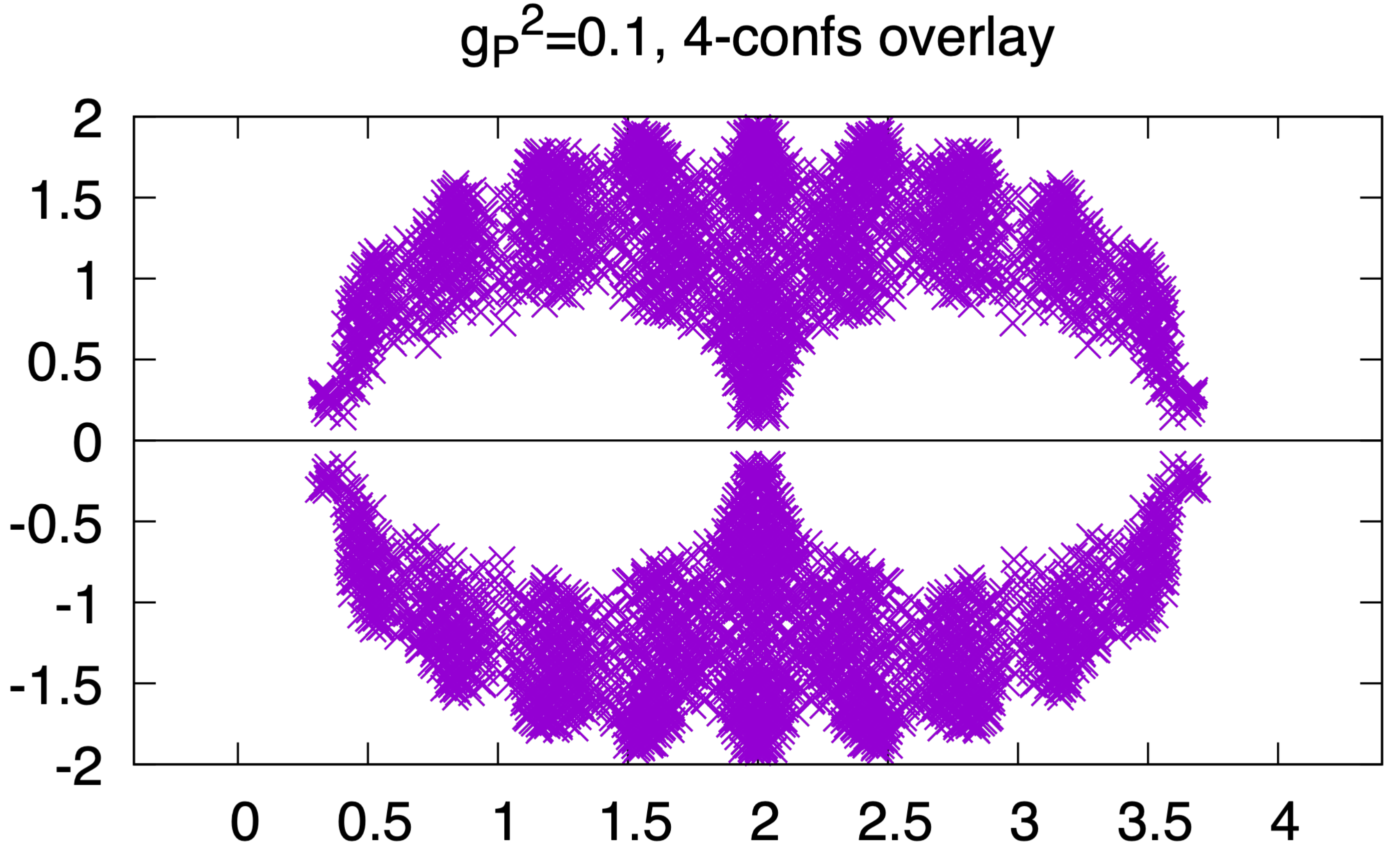
Spectrum of $D = D_W + \sqrt{g_P^2} \sum_{\alpha=1}^{7} \pi^{\alpha}(x) i\gamma_3 \Gamma^{\alpha}$

preliminary



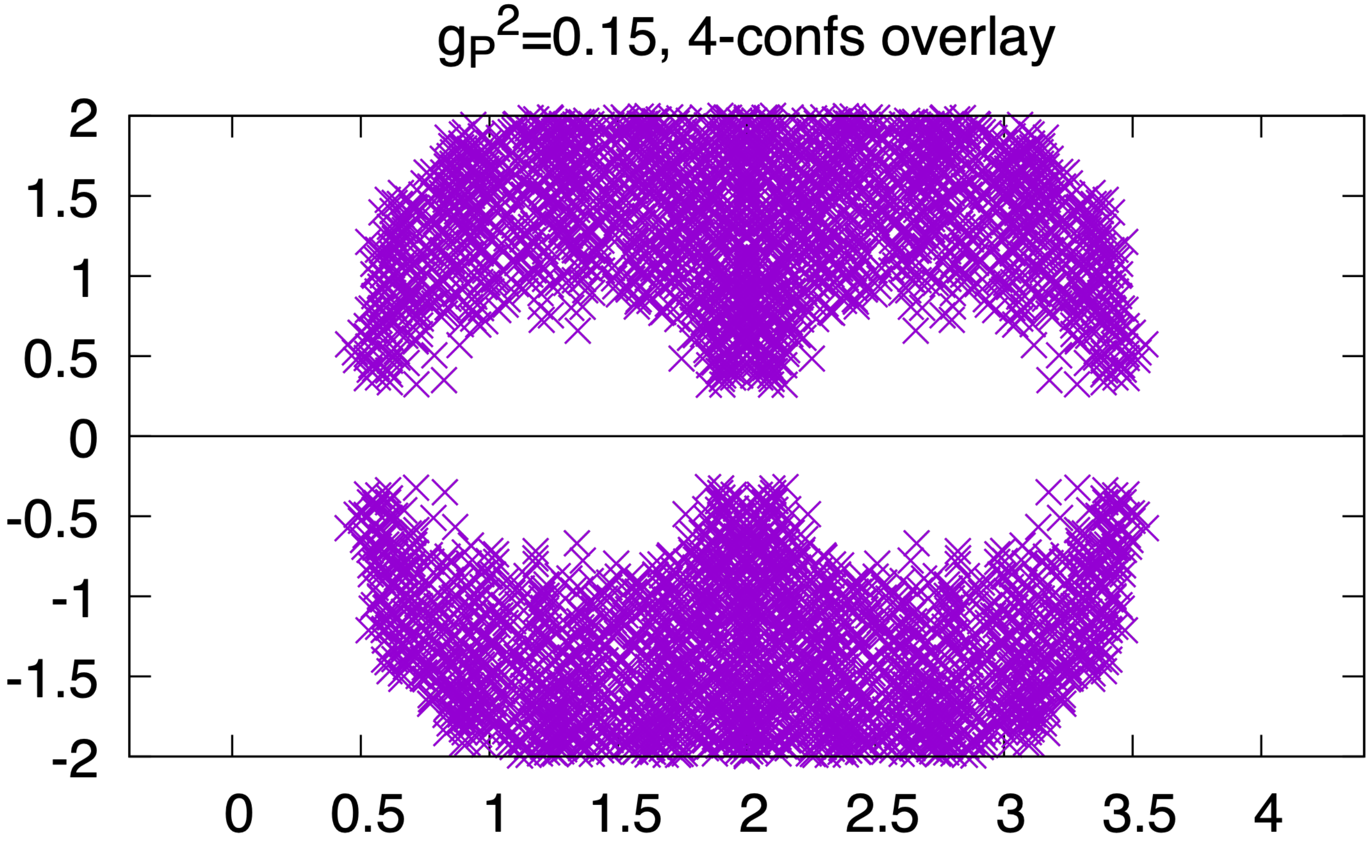
Spectrum of $D = D_W + \sqrt{g_P^2} \sum_{\alpha=1}^{7} \pi^{\alpha}(x) i\gamma_3 \Gamma^{\alpha}$

preliminary



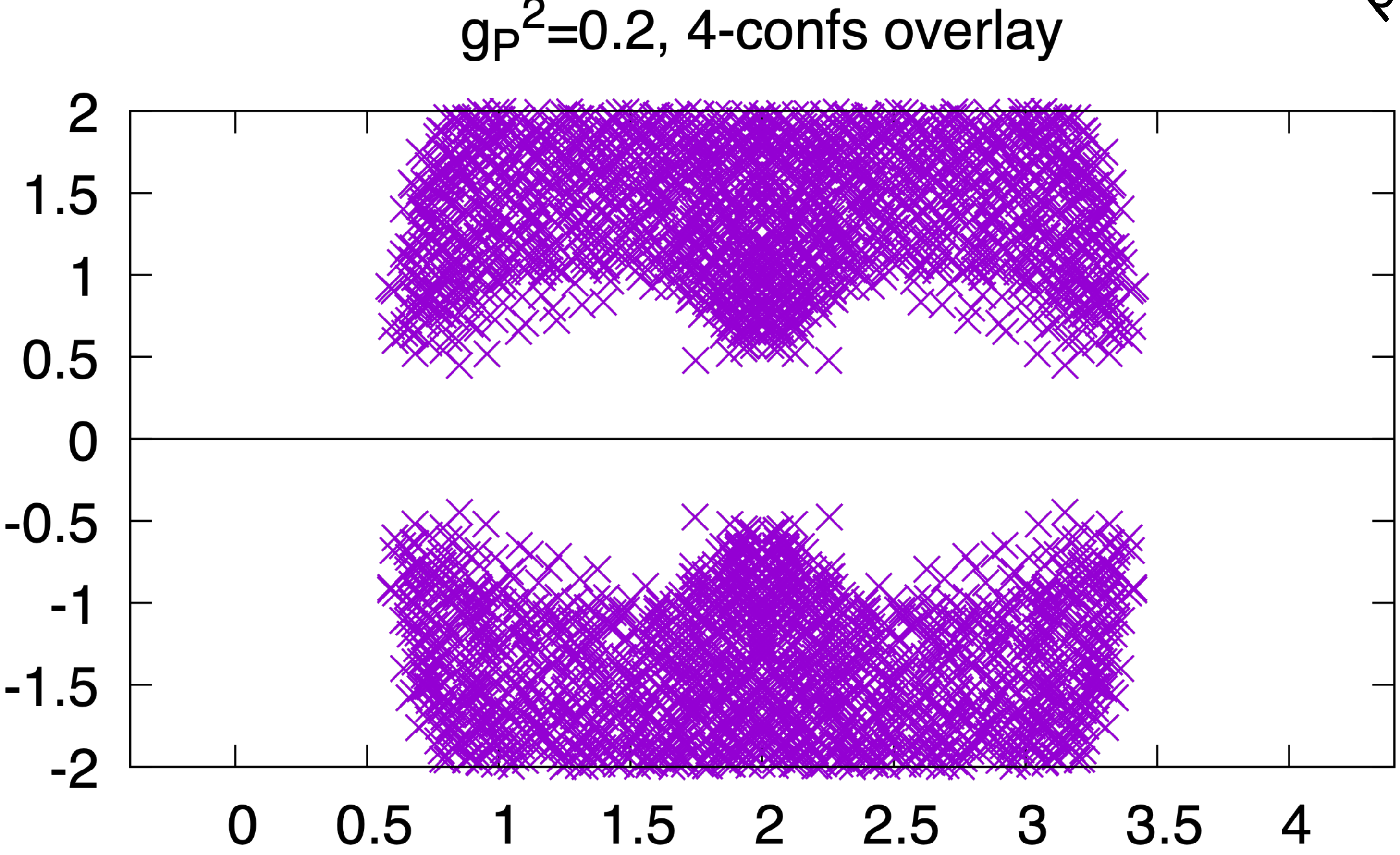
Spectrum of $D = D_W + \sqrt{g_P^2} \sum_{\alpha=1}^{7} \pi^\alpha(x) i\gamma_3 \Gamma^\alpha$

preliminary



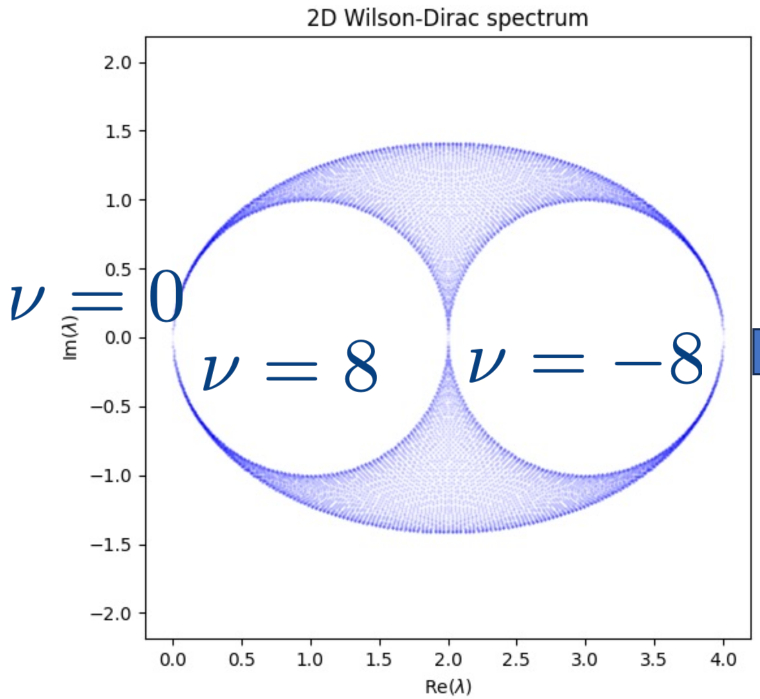
Spectrum of $D = D_W + \sqrt{g_P^2}\sum_{\alpha=1}^{7}\pi^\alpha(x) i\gamma_3\Gamma^\alpha$

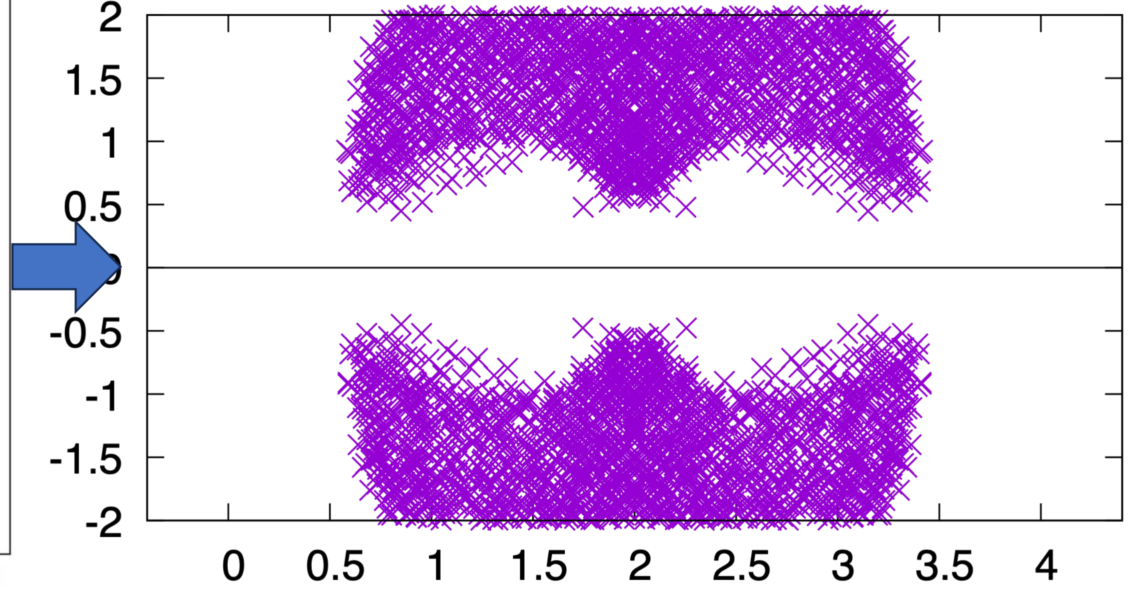
preliminary



Spectrum of $D = D_W + \sqrt{g_P^2} \sum_{\alpha=1}^{7} \pi^\alpha(x) i\gamma_3\Gamma^\alpha$

preliminary



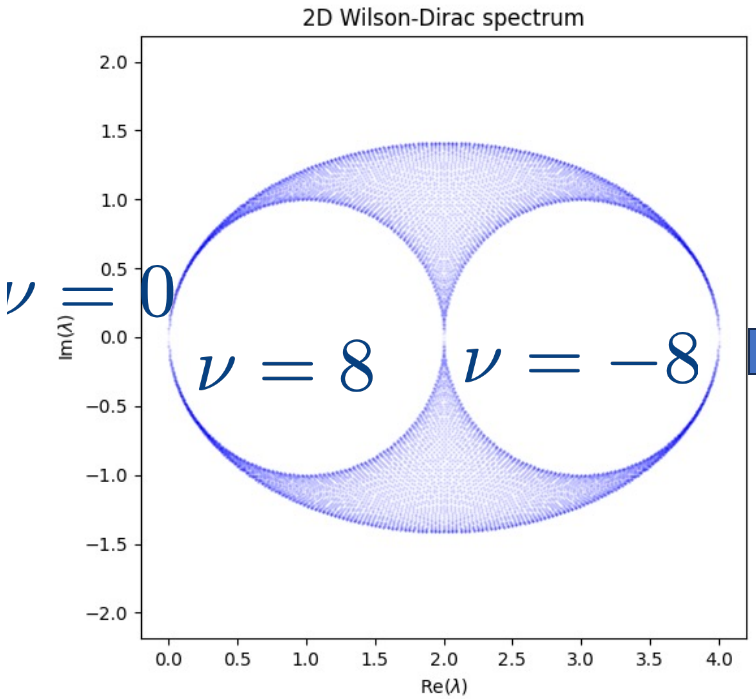


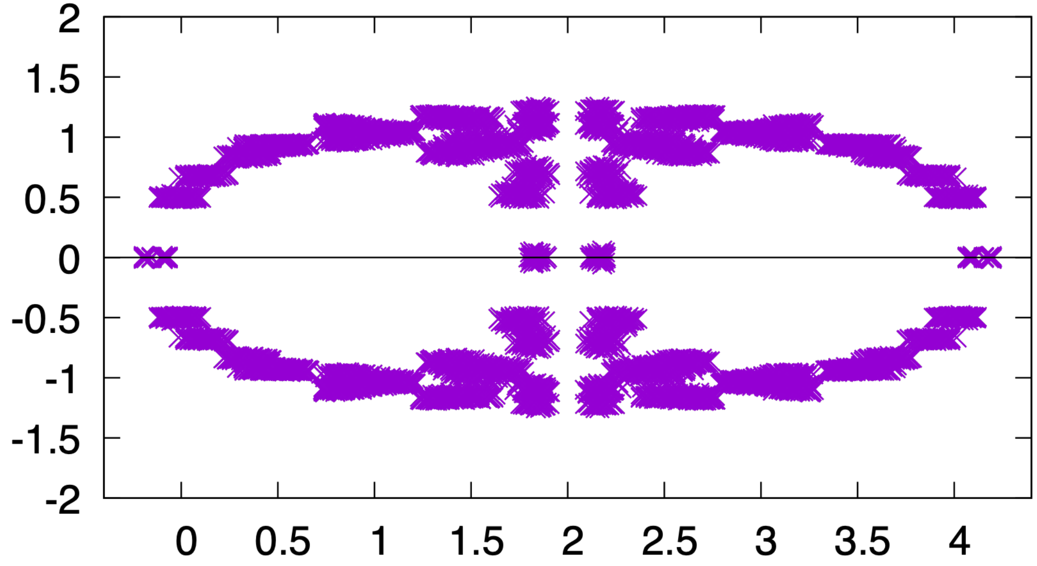
Gap opens for $g_P^2 > 0$!

Note : Yukawa term is anti-Hermitian.

Spectrum of $D = D_W + \sqrt{g_P^2}\sum_{\alpha=1}^{7}\pi^{\alpha}(x)i\gamma_3\Gamma^{\alpha}$

preliminary



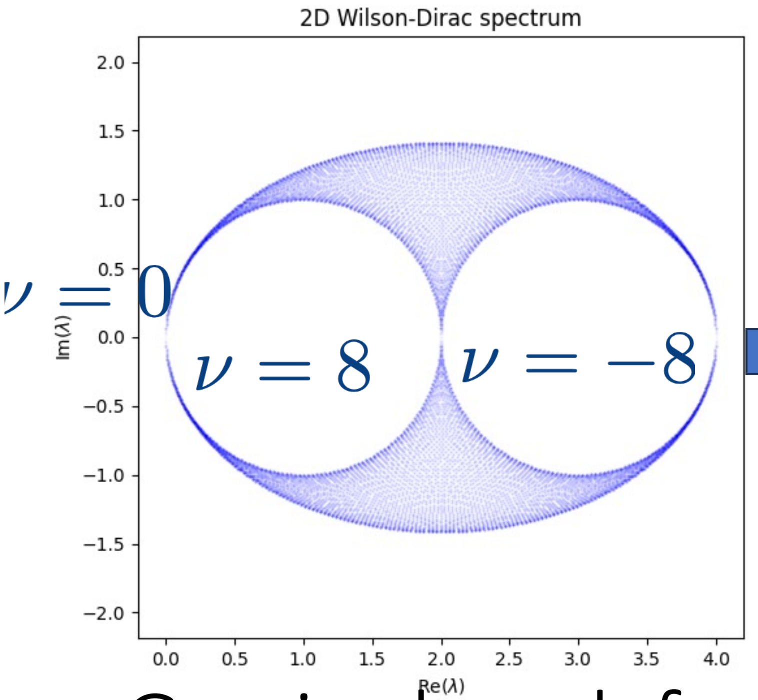


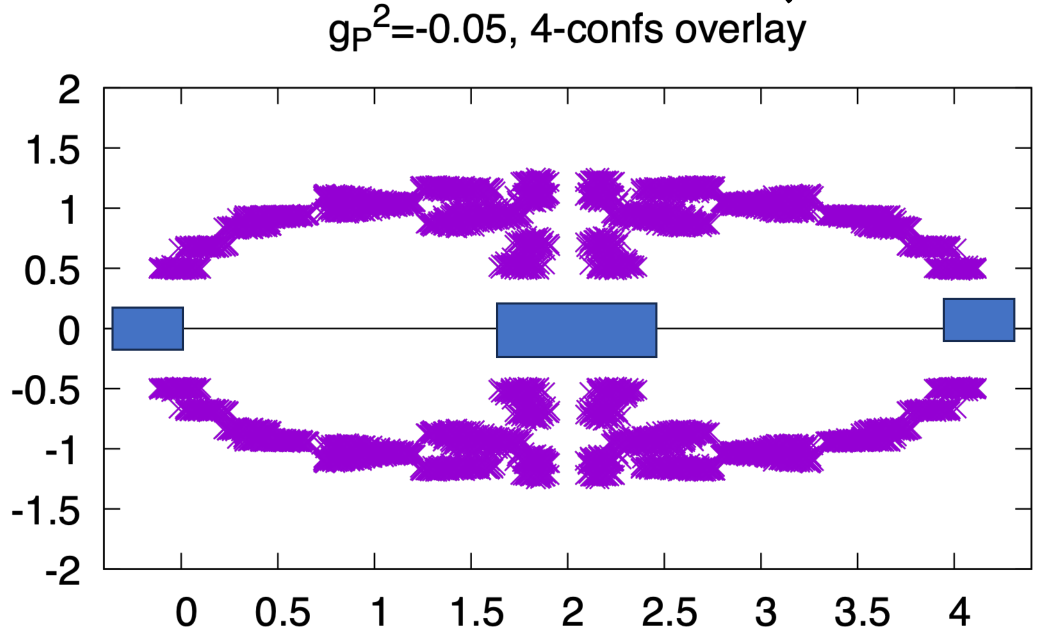
Gap is closed for $g_P^2 < 0$!

Note : Yukawa term is Hermitian.

Spectrum of $D = D_W + \sqrt{g_P^2} \sum_{\alpha=1}^{7} \pi^\alpha(x) i\gamma_3 \Gamma^\alpha$

preliminary





Gap is closed for $g_P^2 < 0$!

Aoki phase appearing?

Chern number is not well-defined.

Possible phase diagram

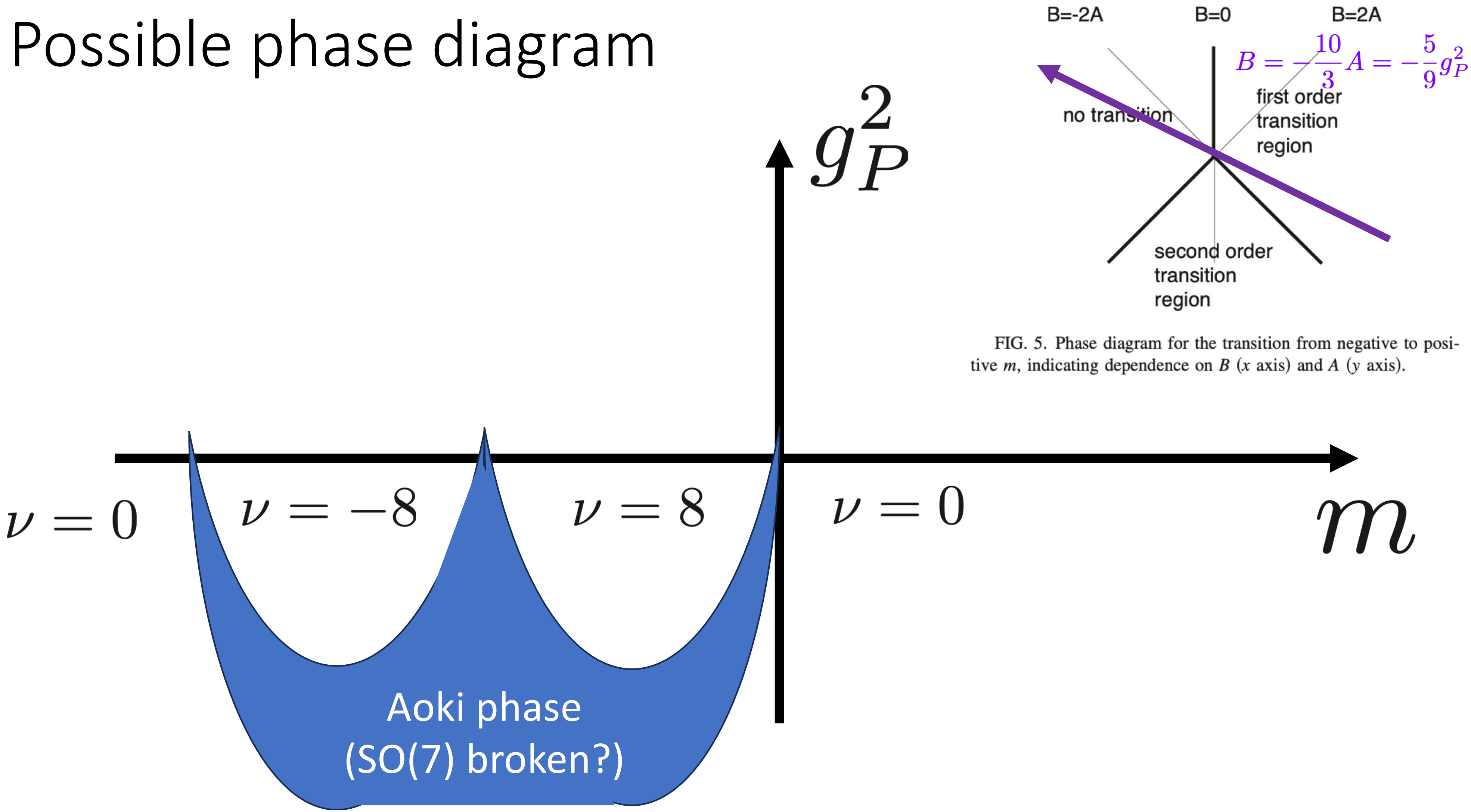
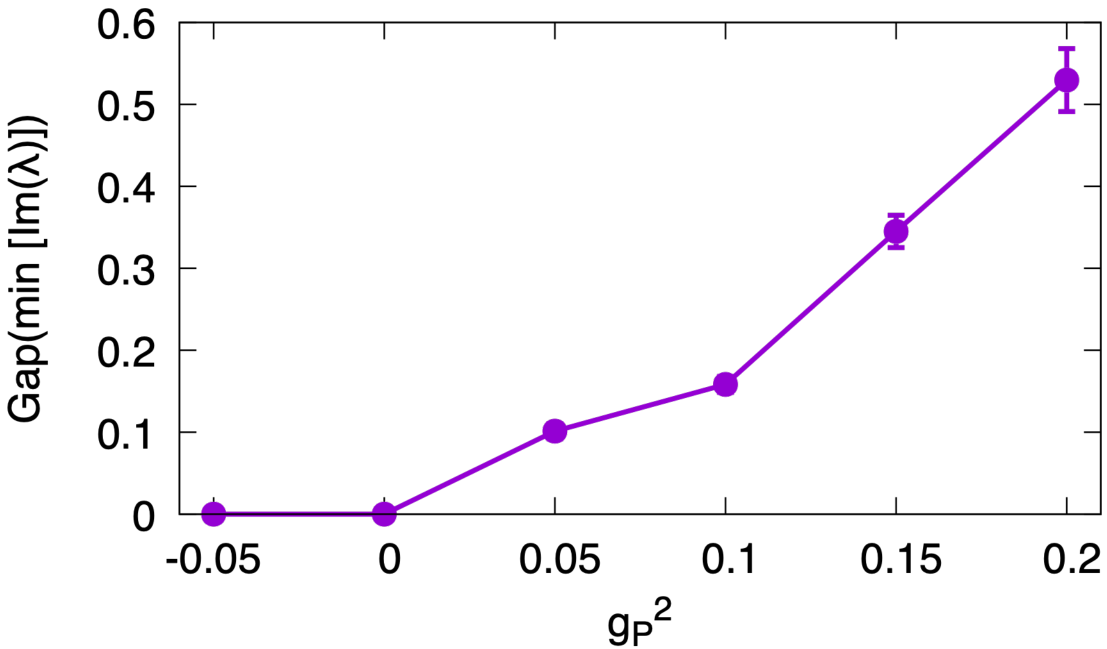


FIG. 5. Phase diagram for the transition from negative to positive m, indicating dependence on B (x axis) and A (y axis).

Yukawa coupling vs. gap

preliminary

$$\min(|\mathrm{Im}(\mathrm{eigenvalue})|)$$



Fidkowski-Kitaev 2010 Fig. 5

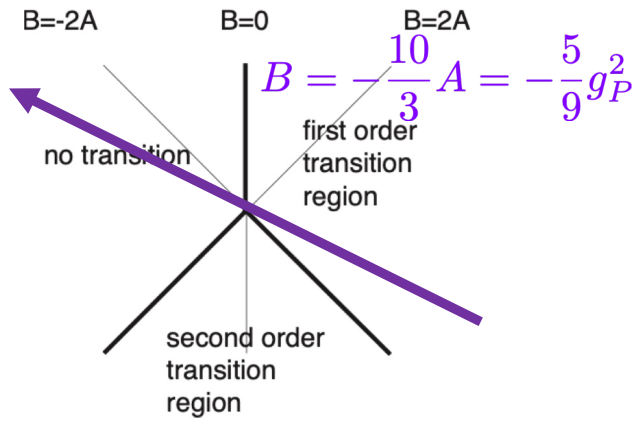


FIG. 5. Phase diagram for the transition from negative to positive m, indicating dependence on B (x axis) and A (y axis).

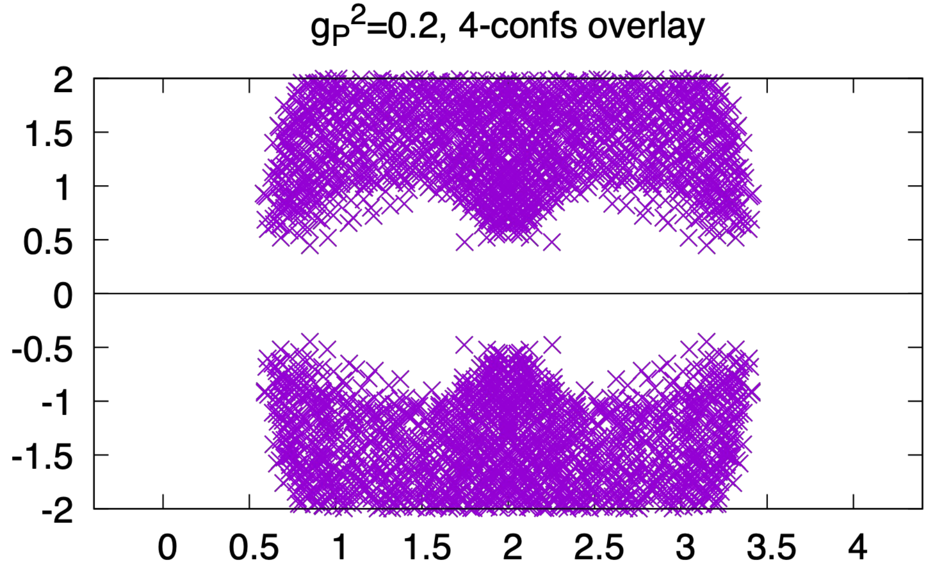
Todo

1) Large volume scaling
2) Continuum limit

Standard mass vs. SMG

$$D_W + m$$

Standard mass
= shift in the real direction



SMG = gap in the
imaginary direction

Contents

Domain-wall fermion = Majorana chain with edges

$$D = D_W + \sqrt{g_P^2} \sum_{\alpha=1}^{7} \pi^\alpha(x) i\gamma_3 \Gamma^\alpha + m\mathrm{sgn}(x - x_1)\mathrm{sgn}(x - x_2)$$

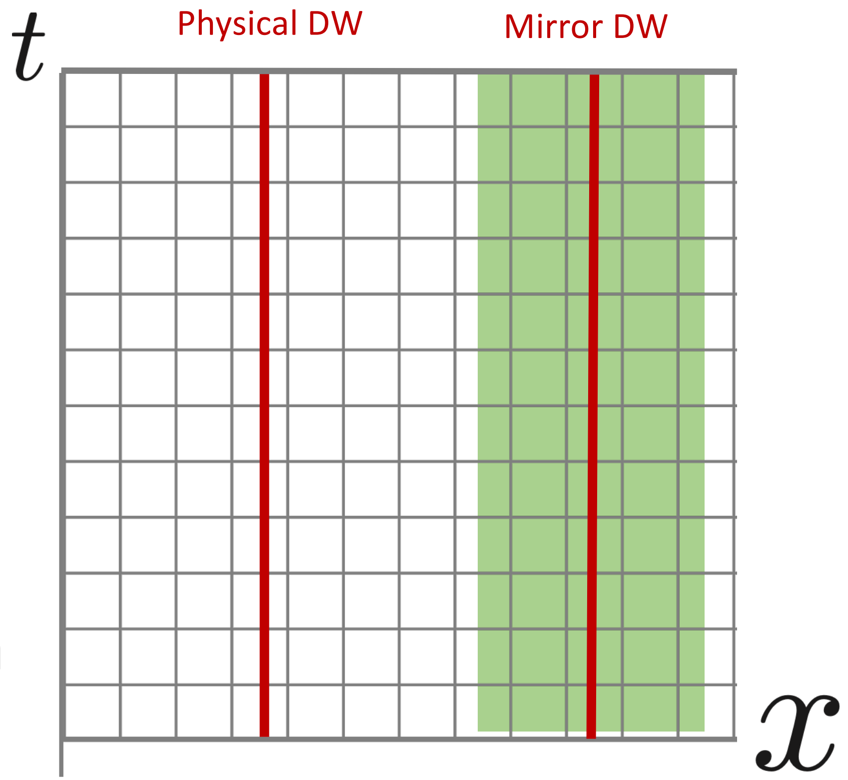
$x_1 < x < x_2$: $\nu = 8$ phase

otherwise : $\nu = 0$ phase

we set L=12 periodic lattice,

$$x_1 = 3.5, \quad x_2 = 9.5$$

and couple Yukawa interaction only at $x = 8, 9, 10, 11$



Edge modes cannot have mass term.

$$D = D_W + \sqrt{g_P^2} \sum_{\alpha=1}^{7} \pi^\alpha(x) i\gamma_3 \Gamma^\alpha + m\mathrm{sgn}(x - x_1)\mathrm{sgn}(x - x_2)$$

The edge-localized modes

$$\chi_i^{\mathrm{edge}} = \frac{1 \pm \sigma_1}{2} \exp(-m|x - x_{1,2}|)\phi_i(t)$$

cannot have the mass terms.

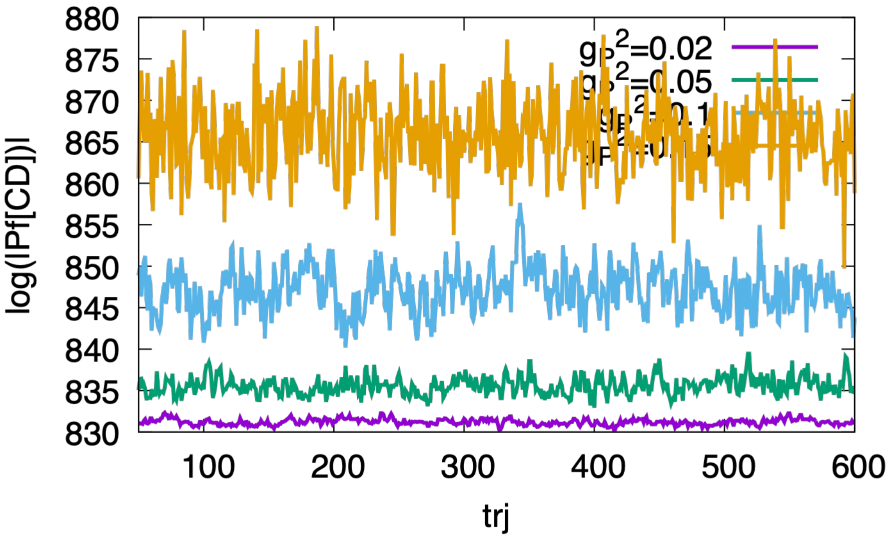
$$(\chi_i^{\mathrm{edge}})^T C m \chi_i^{\mathrm{edge}} = 0.$$

$(\chi_i^{\mathrm{edge}})^T m \chi_j^{\mathrm{edge}}$ breaks Rt symmetry.

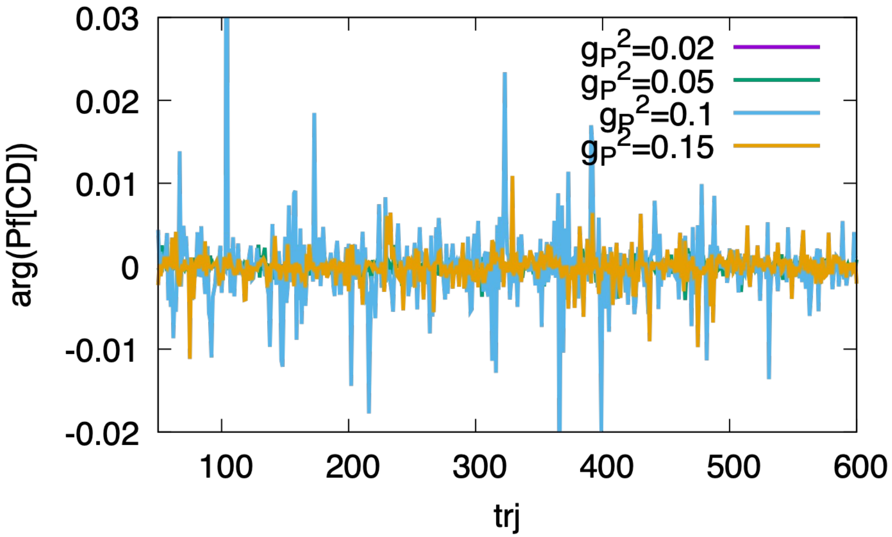
$$R_t \chi(x, t) R_t^{-1} = i\sigma_1 \chi(x, -t)$$

HMC history of Pfaffian with domain-walls

Log of |Pfaffian|



phase of Pfaffian (effect is < 1%)

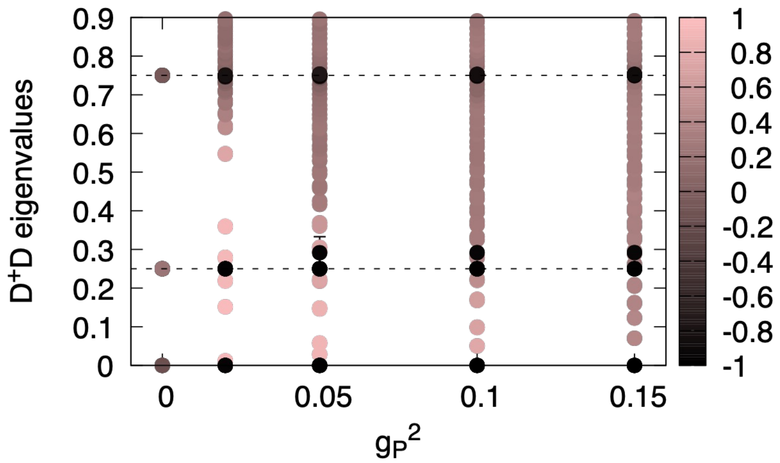


We simply ignore the phase in the preliminary analysis.

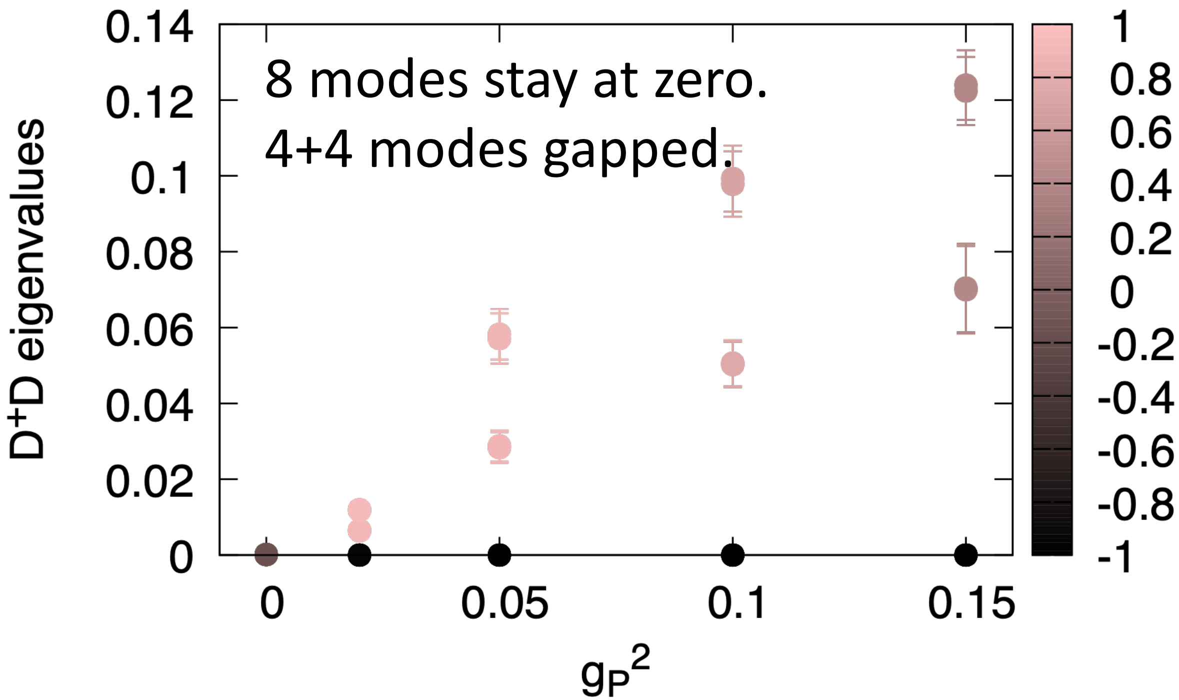
Spectrum of $D^\dagger D$

preliminary

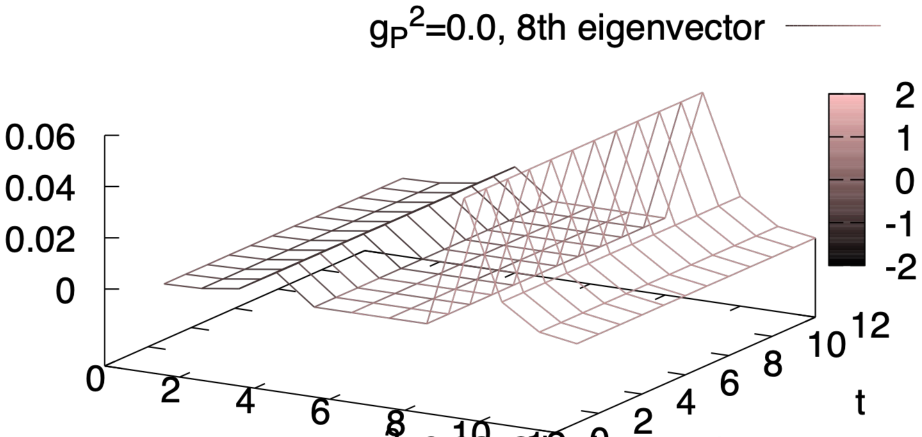
$$D = D_W + \sqrt{g_P^2} \sum_{\alpha=1}^{7} \pi^\alpha(x) i\gamma_3 \Gamma^\alpha + m\mathrm{sgn}(x - x_1)\mathrm{sgn}(x - x_2)$$

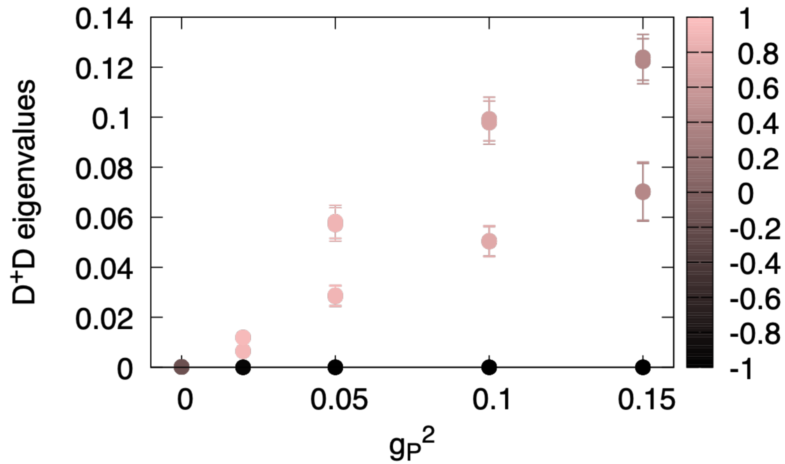


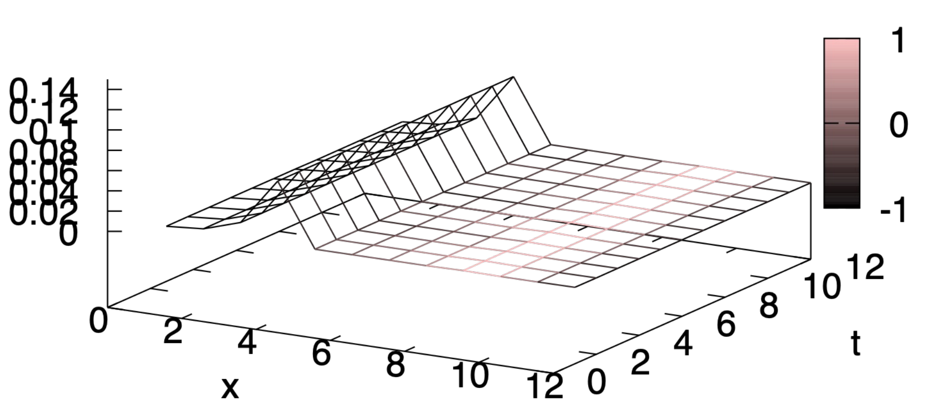
Zoomed at Near zero modes

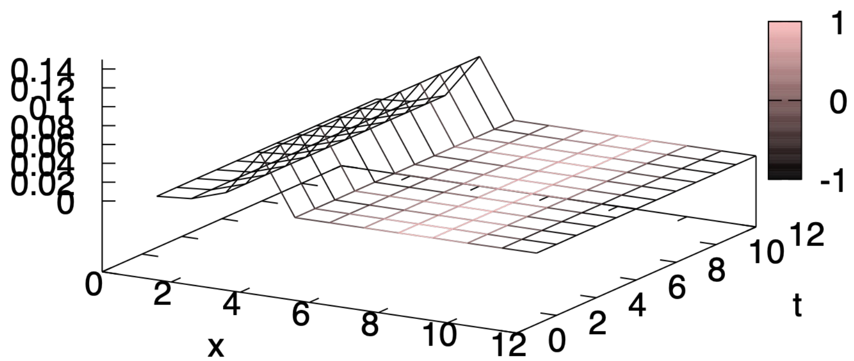


8^{th} eigenmode = physical gapless fermion (at p=0)

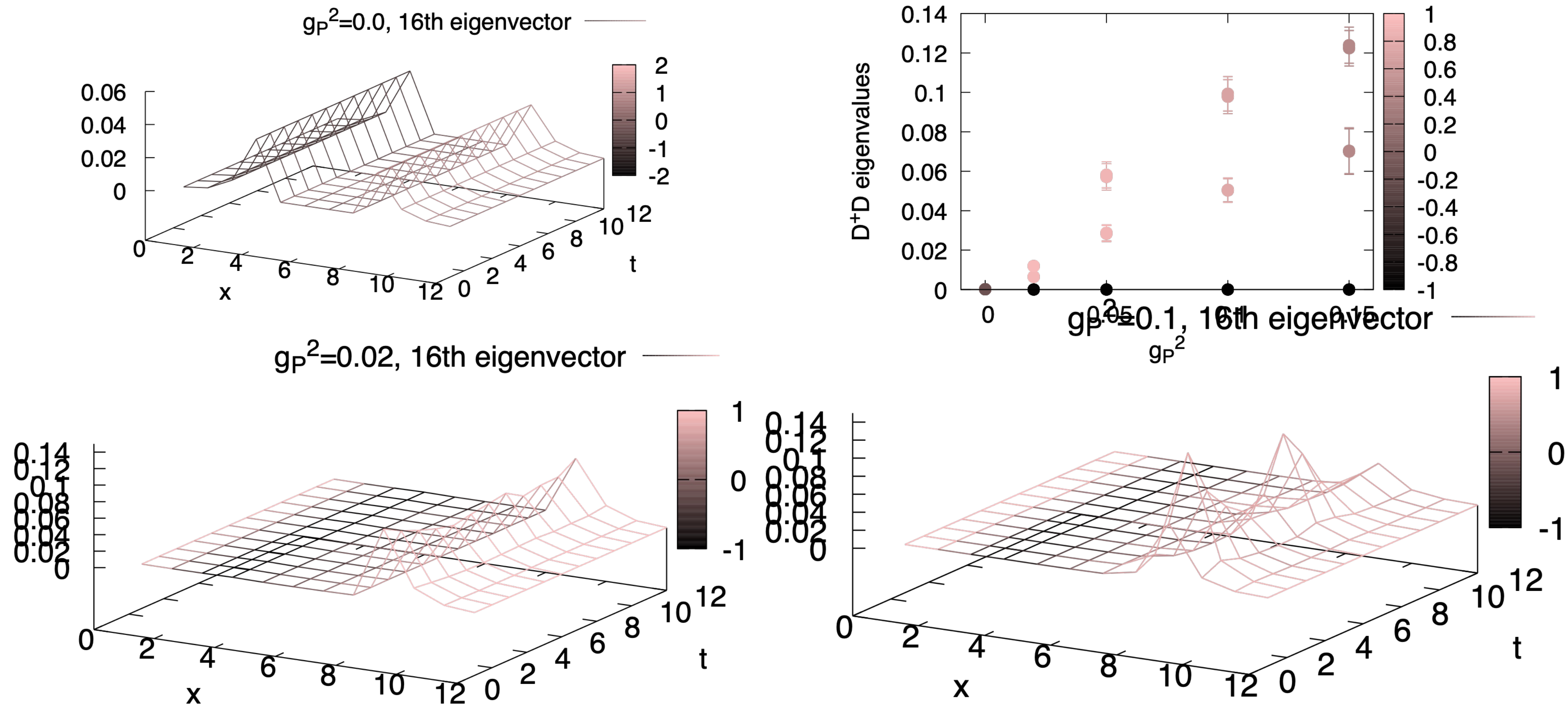






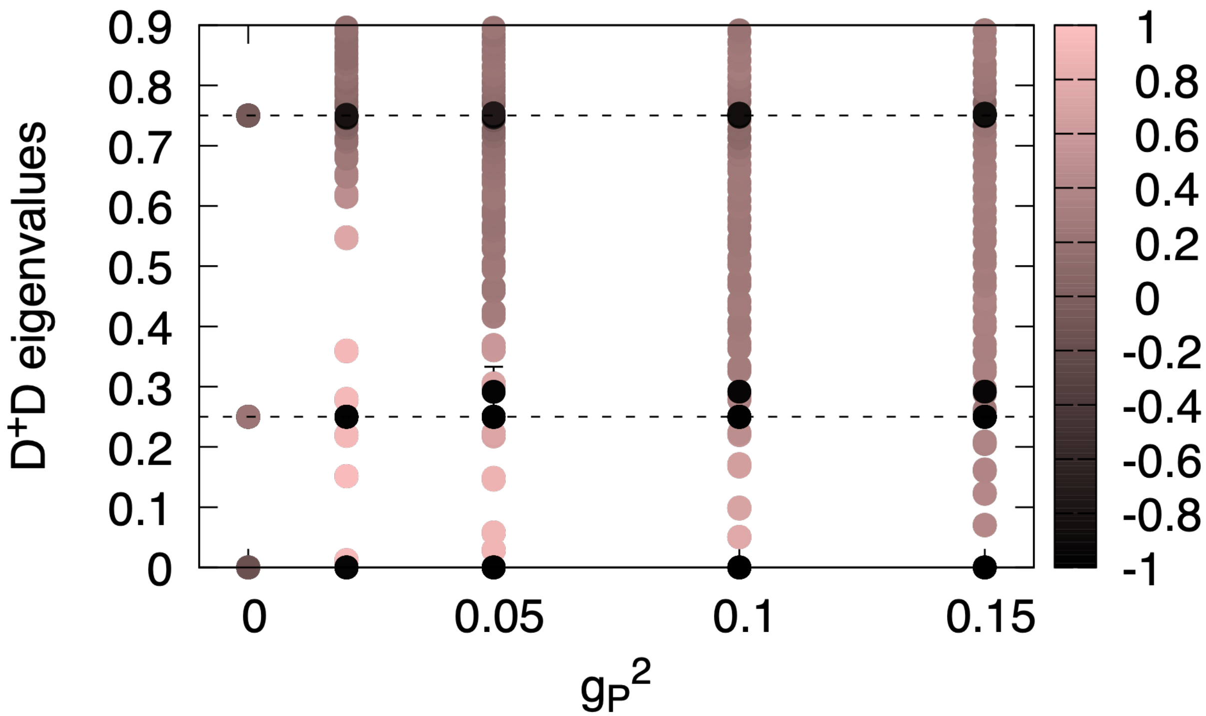


16^{th} eigenmode : mirror fermions are gapped !



Question

What determines the gap size?



Lowest modes are increasing but some higher modes are decreasing.

Can the gap increase monotonically with g_P^2?

If not, what determines the gap size?

Summary

We have simulated the path-integral version of Fidkowski-Kitaev model with and without domain-wall mass term.

1. The gap of the theory can be examined by the Dirac-Yukawa operator with Gaussian random Higgs fields.
2. Phase quenching + reweighting works (phase effect < 1%)
3. For $g_P^2 > 0$ the gap opens (mod Z_8 sectors are connected)!
4. For $g_P^2 < 0$ the gap remain closed (Aoki-phase appears?).
5. For domain-wall fermions only mirror edge modes can be gapped.

Dirac gap and mass

Assumption $D^\dagger D > M_0^2$

$$D = D_W + \sqrt{g_P^2} \sum_{\alpha=1}^{7} \pi^\alpha(x) i\gamma_3 \Gamma^\alpha + m$$

$$\begin{aligned}
\langle \chi(x)\chi(y)\rangle &= \langle x|[D^{-1}C^{-1}]|y\rangle = \langle x|[(D^\dagger D)^{-1} D^\dagger C^{-1}]|y\rangle \\
&= \langle x|[(D^\dagger D)^{-1}] \int dz|z\rangle\langle z|[D^\dagger C^{-1}]|y\rangle \\
&\sim \langle x|[(D^\dagger D)^{-1}]|y\rangle (const. + \text{local corrections})
\end{aligned}$$

$$\begin{aligned}
\langle x|[(D^\dagger D)^{-1}]|y\rangle &= \int_0^\infty du \langle x|e^{-u(D^\dagger D)}|y\rangle \\
&< C \int_0^\infty du \frac{1}{t^{D/2}} \exp\left(-uM_0^2 - |x-y|^2/4u\right) \to \exp(-M_0|x-y|).
\end{aligned}$$