Towards a nonperturbative formulation of non-Hermitian quantum physics

resonance, complex scaling, and resurgence analysis

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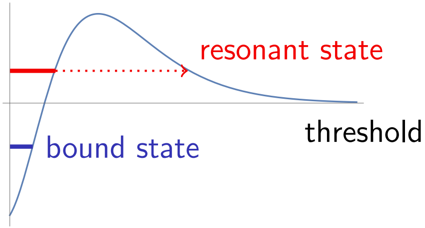
w/ Shoya Ogawa (Collaboration with nuclear physicist)

- PTEP **2025**, no.10, 103B01 (2025) [arXiv:2503.18741 [hep-th]].
- JHEP **10**, 49 (2025) [arXiv:2505.02301 [hep-th]].
- JPhysA **59**, 225306 (2026) [arXiv:2508.09211 [quant-ph]].
- JPhysA **59**, 145202 (2026) arXiv:2510.11766 [quant-ph].

Introduction to “Resonance”

- (Usual) Quasi-stationary state in quantum system
  - There are some local minima (vacua); one decays to an other.
  - E.g., double well potential, periodic cosine well, . . .
  - After decaying, finially stable ground state.
  - Eventually bound state beyond perturbation theory.
- Unstable state after decay: Resonant state



  - Simply say, plane wave in asymptotic region
    - ⋆ But, discrete $k = k_R - ik_I \in \mathbb{C}$.
  - Similar/related to scattering theory, nuclear physics, open system, and non-Hermitian QM

Resonant state and S-matrix

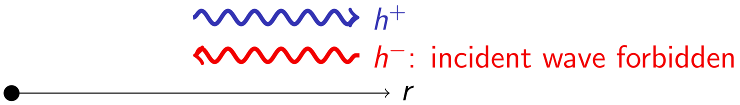
- Let φ_l be regular solution of radial Schrödinger equation
  - φ_l is subject to

$$\varphi_l(k,r) \overset{r\to 0}{\to} j_l(kr) \qquad j_l\text{: Spherical Bessel function}$$

  with angular momentum l.

- At $r \to \infty$, linear combination of Hankel functions h_l:

$$\varphi_l(k,r) \to \mathcal{F}_l(k) h_l^-(kr) - \mathcal{F}_l^*(k) h_l^+(kr) \quad \mathcal{F}_l\text{: Jost function}$$



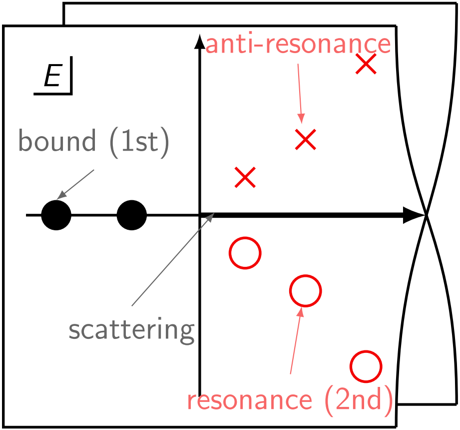
- S-matrix is written in terms of Jost function

$$S_l(k) = \frac{\mathcal{F}_l^*(k)}{\mathcal{F}_l(k)} \to \infty \qquad \text{pole singularity}$$

- Some kind of scattering of multiple particles

Distribution of each state

- Riemann surface of complex E-plane



- Complex energy $E = E_r - i\frac{\Gamma}{2}$ in the 4th quadrant (Resonant energy E_r, decay width Γ)

Subtleties of resonance physics

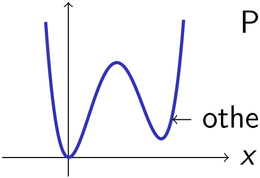
- Resonance appears quite universally!
  - cf. Black hole physics [Miyachi–Namba–Omiya–Oshita '25]
  - Non-normalizable (divergence of norm)
  - Completeness? What is the wave function itself?
- We aim to understand it in not only math but also phys senses

- There is a large gap between phenomenological success & math
  - Complex scaling method: $r \to re^{i\theta}$ for large enough θ.
  - Mathematically well-defined by Aguilar–Balslev–Combes (ABC) theorem [AC '71, BC '71] (as I will explain later).
  - Physics depends on θ explicitly, but computationally successful.
- Quite well-established definition/regularization (historically [$>$ 50years], experimentally, monumentally)

- To make it transparent, consider the complex-x plane
- Focus on analytic solutions of diff eq w.r.t. $x \in \mathbb{C}$

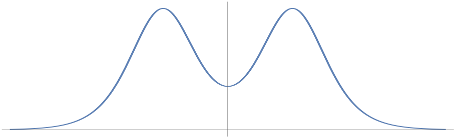
Resurgence approach to quasi-stationary states

- E.g., for double well potential, perturbation theory suffers from



Perturbative vac decays w/ decay rate Γ
$\rightarrow$Resurgent asymptotic series

- Complex energy of resonance: $E = E_r - i\frac{\Gamma}{2}$
  (Resonant energy E_r, decay width Γ)

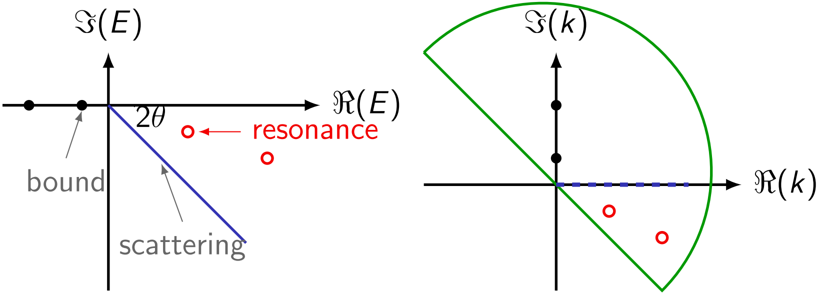


E.g., $V(x) = \frac{U_0}{\cosh^2 \beta(x-a)} + \frac{U_0}{\cosh^2 \beta(x+a)}$

- Can resurgence theory overcome difficulties as non-perturbative formulation of QM?
- If so, *essence in "$\forall$QM" = analyticity in resurgence*
  - Quite transparent and precise!!!

Review I: Prescription of complex scaling

- Complex scaling method: Radial direction $r \to re^{i\theta}$ ($k \to ke^{-i\theta}$)



- ABC theorem [See Myo–Kikuchi–Masui–Katō '14]
  1. Resonant wavefunctions are square-integrable.
  2. Bound/resonant *energy* is independent from θ.
  3. Continuous spectrum is rotated by 2θ.
- Deformation of integral contour of Green func.
  - It includes bound, continuous, and resonant states.
  - $\sum_B |\psi_B^\theta\rangle\langle\tilde{\psi}_B^\theta| + \sum_{R_\theta} |\psi_R^\theta\rangle\langle\tilde{\psi}_R^\theta| + \int_{S_\theta} |\psi_S^\theta\rangle\langle\tilde{\psi}_S^\theta| = 1$ (not $*$-ring?)

Review II: Complexification as non-pert. method

- Rough sketch:

  Perturbation theory in quasi-stationary system provides asymptotic series growing factorially, $\sum_n n!\lambda^n$.

1. Computation with complex coupling $\lambda \to e^{\pm i\epsilon}\lambda$
   - Resummation technique $\to$ convergent if $\epsilon > 0$
   - Ambiguity with $\epsilon \to 0 = \pm iO(e^{-1/\lambda})$: non-perturbative
2. Steepest descent around saddle point $x \in \mathbb{C}$ (Lefschetz thimble)
   - $\underbrace{\text{(pert.)}}_{\text{finite}\pm\text{ambiguity}} \mp \underbrace{e^{-1/\lambda}}_{\text{nontrivial saddle}} \text{(pert.)} + \ldots$

- Resurgent (no ambiguity)

- Available toolkit here: Exact WKB analysis [Voros]

More issue: inverted Rosen–Morse potential

- Potential giving resonance

$$V(x) = \frac{U_0}{\cosh^2 \beta(x-a)} + \frac{U_0}{\cosh^2 \beta(x+a)}$$
$$\downarrow$$
$$V(x) = U_0/\cosh^2 \beta x$$

- Exact solution of Schrödinger equation is given by

$$\psi(x) = (1-\xi^2)^{-\frac{ik}{2\beta}} F\left(-\frac{ik}{\beta} - s, -\frac{ik}{\beta} + s + 1, -\frac{ik}{\beta} + 1, \frac{1-\xi}{2}\right)$$

F: Gauss hypergeometric function, $\xi = \tanh \beta x$, $k = \sqrt{2mE}/\hbar$,

$s = \frac{1}{2}(-1 + \sqrt{1 - 8mU_0/\beta^2\hbar^2})$

- Resonant energy

$$E_n^R = \frac{\hbar^2\beta^2}{8m}\left[\sqrt{\frac{8mU_0}{\beta^2\hbar^2} - 1} - i(2n+1)\right]^2$$

- Barrier resonance: localized state at potential bump (?)
  - Not intuitively clear why this is happening.

WKB ansatz

- Consider a Schrödinger equation

$$\left(-\frac{d^2}{dx^2} + \hbar^{-2} Q(x)\right)\psi = 0, \qquad Q(x) = 2[V(x) - E]$$

- Introduce WKB ansatz as a formal power series

$$\psi(x,\hbar) = e^{\int^x S(x',\hbar)dx'}, S(x,\hbar) = \frac{1}{\hbar} S_{-1}(x) + S_0(x) + \hbar S_1(x) + \dots,$$

and substitute this into the equation; we have a recursive equation of S_i: Riccati equation

$$S^2 + \frac{dS}{dx} = \hbar^{-2} Q.$$

- We have two solutions because of the leading-order equation

$$S_{-1}^2 = Q \qquad \Rightarrow \qquad S_{-1} = S_{-1}^{\pm} \equiv \pm\sqrt{Q}.$$

Solutions are $\psi^{\pm} = e^{\int S^{\pm}} \sim e^{\pm \frac{1}{\hbar}\int \sqrt{Q}}$.

Borel resummation

- Borel resummation: summing divergent asymptotic series

$$f(\lambda) \sim \sum_{k=0}^{\infty} f_k \lambda^{k+1} \qquad \text{with } f_k \sim a^k k! \text{ as } k \to \infty$$

$\Downarrow$ Borel transform

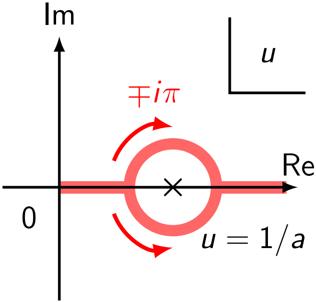
$$B(u) \equiv \sum_{k=0}^{\infty} \frac{f_k}{k!} u^k = \frac{1}{1-au} \qquad \text{(Pole singularity at } u = 1/a\text{).}$$

- The Borel sum is given by

$$f(\lambda) \equiv \int_0^{\infty} du\, B(u) e^{-u/\lambda}.$$

- $a < 0$ (alternating series) $\to$ convergent
- $a > 0 \to$ ill-defined due to the pole

$\Rightarrow$ Imaginary ambiguity $\sim \pm e^{-1/(a\lambda)}$

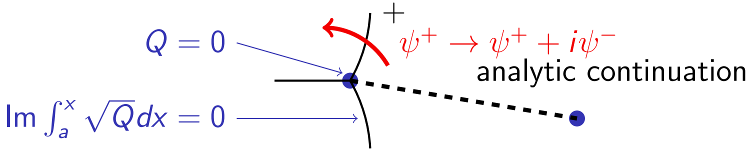


Stokes phenomena, Stokes geometry

- WKB ansatz is also asymptotic series; where is it Borel summable?
- Integral contour includes $\mp \int^x dx' \sqrt{Q}$ for $\psi^\pm$

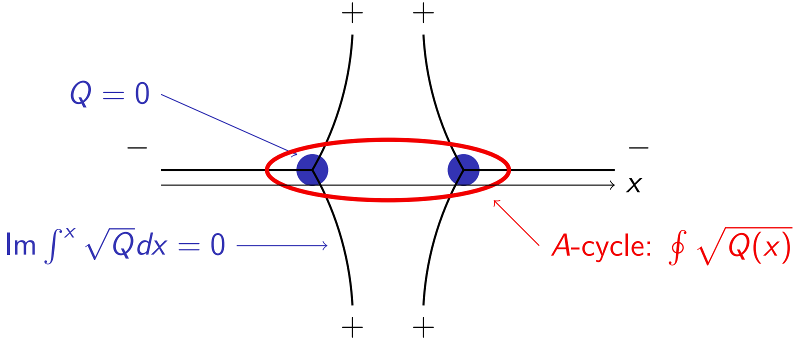
$$\text{Stokes curve}: \left\{ x \middle| \mathrm{Im} \int_a^x dx' \sqrt{Q} = 0, \text{where } Q(x=a)=0 \right\}$$

  - a: turning point.
  - Not on Stokes curve, Borel summable (analytically continuable).
  - Across a Stokes curve, solution suddenly changes.
  - $\psi^\pm$ is dominant with $\mathrm{Re} \int_a^x dx' \sqrt{Q} \gtrless 0$.



Quantization condition from normalizability

- (Usual) QM: normalizable from $x \to -\infty$ to $x \to \infty$
  - E.g., Harmonic oscillator

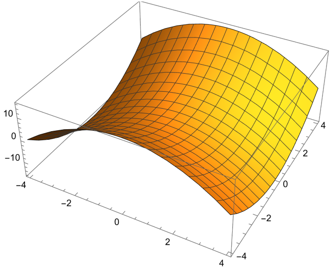


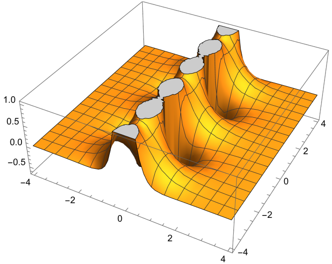
  - $\psi_{\text{final}} = (\text{analytic continuation, Stokes pheno}) \times \psi_{\text{initial}}$
  - Non-normalizable solution vanishes: $\psi_{\text{initial}}^{\text{damp}} \not\to \psi_{\text{finial}}^{\text{grow}}$

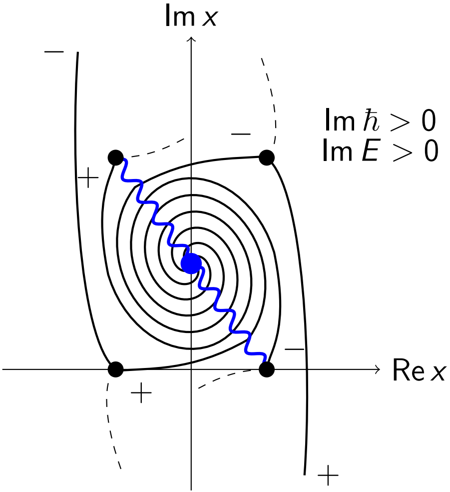
$$1 + A = 1 + e^{\hbar^{-1} \oint \sqrt{Q}} = 0 \qquad \Rightarrow \qquad E = \hbar \left(n + \frac{1}{2} \right)$$

How to see resonant state from exact WKB

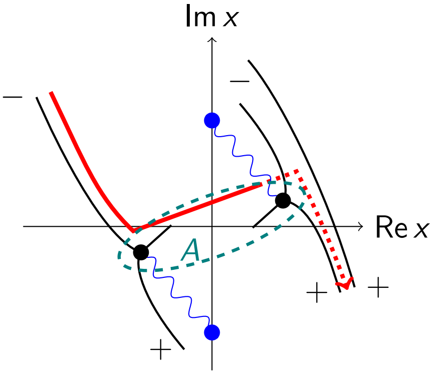
- Resonant state: quasi-stable in complex region
  - $V = -x^2$ vs $V = 1/\cosh^2 x$







Leading order estimate (Rigorous one in next slide)



- Normalizable path:
- Barrier resonant energy from exact WKB analysis
  - A-cycle

$$\int_{-\cosh^{-1}\sqrt{\frac{1}{E}}}^{\cosh^{-1}\sqrt{\frac{1}{E}}} \sqrt{2\left(\frac{1}{\cosh^2 x} - E\right)} = \sqrt{2}\pi(1+\sqrt{E})$$

quantization condition: $1 - A = 0 \qquad E = \left(1 - \frac{in}{\sqrt{2}}\right)^2$

Backup: Rigorous estimate = exact solution

- $A = 1$:

$$\left[\frac{F\left(-\frac{ik}{\beta}-s,-\frac{ik}{\beta}+s+1,-\frac{ik}{\beta}+1,\frac{1-|\xi|}{2}\right)}{F\left(-\frac{ik}{\beta}-s,-\frac{ik}{\beta}+s+1,-\frac{ik}{\beta}+1,\frac{1+|\xi|}{2}\right)}\right]^2 = 1,$$

where $|\xi| = \tanh(\beta \cosh^{-1}\sqrt{U_0/E})$.

- Here, we use the formula of the hypergeometric function as

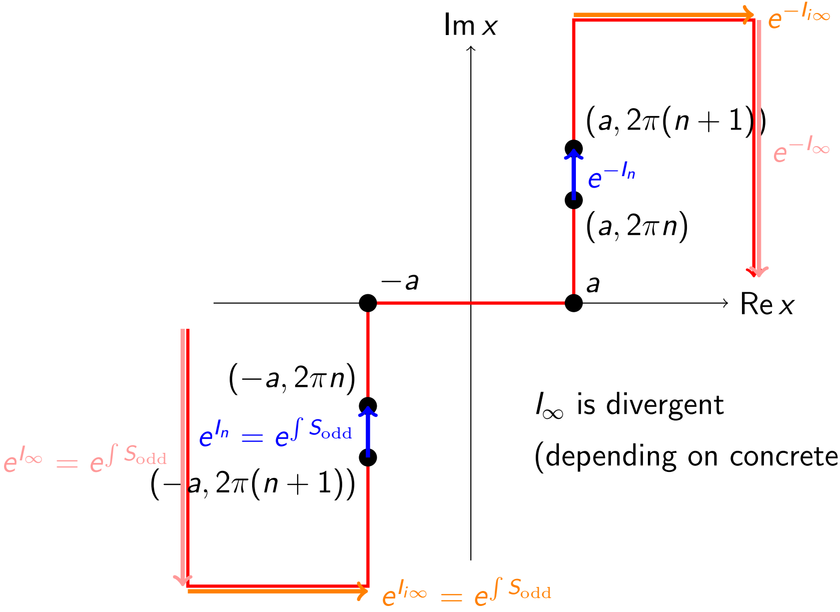
$$F\left(-\frac{ik}{\beta}-s,-\frac{ik}{\beta}+s+1,-\frac{ik}{\beta}+1,\frac{1\pm|\xi|}{2}\right)$$
$$=\mathfrak{A}F\left(-\frac{ik}{2\beta}-\frac{s}{2},-\frac{ik}{2\beta}+\frac{s+1}{2},\frac{1}{2},|\xi|^2\right)\mp\mathfrak{B}F\left(-\frac{ik}{2\beta}-\frac{s-1}{2},-\frac{ik}{2\beta}+\frac{s}{2}+1,\frac{3}{2},|\xi|^2\right),$$

where

$$\mathfrak{A} = \frac{\Gamma\left(-\frac{ik}{\beta}+1\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{ik}{2\beta}-\frac{s-1}{2}\right)\Gamma\left(-\frac{ik}{2\beta}+\frac{s}{2}+1\right)}, \qquad \mathfrak{B} = \frac{\Gamma\left(-\frac{ik}{\beta}+1\right)\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(-\frac{ik}{2\beta}-\frac{s}{2}\right)\Gamma\left(-\frac{ik}{2\beta}+\frac{s+1}{2}\right)}.$$

- $\mathfrak{A} = 0$, $\mathfrak{B} = 0$: odd/even number of nodes

What is problematic in naive path?



I_∞ is divergent

(depending on concrete path)

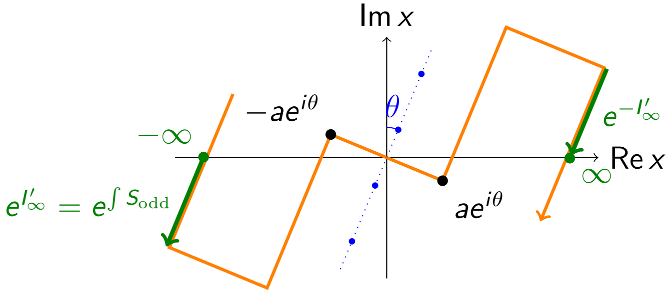
Regularization schemes

- Zel'dovich transformation: $S = e^{-\epsilon x^2}$ ['61, Berggren '68]

$$H\psi = E\psi \to H_S\psi_S = E\psi_S, \qquad \psi_S \equiv S\psi,\ H_S \equiv SHS^{-1}.$$

  - We find $H_S = H - \frac{\hbar^2\epsilon}{m}\left(\frac{d}{dx}x + x\frac{d}{dx}\right)$
  - Now, $e^{I_\infty} \to e^{\int(S_{\rm odd} - 2\epsilon x)}$ becomes finite.

- Complex scaling by θ rotates the Stokes graph as



  - Now, $e^{I_\infty} \to e^{I'_\infty}$ becomes finite.

ABC theorem

1. Resonant wavefunctions are square-integrable.
   - Yes, if the path connects $\mathrm{Im}\, x \pm \infty$ or is regularized. It straightforwardly leads to discrete and complex energies.
2. Bound/resonant energy is independent from θ.
   - Potential and contour are rotated simultaneously; A is invariant.
3. Continuum spectrum is rotated by 2θ.
   - To keep Siegert b.c. (e^{ikx}), $k \to e^{-i\theta} k$ when $x \to e^{i\theta} x$.
   - Due to $k = \frac{\sqrt{2mE}}{\hbar}$, $E \to e^{-2i\theta} E$.

Backup: Rigged Hilbert space with resonance

- We introduce

$$\mathcal{D}_\varepsilon^{\mathrm{R}} = \{x \in \mathbb{C} \,|\, \varepsilon > 0, \lim_{r \to \pm\infty} |x - r| < \varepsilon\}$$

  - $\mathcal{D}_\varepsilon^{\mathrm{R}}$ is the most crucial singular region.
- if $x \in \mathbb{C} \setminus \mathcal{D}_\varepsilon^{\mathrm{R}}$, the Hilbert space is well-defined, $\mathcal{H}_\varepsilon$
  - Def. of norm and inner product is quite different!
  - Resonance is included as "bound" state.
- (Operator algebra) Set of operators $\{A_i\}$ where A_i is defined on $D(A_i) \subset \mathcal{H}_\varepsilon$; then let us introduce the dense subspace

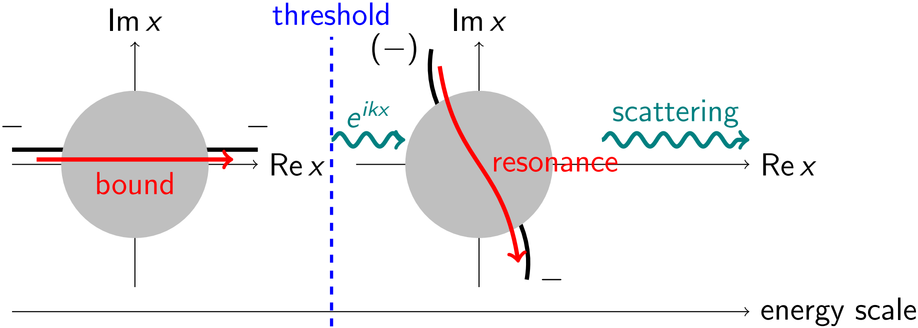
$$\Phi \equiv \cap_i D(A_i) \subseteq \mathcal{H}_\varepsilon.$$

- The range is defined by the limit as $\varepsilon \to 0$, say, $\Phi^\times$
- We find the Gelfand multiplet

$$\Phi \subseteq \mathcal{H}_\varepsilon \subset \Phi^\times,$$

and $(\mathcal{H}_\varepsilon, \Phi)$ is the rigged Hilbert space.

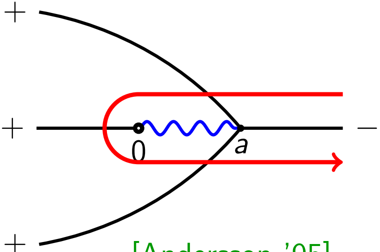
Scattering theory



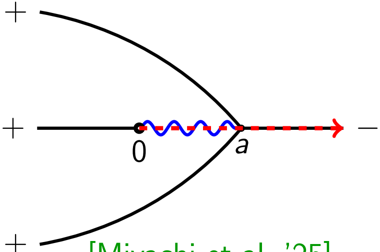
- Below threshold: Bound states
- Above threshold: Resonance & continuum spectra
  - Read transmission/reflection coefficient from monodromy&analytic continuation
  - Transmission coefficient is divergent if E comes across resonance.

Path choice independence in 3D radial equation

- Coulomb Potential $-\frac{e^2}{r}$: Closed loop vs. open path



[Andersson '95]



[Miyachi et al. '25]

  - $r < 0 \leftrightarrow$ Black Hole: closed path inside event horizon.
- Actually, same phenomana occur!
  - Aoki–Kawai–Takei work: taking a local (regular) basis near origin, where path starts at $|r| = \epsilon > 0$ in lateral Borel direction with appropriate small-circle monodromy.
  - Basis = connected subdominant wavefunction along another lateral Borel direction.

Summary

- We develop a unified framework for analyzing quantum mechanical resonances using exact WKB method
  - Serving as a non-perturbative formulation of QM.
  - Incorporating the Zel'dovich regularization, the complex scaling method, and the rigged Hilbert space.
  - Demonstrated by examining inverted Rosen–Morse potential.
- A prelude to non-Hermitian QM
  - Non-unitary similarity transformation of self-adjoint H.
  - Large gap between phys and math.
  - Berry phase of complex-scaled resonance as exceptional point [OM–Ogawa–Onoda '25].
  - CSM to black hole QNMs [Ogawa–Hirose–OM '26a, '26b].
- Future works:
  - Numerical approach.
  - Higher dim (or QFT)???